Majorana modes in topological superconductors

Roman Lutchyn

Microsoft Station Q
• Introduction to topological quantum computing
• Physical realizations of Majorana modes in topological superconductors
• Experimental signatures of Majorana modes
• Quantum computing with Majoranas

Literature:

C. Nayak et al., Rev. Mod. Phys. 80, 1083 (2008)
J. Alicea, Rep. Prog. Phys. 75, 076501 (2012);
B. A. Bernevig, Topological Insulators and Topological Superconductors
S. Das Sarma, M. Freedman, & C. Nayak, Quantum Information 1, 15001 (2015)
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Math + Physics + Computer Science

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Our approach

- **Explore:** Mathematics, Physics, Computer Science, and Engineering required to build and effectively use quantum computers

- **General approach:** Topological quantum computation

- We coordinate with experimentalists and other theorists at:
  - Bell Labs
  - Caltech
  - Columbia
  - Delft
  - ETH
  - Harvard
  - Niels Bohr Institute
  - Princeton
  - Stanford
  - University of Chicago
  - University of Maryland
  - University of Illinois
  - University of California Santa Barbara
  - Weizmann Institute
“A quantum computer – a new type of machine that exploits the quantum properties of information – could perform certain types of calculations far more efficiently than any foreseeable classical computer” – J. Preskill (Caltech).

Many laboratories around the world are trying to harness the power of quantum mechanics!
Why is QC more powerful?

• Superposition

A (classical) **bit** is given by a physical system that can exist in one of two distinct states: 0 or 1

A **qubit** is given by a physical system that can exist in a linear combination of two distinct quantum states: $|0\rangle$ or $|1\rangle$

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

$\alpha, \beta \in \mathbb{C}$

$|\alpha|^2 + |\beta|^2 = 1$

• Entanglement

Quantum states need not be separable. For example:

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}} (|0_A0_B\rangle + |1_A1_B\rangle)$$

$$\neq |\psi_A\rangle \otimes |\phi_B\rangle$$

This is the property that enables quantum state teleportation and “spooky action at a distance.”
Quantum Parallelism

Computations can be performed in superposition!

1 qubit

\[ |\psi_i\rangle = \alpha |0\rangle + \beta |1\rangle \]

Logical gate (Unitary evolution)

\[ |\psi_f\rangle = \alpha' |0\rangle + \beta' |1\rangle \]

2 qubits

\[ |\psi_i\rangle = c_{00} |00\rangle + c_{10} |10\rangle + c_{01} |01\rangle + c_{11} |11\rangle \]

The same clock cycle!

\[ |\psi_f\rangle = c'_{00} |00\rangle + c'_{10} |10\rangle + c'_{01} |01\rangle + c'_{11} |11\rangle \]

Use the exponential advantage of \(2^n\) over \(n\) in going from \(n\) ‘bits, classical 2-level system, to \(n\) ‘qubits, the quantum 2-level system:

EXPOENTIAL SPEED-UP!
A qubit = quantum two-level system

\[ H = \vec{B}(t) \cdot \vec{S} \]

Energy splitting - \( B_z \)

Rotation about \( x \)-axes - \( B_x(t) \)

\( |0\rangle, |1\rangle \) are eigenstates of \( S_z \)

\[ |\Psi\rangle = |a||0\rangle + |b|e^{i\phi}|1\rangle \]

• Bloch vector
Experimental realization of qubits

Microscopic degrees of freedom

- ion trap
- diamond-based
- silicon-based

Collective degrees of freedom

- superconducting qubits
- topological qubits
Quantum Decoherence

We would like to use a natural resource - many-body Hilbert space. However, quantum phenomena do not occur in a Hilbert space. They occur in a laboratory!

Quantum decoherence – the loss of information from a system into the environment.

Classical computers: store multiple copies
Quantum computers: situation is more complex

Nevertheless, quantum error correction is theoretically possible but VERY HARD in practice – error rate $< 10^{-4}$
Topological phases of matter have topological degrees of freedom which **decouple** from local operations.

Quantum information encoded in topological degrees of freedom is “**topologically protected**” from noise (errors are exponentially suppressed).
Examples of topological phases

Quantized Hall conductivity:

\[ J_y = \sigma_{xy} E_x \]
\[ \sigma_{xy} = n \frac{e^2}{h} \]

Integer accurate to \(10^{-9}\)
“Examples” of non-Abelian Topological Phases

2D $p_x + i p_y$ Paired Systems:

- Moore-Read “Pfaffian” Fractional Quantum Hall state
  Moore & Read (1991)
- $p_x + i p_y$ superfluids and superconductors
  Read & Green (2000); Volovik (2000)
Topological quantum computation depends on

- the existence of non-Abelian topological phase
- the ability to manipulate quasiparticle excitations (anyons) in these phases

Candidate systems:

Fractional QH states  topological superconductors  spin systems
Dimensionality and Quantum Statistics

Quantum statistics: behaviour of $\Psi(x_1, x_2, \ldots)$ under exchange of particles

Abelian anyons

$$\psi(x_2, x_1) = e^{i\pi \theta} \cdot \psi(x_1, x_2)$$

fractional phase

Non-Abelian anyons

$$\psi(x_1 \leftrightarrow x_3) = M \cdot \psi(x_1, \ldots, x_n)$$
$$\psi(x_2 \leftrightarrow x_3) = N \cdot \psi(x_1, \ldots, x_n)$$

In general $M$ and $N$ do not commute!

In 3+1 dimensions, the only statistics are

$$T^2 = 1 \quad \text{bosonic/fermionic} \quad \Psi_a \rightarrow \pm \Psi_a$$

In 2+1 dimensions, the situation is different:

$$T \neq T^{-1} \quad \text{Wilczek'82}$$
$$\text{Leinaas & Myrheim'77}$$

Abelian statistics

$$\Psi_a \rightarrow e^{i\theta} \Psi_a$$

example: QH state $\nu = \frac{1}{3}$

non-Abelian statistics

$$\Psi_a \rightarrow T_{ab} \Psi_b$$

example: QH state $\nu = \frac{5}{2}$

Solid-state Majoranas obey non-Abelian statistics
Solid-state Majoranas

Consider creation/annihilation operators for a spinless fermion

\[ c^\dagger \quad c \quad \{ c, c^\dagger \} = 1 \]

These can be rewritten in terms of Majorana fermion operators

\[ a_1 = a_1^\dagger = c + c^\dagger \]
\[ a_2 = a_2^\dagger = i(c^\dagger - c) \]

What happens if we have many Majoranas?
Quantum computation with Majoranas

Dirac fermion operator \( c = a_1 + i a_2 \)

Two degenerate states \( |0\rangle \) and \( c^\dagger |0\rangle \) \( \rightarrow \) 1 qubit

\( 2N \) Majoranas = \( N \) qubits

\( \text{Kitaev’03} \)

4 degenerate states

\( c_A = a_1 + i a_2 \) \( \rightarrow \) \( |0_A, 0_B\rangle \), \( |0_A, 1_B\rangle \)

\( c_B = a_3 + i a_4 \) \( \rightarrow \) \( |1_A, 0_B\rangle \), \( |1_A, 1_B\rangle \)

Quantum state changes!

\( |0_A, 0_B\rangle \rightarrow \frac{1}{\sqrt{2}} (|0_A, 0_B\rangle + |1_A, 1_B\rangle) \)
In 1938 one of the world's greatest scientists withdrew all his money and disappeared during a boat trip from Palermo to Naples. Whether he killed himself, was murdered or lived on under a different identity is still not known.

Kouwenhoven’s group is trying to find Majorana in the lab.
Proposed physical realizations of solid-state Majoranas

\[ \nu = \frac{5}{2} \text{ FQHE} \]

2D \( p_x + ip_y \) SC

TI/SC

chiral p-wave \( \Delta(p) = \Delta_0 (\hat{p}_x + i\hat{p}_y) \)

SM/SC

RKKY systems
BCS Theory of Superconductivity

Key idea:
- Electrons form pairs (Cooper pairs)

\[ H_{BCS} = \sum_k \xi_k c_k^{\dagger} c_k + \sum_{k,k'} V_{k,k'} c_k^{\dagger} c_{-k}^{\dagger} c_{-k'} c_{k'} \]

- Triplet OP \( \Delta_k = -\sum_{k'} V_{k,k'} \langle c_{k'} c_{-k'} \rangle \)  
  assume \( V_{k,k'} = -V_0 e^{i(\phi_k - \phi_{k'})} \)

Solution \( \Delta_k = e^{i\phi_k} \sum_{k'} V_0 e^{i\phi_{k'}} \langle c_{k'} c_{-k'} \rangle = \Delta_0 (k_x + ik_y) \)

\[ H_{MF} = \sum_k \left( \xi_k c_k^{\dagger} c_k + \Delta_k c_k^{\dagger} c_{-k}^{\dagger} + \Delta_k^{*} c_{-k} c_k \right) \]
Bogoliubov-de-Gennes formalism

\[ H_{\text{MF}} = \frac{1}{2} \sum_k \Psi_k^\dagger H_{\text{BdG}}(k) \Psi_k \]

where \( \Psi_k = (c_k, c_{-k}^\dagger) \)

Bogoliubov-de-Gennes Hamiltonian

\[ H_{\text{BdG}} = \begin{pmatrix} h_0 & \Delta \\ \Delta^\dagger & -h_0^T \end{pmatrix} \]

Bogoliubov transformation

\[ \gamma_k = u_k c_k + v_k c_{-k}^\dagger \]

Built-in anti-unitary particle-hole symmetry \( \Theta = \tau_x K \)

\[ \Theta H_{\text{BdG}}(k) \Theta^{-1} = -H_{\text{BdG}}(-k) \]

which relates positive and negative energy solutions

\[ \Psi_{-E} = \Theta \Psi_E \]
Superconductors are natural hosts for Majoranas

Bogoliubov quasiparticle

\[ \gamma_n = \int dx \left( u_n(x)\psi(x) + v_n(x)\psi^\dagger(x) \right) \]

equal superposition of a particle and a hole

\[ u_0 = v_0^* \]

Majorana fermion

\[ a = a^\dagger \]

Look for **ZERO** energy states!

\[ \begin{array}{c}
E = 0 \\
E = 2\Delta_0
\end{array} \]

Empty 

Occupied

Bound states in vortices

Midgap states at the interfaces
Topological protection of zero-energy mode

Bogoliubov-de-Gennes equations

\[
\begin{pmatrix}
 h_0 & \Delta \\
 \Delta^\dagger & -h_0^T
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
= E
\begin{pmatrix}
u \\
v
\end{pmatrix}
\]

Particle-hole symmetry:

If \( \begin{pmatrix}
u \\
v
\end{pmatrix} \) is a solution with \( E \)

If \( \begin{pmatrix}
v^* \\
u^*
\end{pmatrix} \) is a solution with \(-E\)

For spinless fermions particle-hole symmetry guarantees Majorana mode at \( E = 0 \)

Two topological classes of BdG Hamiltonians

2N + 1 solutions

2N solutions

\( E \)

\( E = 0 \)

\( -E \)

Read and Green, PRB’00

This argument breaks down if levels are spin-degenerate
Toy model: 1D Majorana wire

Spinless fermion with p-wave pairing

\[ H = -\mu \sum_{j=1}^{N} c_j^\dagger c_j - \sum_{j=1}^{N-1} (tc_j^\dagger c_{j+1} + |\Delta|e^{i\phi} c_j c_{j+1} + h.c.) \]

Kitaev (2000)

Two topologically distinct phases:

**trivial**: \( t = 0 \) and \( \Delta = 0 \) and \( \mu < 0 \)

\[ c_j = \gamma_{j,A} + i\gamma_{j,B} \]

**non-trivial**: \( \mu = 0 \) and \( |t| = |\Delta| \)

\[ H = it \sum_{j=1}^{N-1} \gamma_{B,j}\gamma_{A,j+1} \]

GS degeneracy:

\( i\gamma_{1,B}\gamma_{N,A} |\Psi_{e/o}\rangle = \pm |\Psi_{e/o}\rangle \)
Introduction to bosonization

• Free fermions

\[ H = \int dx \psi^\dagger(x) \left( -\frac{\hbar^2}{2m} \partial_{xx} - E_F \right) \psi(x) \]

\[ \psi(x) = e^{ik_F x} \psi_R(x) + e^{-ik_F x} \psi_L(x) \]

\[ H = v \int dx \left[ \psi^\dagger_R(x) (-i\partial_x) \psi_R(x) - \psi^\dagger_L(x) (-i\partial_x) \psi_L(x) \right] \]

• Introduce bosonic variables:

\[ \psi_{R/L}(x) \propto e^{i(\pm \phi - \theta)} \quad [\partial_x \phi(x), \theta(x')] = \pi i \delta(x - x') \]

\[ H_{44} = \frac{uv}{24\pi} \int dx \left[ (\partial_x \phi^2)^2 + \frac{1}{K} (\partial_x \phi) \phi^2 \right] \]
• Effect of interactions in Majorana wires

• Consider a 1D spinless p-wave superconductor with interactions

\[ \Delta_P \psi_R^\dagger(x) \psi_L^\dagger(x) + h.c. \]

\[ H = \int_{-L/2}^{L/2} dx \left( \frac{\nu}{2\pi} \left[ K (\partial_x \theta)^2 + K^{-1} (\partial_x \phi)^2 \right] \right) \]

\[ \frac{d\Delta_P}{dl} = (2 - K^{-1}) \Delta_P \quad \text{For } K > 1/2 \rightarrow \Delta_P \text{ flows to strong coupling} \]

There are two degenerate ground states: \( \theta = 0 \) and \( \theta = \pi \)

Spontaneously broken \( \mathbb{Z}_2 \) symmetry in bosonic variables corresponds to ground-state degeneracy in fermionic variables.

Fermion parity operator \((-1)^{N_F}\) transforms \( \theta \rightarrow \theta + \pi \)

\[ |\text{even/odd}\rangle \equiv |\theta = 0\rangle \pm |\theta = \pi\rangle \]
Degeneracy splitting for finite-length wires

At strong coupling $y_{\Delta}(l^*) \sim 1 \rightarrow a \sim \xi$

$$S = \frac{vK}{2\pi} \int d\tau \int_0^L dx \left( (\partial_x \theta)^2 + v^{-2} (\partial_\tau \theta)^2 \right) - \frac{y_{\Delta}}{\xi^2} \cos 2\theta$$

Splitting between two degenerate ground states $|\theta = 0\rangle$ and $|\theta = \pi\rangle$:

$$\delta E \propto \exp(-S_{\text{inst}}[l^*])$$

Lowest action instanton is translationally invariant

$$\delta E \propto \exp\left(-\frac{4\sqrt{K}L}{\pi \xi}\right)$$
Effect of disorder in Majorana wires

Binomial distribution of $\mu(x)$

$$\text{Probability } p^l(1-p)^{2l} = e^{-l/l_d}$$

$$\nu(\varepsilon) = \int dl \delta(\varepsilon - e^{-l/x})e^{-l/l_d}$$

$$= e^{-1+l_d/x}$$

1D – Motrunich et al’01

Topological SC
(localized Majorana modes)

$$\frac{1}{2\tau} = \Delta$$

Insulator
(no localized Majorana modes)

0

$\nu(\varepsilon)$

$-\Delta \ 0 \ \Delta \ \varepsilon$
Interplay of disorder & interactions in Majorana wires

\[ S = \int_0^\beta d\tau \int_0^L dx \left( \frac{v}{2\pi} \left[ K (\partial_x \theta)^2 + K^{-1} (\partial_x \phi)^2 \right] - \frac{\Delta_P}{2\pi a} \cos 2\theta \right) \]

Short-range disorder action in bosonic variables reads

\[ S_{\text{imp}} = \int_0^\beta d\tau \int dx \ V_{\text{imp}} \ \delta(x) \Psi^\dagger(x) \Psi(x) \]

\[ S_{\text{dis}} = \int dx d\tau \left[ \xi(x) \frac{e^{-i2\phi(x)}}{2\pi a} + \text{H.c.} \right] \]

disorder competes with p-wave superconductivity!
Interplay of disorder & interactions in Majorana wires

Gaussian disorder \( \langle \xi (x) \xi (x') \rangle = D_b \delta(x - x') \)

Use replica method: \( \langle F \rangle_\xi = -T \langle \log Z \rangle_\xi = -T \lim_{n \to 0} \left( \frac{Z^n - 1}{n} \right)_\xi \)

\[ S = \sum_{a,b=1}^{n} \left[ \delta_{ab} S_{0,a} - \frac{D_b}{(2\pi a)^2} \int dx d\tau d\tau' \cos(2 [\phi_a (x, \tau) - \phi_b (x, \tau')]) \right] \]

Perturbative RG equations

\[ \frac{dy_\Delta (\ell)}{d\ell} = [2 - K^{-1} (\ell)] y_\Delta (\ell) \]
\[ \frac{dy_b (\ell)}{d\ell} = [3 - 2K (\ell)] y_b (\ell) \]

Topological phase diagram for disordered Majorana wires

TP phase boundary

\[ y_b(l_{\text{max}}) = y_\Delta(l_{\text{max}}) = 1 \]

\[ \frac{1}{\tau E_F} \sim \left( \frac{\Delta_P}{E_F} \right)^\nu \]

\[ \nu = \frac{3 - 2K}{2 - K^{-1}} \]

\[ K = 1 \rightarrow \frac{1}{\tau} \sim \Delta_P \]

Lobos, Lutchyn & Das Sarma, PRL’12
• Introduction to topological quantum computing
• Physical realizations of Majorana modes in topological superconductors
• Experimental signatures of Majorana modes
• Quantum computing with Majoranas

Literature:

C. Nayak et al., Rev. Mod. Phys. 80, 1083 (2008)
J. Alicea, Rep. Prog. Phys. 75, 076501 (2012);
B. A. Bernevig, Topological Insulators and Topological Superconductors
S. Das Sarma, M. Freedman, & C. Nayak, Quantum Information 1, 15001 (2015)
• Physical realizations: ways to avoid fermion doubling

Different routes to realize a non-degenerate Majorana zero-energy modes:

• Chiral p-wave superconductors

• Topological insulator/s-wave superconductor

• Spin-orbit coupled semiconductor/s-wave superconductor

• Magnetic atom chain/s-wave superconductor
Sr$_2$RuO$_4$ - topological superconductor?

$T_c \sim 1 \text{ K} -$ strongly varies with disorder
- unconventional superconductor
- spin-triplet p-wave pairing

Order parameter: $\Delta(p) = \Delta_0(p_x + i p_y)$

Time-reversal symmetry $\Theta$: $p \rightarrow -p$ and $i \rightarrow -i$

$\Theta \Delta(p) \propto \Delta_0(p_x - i p_y)$

Order parameter breaks $T$ (and $P$) symmetry!

Topological index (Chern number) $n=2$ because of spin
Evidence for TRS-breaking in Sr$_2$RuO$_4$

Polar Kerr effect - “smoking gun” signature for TRSB

Rotation of the polarization plane

\[ n_R \neq n_L \]

Transmission - Faraday effect

Reflection - Kerr effect

\[ \Theta_K = \text{Im} \left[ \frac{4\pi \sigma_{xy}(\omega)}{n(n^2-1)\omega d} \right] \]

Calculate Hall conductivity $\sigma_{xy}$ at $B = 0$

(Courtesy of A. Kapitulnik)

Xia et al., PRL 97, (2006)
Anomalous Hall conductivity of a chiral SC

Metals/topological insulators

Linear response:
\[ \sigma_{xy} = \lim_{\omega \to 0} \frac{i}{\omega \beta} \sum_{n,k} j_x(k)G(k, i\nu_n + \omega) j_y(k)G(k, i\nu_n) \]

Chiral p-wave superconductor:

Calculate gauge-invariant effective action for E-M field at high \( \omega \)

Lutchyn, Nagornykh, Yakovenko, PRB’08; Roy and Kallin, PRB’08

\[ Z = \int D\Psi^\dagger D\Psi DAe^{-S[\Psi^\dagger, \Psi, A]} \quad \Rightarrow \quad j = -\frac{\delta S(A)}{\delta A} \]

Effective action has two types of terms

respect TRS: \( S_0 \)

break TRS: \( S_{\text{TRSB}} \)
Anomalous Hall conductivity

Time-reversal-symmetry-breaking effective action

\[ S_{\text{TRSB}} = \Theta \int dt \, d^2r \left( A_0 + \partial_t \Phi / 2e \right) \left( \partial_y A_x - \partial_x A_y \right) \]

\[ \Theta = \frac{e^2}{2h} \]

\( S_{\text{TRSB}} \) is similar to Chern-Simons action \( \text{But } U(1) \text{ symmetry is broken} \)

\[ \sigma_{xy}(\omega,q) = \frac{e^2}{2h} \frac{v_F^2 q^2 / 2}{\omega^2 - v_F^2 q^2 / 2} \]

Galilean invariance

Hall conductivity vanishes at \( q \to 0 \) \( \text{(Read and Green, PRB’00)} \)

Need to consider disorder!

Lutchyn, Nagornykh, Yakovenko, PRB’08;
Disorder-induced Hall conductivity

Disorder-scattering diagrams

non-Gaussian disorder

Red – imaginary
Blue – real part

Prediction for Kerr angle

Gaussian disorder

$\Theta_K \sim 40$ mrad

Kapitulnik, N. J. of Phys.’09

Goryo, PRB’08; Lutchyn, Nagornykh, Yakovenko, PRB’09
Evidence for TRS in strontium ruthenate

\( T_c \sim 1 \text{ K}, \) strongly varies with disorder: unconventional SC spin-triplet pairing (spin susceptibility)

- TRS breaking:

  - \( \mu \text{SR} \) measurements (Columbia’98)
  - Kerr effect (Stanford’06)
  - Josephson interferometry (UIUC’06)
  - scanning SQUID microscopy (Stanford’10): no evidence for edge supercurrent
Evidence for half-quantum vortices

Triplet OP: \[ \Delta_{\alpha\beta}(k) = \sum_{\mu} d_{\mu}(k)(\sigma_{\mu}i\sigma_y))_{\alpha\beta} \]

\[ \Delta(k)e^{i\Phi} = (e^{i\phi_k} |\uparrow\uparrow\rangle + e^{-i\phi_k} |\downarrow\downarrow\rangle) e^{i\Phi} \]

Half-quantum vortices correspond to 
\[ \Phi \rightarrow \Phi + \pi \text{ and } \phi_k \rightarrow \phi_k + \pi \]

Problem: Half-quantum vortices are unstable in bulk superconductors

There is evidence for HQVs in mesoscopic samples: magnetometry measurements by Jang et al., 2011

Problem: small mini-gap energy scale \( \Delta E \sim \frac{\Delta^2}{E_F} \): hard to resolve the spectrum

Overall, it seems that the observation of Majorana in SRO will be quite challenging

Budakian’s group (UIUC)
Topological insulator – new phase of matter

Conventional band insulator

Topological insulators:

2D Quantum Hall insulator
K. von Klitzing, G. Dorda, M. Pepper’81
Time reversal symmetry is broken

2D Quantum spin Hall insulator
Kane and Mele’05
Bernevig and Zhang’05
Time reversal symmetry is preserved

Number of crossings $\rightarrow$ $Z_2$(even-odd) topological invariant
3D topological insulators

band insulators with strong spin-orbit coupling

\[ H = \sigma \cdot \mathbf{p} - \mu \]

Topological metal = $\frac{1}{4}$ graphene

Dirac Hamiltonian

Fermion doubling problem is avoided because Dirac cones are spatially separated

Moore and Balents, PRB’07
Fu, Kane and Mele, PRL’07
Roy, PRB’09

Y. Xia et al., Nat. Phys.’09

Fu, Kane, PRL’08
Topological insulator/superconductor heterostructure

Andreev scattering processes

\[ A_{\uparrow \downarrow}(\omega) = t^2 F(\omega) \]
\[ A_{\uparrow \downarrow}(0) \equiv \Delta \]

Dirac surface state
(proximity induced s-wave pairing)
(no spin degeneracy)

Superconductor-magnet domain wall at the edge of 2D QSHI

- HgTe quantum wells
- InAs/GaSb quantum wells

Fu and Kane, PRL’08

Fu and Kane, PRL’09
Majorana modes at TI/SC interface

\[ H_{\text{BdG}} = (-i\nu \vec{\sigma} \vec{\nabla} - \mu)\tau_z + \Delta_0(r)(\tau_x \cos \theta + \tau_y \sin \theta) \]

Dirac equation + s-wave vortex → Majorana modes

Explicit solution:

\[ \Psi = \begin{pmatrix} u_\uparrow \\ u_\downarrow \\ v_\downarrow \\ -v_\uparrow \end{pmatrix} = \begin{pmatrix} J_0(\frac{\mu}{\nu}r) \\ iJ_1(\frac{\mu}{\nu}r)e^{i\theta} \\ J_1(\frac{\mu}{\nu}r)e^{-i\theta} \\ -iJ_0(\frac{\mu}{\nu}r) \end{pmatrix} e^{-i\frac{\pi}{4} - \int_0^r dr' \Delta(r')} \]

\[ a = \sum_{r, \sigma} (u_{r\sigma} c_{\sigma} + v_{r\sigma} c_{\sigma}^\dagger) \]

\[ \Rightarrow u_\uparrow = v_\uparrow^* \quad \text{and} \quad u_\downarrow = v_\downarrow^* \]

Majorana fermion \( a = a^\dagger \)
Challenges

3D Topological Insulator Majorana proposals

• bulk conduction
• proximity-induced gap
• surface disorder

2D Topological Insulator Majorana proposals

• bulk conduction
• proximity-induced effect
• certain edge properties (e.g., temperature and magnetic field dependence) are not understood
Semiconductor/superconductor heterostructures

Rather than looking for topological superconductors in nature, we could try to engineer suitable Hamiltonians.

Necessary ingredients:

- strong spin-orbit interaction + large magnetic field to avoid spin degeneracy
- superconducting proximity effect

2D: Majoranas “live” in vortices

Sau, Lutchyn, Tewari, Das Sarma, PRL (2010)
Alicea, PRB(2010)

1D: Majoranas “live” at the ends of wires

Lutchyn, Sau, Das Sarma, PRL(2010)
Oreg, Refael, von Oppen, PRL(2010)
Rashba spin-orbit coupling

2D quantum well

\[ H = \frac{\vec{p}^2}{2m^*} - \mu + \frac{\alpha_R}{\hbar} \hat{e}_z \cdot (\vec{\sigma} \times \vec{p}) \]

Rashba spin-orbit coupling
Semiconductor with spin-orbit interaction

\[ H_0 = \begin{pmatrix} \frac{p^2}{2m} - \mu & \alpha i (p_x - i p_y) \\ -\alpha i (p_x + i p_y) & \frac{p^2}{2m} - \mu \end{pmatrix} \]

- Semiconductor with Rashba interaction
- Fermi sea
- Spin orientation changes around Fermi surface
- Single Fermi surface!
Spinless $p+ip$ superconductivity in semiconductor/superconductor heterostructures

\[ H = \Psi^\dagger_\lambda \left( \frac{p^2}{2m} - \mu + V_z \sigma_z + \alpha \hat{z} (\sigma \times p) \right)_{\lambda \lambda'} \Psi_{\lambda'} + \Delta_{\text{ind}} \Psi^\dagger_\uparrow \Psi^\dagger_\downarrow + h.c. \]

Diagonalize $H_0$

\[ H_{\text{SC}} = \begin{cases} \Delta_{--}(p) \Psi^\dagger_-(p) \Psi^\dagger_-(p) \\ \Delta_{-+}(p) \Psi^\dagger_+(p) \Psi^\dagger_+(p) \end{cases} \]

\[ \Delta_{--}(p) \propto \Delta_0 \frac{p_x + ip_y}{|p|} \]

Rigorous proof: calculate topological index (first Chern number)

\[ n = \frac{1}{2\pi} \int_{E<0} F dS \]

\[ n = 1 \text{ for } |V_z| > \sqrt{\mu^2 + \Delta^2} \]

\[ n = 0 \text{ for } |V_z| < \sqrt{\mu^2 + \Delta^2} \]
1D semiconductor nanowires

\[ H_0 = \int_{-L}^{L} dx \psi_\sigma^\dagger(x) \left( -\frac{\partial^2}{2m^*} - \mu + i\alpha \sigma_y \partial_x + V_x \sigma_x \right) \psi_{\sigma'}(x) \]

- single channel nanowire
- spin-orbit coupling
- Zeeman splitting

Magnetic field \( B_x \) opens up gap in the spectrum at \( p_x = 0 \)

InAs, InSb nanowires

- large spin-orbit (\( \alpha \sim 0.1 \text{eVÅ} \))
- large \( g \)-factor (\( g \sim 10 - 50 \))
- good contacts with metals
Majorana quantum wires

Theoretical idea of Majorana wire was introduced by Kitaev (2001)

$$H_{MW} = \int_{-L}^{L} dx \left[ \psi_\sigma^\dagger \left( -\frac{\partial_x^2}{2m^*} - \mu + i\alpha \sigma_y \partial_x + V_x \sigma_x \right) \psi_{\sigma'} + \Delta_0^* \psi_\uparrow \psi_\downarrow + \Delta_0 \psi_{\downarrow}^\dagger \psi_{\uparrow}^\dagger \right]$$

Rashba spin-orbit + in-plane field

Proximity-induced superconductivity

$$2L \gg \xi$$

$s$-wave superconductor (Al or Nb)

topologically non-trivial

$$|V_x| > \sqrt{\mu^2 + \Delta_0^2}$$

topologically trivial

$$|V_x| < \sqrt{\mu^2 + \Delta_0^2}$$

Lutchyn, Sau, Das Sarma, PRL(2010)

Oreg, Refael, von Oppen, PRL(2010)

Drive topological phase transition by changing $V_x$ or $\mu$
Density of states across phase transition

Finite-size numerical studies $L_x = 10\mu m$

DoS in topologically non-trivial phase

$$|V_x| > \sqrt{\mu^2 + \Delta_0^2}$$

DoS in topologically trivial phase

$$|V_x| < \sqrt{\mu^2 + \Delta_0^2}$$
Challenges

2D Majorana proposals

- requires two interfaces
- low disorder
- tuning density

1D Majorana wire proposals

- low disorder
- tuning density
- large SO coupling and g-factor
Yu-Shiba-Rusinov states

\[ H_{imp} = J_0 \, \mathbf{S} \cdot \mathbf{s} \, \delta(r) \]

\[ E = \pm \Delta \, \frac{1 - \tilde{J}_0^2}{1 + \tilde{J}_0^2} \]

\[ \tilde{J}_0 = \pi \, v_F \, J_0 \, |S| \]

A. I. Rusinov, JETPL 9, 85 (1969)
MBS in ferromagnetic impurity chains

- SC with Rashba SOC

• \( E = \Delta_s \)
• \( E = 0 \)
• \( E = -\Delta_s \)

\[ \Delta_{top} \propto SOC \]

for \( S// x \) or \( z \)
Observation of Majorana fermions in ferromagnetic atomic chains on a superconductor

Stevan Nadj-Perge, Ilya K. Drozdov, Jian Li, Hua Chen, Sangjun Jeon, Jungpil Seo, Allan H. MacDonald, B. Andrei Bernevig, Ali Yazdani

Introduction to topological quantum computing

Physical realizations of Majorana modes in topological superconductors

Experimental signatures of Majorana modes

Quantum computing with Majoranas

Literature:

C. Nayak et al., Rev. Mod. Phys. 80, 1083 (2008)

J. Alicea, Rep. Prog. Phys. 75, 076501 (2012);


B. A. Bernevig, Topological Insulators and Topological Superconductors

S. Das Sarma, M. Freedman, & C. Nayak, Quantum Information 1, 15001 (2015)
Experimental progress in nanowires

- **Zero-bias tunneling conductance:**
  - Mourik et al. (Delft), Science 336, 1003 (2012)
  - Deng et al. (Lund/Peking), Nano Lett. 12, 6414 (2012)
  - Das, A. et al. (Weizmann), Nature Phys. 8, 887 (2012)
  - Finck et al. (UIUC), PRL 110, 126406 (2013)
  - Churchill et al. (Harvard), PRB 87, 241401(R) (2013)

- **Coulomb blockade experiments**
  - Albrecht et al., Nature 531, 206 (2016)

- **Fractional (4pi-periodic) ac Josephson effect:**
  - Rokhinson et al., Nature Phys. 8, 795 (2012)
Zero-bias tunneling conductance

Andreev reflection

\[ \frac{dI}{dV} \left( \frac{2e^2}{h} \right) \text{ topological} \quad \text{and} \quad \text{non-topological} \]

\[ T = 0 \]

\[ eV/\Gamma \]

\[ I = \frac{2e}{h} \int dE \left[ f_F(E - eV) - f_F(E) \right] |S^{\text{ph}}(E)|^2 \]

\[ \begin{cases} S(0) = \left( \begin{array}{cc} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{array} \right) & \text{or} & \left( \begin{array}{cc} 0 & e^{i\beta} \\ e^{-i\beta} & 0 \end{array} \right) \end{cases} \]

\[ T \gg eV \]

\[ S(0) = \sigma_x S^*(0) \sigma_x \]

\[ \frac{\tau}{\Gamma}, T \gg \Gamma \]

\[ \frac{\pi^2 T^2}{3\Gamma^2}, T \ll \Gamma \]

\[ E = 0: \text{unitarity} \]

\[ G = \frac{2e^2}{h} \begin{cases} 1 & S(0) = \left( \begin{array}{cc} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{array} \right) \end{cases} \]
Measurements of zero-bias conduction peak

InSb nanowires:
Delft (2012)
Lund/Peking (2012)
Harvard/NBI (2013)

InAs nanowires:
Weizmann (2012)
UIUC (2012)
Zero bias tunneling conductance

Mourik et al. (Delft), Science, 2012

Summary of experimental findings:

- High mobility, quasi-ballistic InSb nanowires; strong SOI, induced SC
- ZBP onsets at $B \sim 100$ mT, $E_z \sim 150 \mu$eV, so $E_z \sim \Delta$
- ZBP remains stuck to zero bias over a significant range of $B$ ($\Delta E_z \sim 0.5-1.5$ meV)
- ZBP persists over large gate ranges for all gates, but gate tuning is required
- ZBP vanishes when $B$ is aligned with $B_{SO}$ which shows that SOI is a necessary ingredient
- ZBP robust in both gate and $B$, not observed in two N-NW-N devices which shows that superconductivity is a necessary ingredient

- Quantization of the conductance

Stanescu, Lutchyn, Das Sarma, PRB’11
Epitaxial interface between Al and InAs

Krogstrup et al. (NBI), Nature Mat. 14, 400 (2015)
Zero-bias tunneling conductance (2015)
• Universal transport signatures of Majorana modes

![Diagram](image.png)

\[ S = S_{\text{LL}} + S_{\text{imp}} \]

\[ S_{\text{imp}} = \int_{t,x} \delta(x) \left[ \lambda_M \cos \Theta + \lambda_N \cos 2\Phi \right] \]

- coupling to Majorana
- electron backscattering

Proceed a la Kane-Fisher (1992): integrate out bulk modes in nanowire and derive boundary theory for TSC/LL junction
Universal transport signatures of Majorana modes

\[ \frac{d\lambda_N}{dl} = (1 - 2K)\lambda_N \quad \frac{d\lambda_M}{dl} = \left(1 - \frac{1}{2K}\right)\lambda_M \]

Fidkowski, Alicea, Lindner, Lutchyn & Fisher, PRB’12
Conductance in the low energy limit

\[ G = \frac{2e^2}{h} - \lambda_N(0) \left( \frac{\max\{T, eV\}}{\Lambda} \right)^{4K-2} \]
• Conductance in the high energy limit

\[ \frac{G}{2e^2/h} \]

\[ G \propto \left( \frac{T}{\Lambda^*} \right)^{\frac{1}{K} - 2} f_1(K) \]

\[ G \propto \left( \frac{eV}{\Lambda^*} \right)^{\frac{1}{K} - 2} f_2(K) \]

- Non-interacting limit
  - \( f_1(K=1) \neq 0 \rightarrow G \propto T^{-1} \)
  - \( f_2(K=1) = 0 \rightarrow G \propto V^{-2} \)
Coulomb blockade experiments with Majorana wires

- s-wave superconductor

InAs nanowire

Non-topological phase
($B = 0$)

Non-topological phase
($B_c > B > 0$)

Topological phase
($B > B_c$)

Majorana zero-energy state
Experiments on proximitized nanowires (N-S-N)

Short wires (L~250nm)  
(Higginbotham et al., Nat Phys. 2015)

Long wires (L~1.5mm)  

Need quantitative theory of thermodynamic and transport properties:


Exponential splitting of Majorana zero modes

Albrecht et al., Nature (2016)
Broadening of charge steps depends on quantum charge fluctuations and encodes information about the state of the system.

Quantum Charge Fluctuations and the Polarizability of the Single-Electron Box


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We measure the average charge on the island of a single-electron box, with an accuracy of two thousandths of an electron. Thermal fluctuations alone cannot account for the dependence of the average charge on temperature, on external potential, or on the quasiparticle density of states in the metal from which the box is formed. In contrast, we find excellent agreement between these measurements and a theory that treats the quantum fluctuations of charge perturbatively.

FIG. 2. (a) Coulomb staircases at \( T = 500 \) mK (dotted line) and \( T = 100 \) mK (solid line). The charging energy \( E_C \) is extracted by fitting to the 500 mK data a theoretical staircase (not shown) broadened only by thermal fluctuations. The 100 mK staircase is compared to the thermal fluctuation theory with no adjustable parameters (dashed line). (b) The residuals...
The single Cooper-pair box as a charge qubit

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![Graphs](image-url)
Coulomb blockade peaks with magnetic field

- Non-topological phase ($B = 0$)
- Non-topological phase ($B_c > B > 0$)
- Topological phase ($B > B_c$)
Weak tunneling limit: spinful case

\[ H = \sum_{k,\sigma} \xi_k a_{k\sigma}^\dagger a_{k\sigma} + t \sum_{k,p,\sigma} \left( a_{k\sigma}^\dagger a_{p\sigma} + h.c. \right) + H_{BCS} + E_C\left( Q/e - N_g \right)^2 \]

\[ \Delta \gg E_C \quad \text{Schrieffer-Wolff transformation} \]

\[ H = \sum_{k,\sigma} \xi_k a_{k\sigma}^\dagger a_{k\sigma} + \frac{t_A}{\Omega} \sum_{k_1,k_2,\sigma} (\sigma a_{k_1 \sigma}^\dagger a_{k_2 - \sigma}^\dagger |2n\rangle \langle 2n+2| + h.c.) + \frac{r}{2\Omega} \sum_{k_1,k_2,\sigma} a_{k_1 \sigma}^\dagger a_{k_2 \sigma} (|2n\rangle \langle 2n| - |2n+2\rangle \langle 2n+2|) \]

Garate’11

Charge Kondo problem \( T_K \sim E_C \exp\left(-\frac{\pi^2}{g}\right) \)

\( E_{GS} \)

\[ \frac{q}{e} = -\frac{e}{2E_C} \left\langle \frac{\partial H}{\partial N_g} \right\rangle \]

\( T_K \)

\( N_g \)

even

even

0.5

1

1.5

2
• Weak tunneling limit: spinful case (B=0)

\[ H = \sum_{k,\sigma} \xi_k a_{k\sigma}^\dagger a_{k\sigma} + t \sum_{k,p,\sigma} \left( a_{k\sigma}^\dagger a_{p\sigma} + h.c. \right) + H_{BCS} + E_C(Q/e - N_g)^2 \]

\[ \Delta < E_C \]

**Lutchyn, Flensberg & Glazman (2016)**

Is \( q(N_g) \) analytic at even-odd transition?

Yes, similar to mixed valence to Kondo regime crossover of Anderson model.

Lutchyn, Flensberg & Glazman (2016)
Weak tunneling limit: spinless case

$$H = \sum_k \xi_k a_k^\dagger a_k + t \sum_{k,p} \left( a_k^\dagger a_p + h.c. \right) + H_{BCS} + E_C(Q/e - N_g)^2$$

Is there any difference in $Q(N_g)$ between Majorana and normal islands?
Two terminal conductance

Cooper pair tunneling

Incoherent single electron tunneling

Resonant tunneling through Majoranas

\[ G_{\text{max}}(T) \approx \frac{e^2}{h} \frac{g_L g_R}{g_L^2 + g_R^2} \]

\[ G_{\text{max}}(T) \approx \frac{e^2}{h} \frac{g_L g_R}{g_L + g_R} \frac{\delta}{T} \]

\[ G_{\text{max}}(T) = \frac{e^2}{h} \frac{g_L g_R}{g_L + g_R} \frac{|\Delta|}{4T} \]

van Heck, RL & Glazman (2016)
Fractional ac Josephson effect

Andreev bound states

$E$

$\Delta_L = \Delta e^{i\varphi}$

$\Delta_R = \Delta_0$

Short junction limit

$L \ll \xi$

particle-hole symmetry protects true level crossing at $\varphi = \pi$

Lutchyn, Sau, Das Sarma, PRL’10
Josephson current through heterostructure

\[ I_\pm(\varphi) \propto \frac{\partial E_\pm(\varphi)}{\partial \varphi} \]
Fractional ac Josephson effect is another signature of topological SC

(Kitaev, arXiv’00; Kwon, Sengupta, Yakovenko’04; Fu, Kane’09)

**Josephson current**

\[ I(\varphi) \propto \frac{\partial E(\varphi)}{\partial \varphi} \propto \sin\left(\frac{\varphi}{2}\right) \]

\[ \tau_r \propto \exp(\Delta/T) \]

**Ensemble-averaged current**

\[ \langle I(\varphi) \rangle = I(\varphi) \tanh\left(\frac{\epsilon(\varphi)}{2T}\right) \]

2π-periodicity is restored
Quantum Computing with Majoranas

Majorana nanowire networks

Alicea et al., Nature Physics’11

Topological quantum buses

Bonderson & Lutchyn, PRL’11
Jiang, Kane & Preskill, PRL’11
Manipulating Majoranas

Flux control

Electrostatic control

Hyart et al., (2013)

Aasen et al. (2015)
Quantum gates in Ising TQFT

Braiding Majoranas

\[ \gamma_1 \rightarrow -\gamma_2 \]
\[ \gamma_2 \rightarrow \gamma_1 \]

Operator implementing this evolution

\[ U_{ij} = \frac{1 + \gamma_i \gamma_j}{\sqrt{2}} \]

\[ |n_1, n_2\rangle \rightarrow U_{12} |n_1, n_2\rangle = e^{i \frac{\pi}{4} (1 - 2n_1)} |n_1, n_2\rangle \]
\[ |n_1, n_2\rangle \rightarrow U_{23} |n_1, n_2\rangle \]
\[ = \frac{1}{\sqrt{2}} [ |n_1, n_2\rangle + i(-1)^{n_1} |1 - n_1, 1 - n_2\rangle ] \]
\[ |n_1, n_2\rangle \rightarrow U_{34} |n_1, n_2\rangle = e^{i \frac{\pi}{4} (1 - 2n_2)} |n_1, n_2\rangle . \]
Protected Exchange Gates

\[ P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \]

\[ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]
• Majorana Quasiparticle Gates

Braiding and interferometric measurements of Majorana quasiparticles gives the Clifford gates.

This is not computationally universal gate set.

But it only requires supplementation by

\[ R(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \quad \theta \neq \frac{n\pi}{2} \]

like, such as

\[ T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \]
“Magic state distillation” • Bravyi & Kitaev (2005)

Given topologically protected Clifford gates, the unprotected, noisy gate

\[
T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}
\]

can be purified (error-corrected) with a remarkably high error threshold of \(~ 14\%\)!

These unprotected gates can be generated by

• Proximity induced energy splitting
• Interference processes
• Importing them from other systems…