CHAPTER 2 REVIEW QUESTIONS

Complete the following review questions using the techniques outlined in this chapter. Then, see Chapter 8 for answers and explanations.

1. Consider the sequence \((x_n)\) whose terms are given by the formula

\[
x_n = \frac{\cos mx \sin^2 n}{\sqrt{n}}
\]

for each integer \(n \geq 1\). Given that this sequence converges, what is its limit?

(A) 0  (B) 1  (C) log 2  (D) \(\sqrt{2}\)  (E) \(\sqrt{e}\)

2. Let \((x_n)\) be the sequence with \(x_1 = 2\) and \(x_n = \sqrt{5x_{n-1} + 6}\) for every integer \(n \geq 2\). Given that this sequence converges, what is its limit?

(A) 4  (B) 6  (C) 8  (D) 10  (E) 16

3. Let \([x]\) denote the greatest integer \(\leq x\). If \(n\) is a positive integer, then

\[
\lim_{x\to n^-} ([x] - [x]) - \lim_{x\to n^-} ([x] - [x]) = ?
\]

(A) -2  (B) 0  (C) 2  (D) 2n - 1  (E) 2n

4. Evaluate the following limit:

\[
\lim_{x\to 0} \frac{\arcsin x - x}{x^2}
\]

(A) 0  (B) \(\frac{1}{6}\)  (C) \(\frac{1}{3}\)  (D) \(\frac{1}{2}\)  (E) 1

5. The curve whose equation is

\[2x^2 + 3x - 2xy - y = 6\]

has two asymptotes. Identify these lines.

(A) \(x = -1\) and \(y = -2\)  (B) \(x = -2\) and \(y = 1\)  (C) \(x = -\frac{1}{2}\) and \(y = x\)

(D) \(x = -\frac{1}{2}\) and \(y = x + 1\)  (E) \(x = \frac{1}{2}\) and \(y = 1 - x\)
6. If the function

\[ f(x) = \begin{cases} \frac{x^2 - 6x + 8}{x^3 - 2x^2 + 2x - 4} & \text{if } x \neq 2 \\ k & \text{if } x = 2 \end{cases} \]

is continuous everywhere, what is the value of \( k \)?

(A) 1  (B) \( \frac{1}{2} \)  (C) \( \frac{1}{8} \)  (D) \( -\frac{1}{3} \)  (E) \(-1\)

7. Evaluate the following limit:

\[ \lim_{x \to 0} \left[ \frac{1}{x^2} \int_{0}^{x} \frac{t + t^2}{1 + \sin t} \, dt \right] \]

(A) \( \frac{1}{2\pi} \)  (B) \( \frac{1}{\pi} \)  (C) \( \frac{1}{2} \)  (D) 1  (E) \( \frac{\pi}{2} \)

8. Determine the domain of the following function:

\[ f(x) = \arcsin (\log \sqrt{x}) \]

(A) \([0, \frac{1}{e^2}]\)  (B) \([\frac{1}{e^2}, 1]\)  (C) \([e, e^2]\)  (D) \([\frac{1}{e^2}, e^2]\)  (E) \([1, e^2]\)

9. Evaluate the derivative of the following function at \( x = e \):

\[ f(x) = \arcsin (\log \sqrt{x}) \]

(A) \( \frac{1}{e\sqrt{3}} \)  (B) \( \frac{e}{\sqrt{2}} \)  (C) \( \frac{\pi e}{2} \)  (D) \( \sqrt{2e} \)  (E) \( \frac{3e}{\sqrt{2}} \)

10. For what values of \( m \) and \( b \) will the following function have a derivative for every \( x \)?

\[ f(x) = \begin{cases} x^2 + x - 3 & \text{if } x \leq 1 \\ mx + b & \text{if } x > 1 \end{cases} \]

(A) \( m = 3, b = -2 \)  (B) \( m = -2, b = -3 \)  (C) \( m = 1, b = -4 \)

(D) \( m = -2, b = 1 \)  (E) \( m = 3, b = -4 \)
11. If \( f(x) \) is a function that's differentiable everywhere, what is the value of this limit?

\[
\lim_{h \to 0} \frac{f(x+3h^2) - f(x-h^2)}{2h^2}
\]

(A) \( 4f'(x) \)  
(B) \( 2f'(x) \)  
(C) \( f'(x) \)  
(D) \( \frac{1}{2} f'(x) \)  
(E) The limit does not exist.

12. What is the equation of the tangent line to the curve \( y = x^3 - 3x^2 + 4x \) at the curve’s inflection point?

(A) \( y = 2x - 3 \)  
(B) \( y = x - 1 \)  
(C) \( y = x + 1 \)  
(D) \( y = 3x - 2 \)  
(E) \( x + y = 1 \)

13. What is the slope of the tangent line to the curve \( xy(x+y) = x + y^4 \) at the point \( (1, 1) \)?

(A) \( 2 \)  
(B) \( 1 \)  
(C) \( 0 \)  
(D) \( -1 \)  
(E) \( -2 \)

14. If \( f(x) = 2|x-1| + (x-1)^2 \), what is the value of \( f'(0) \)?

(A) \( 4 \)  
(B) \( 2 \)  
(C) \( 0 \)  
(D) \( -2 \)  
(E) \( -4 \)

15. If

\[
f(x) = \frac{e^x \arccos x}{\cos x}
\]

then the slope of the line tangent to the graph of \( f \) at its \( y \)-intercept is

(A) \( -\frac{\pi}{2} \)  
(B) \( -1 \)  
(C) \( \frac{\pi}{2} - 1 \)  
(D) \( 1 \)  
(E) \( \frac{\pi}{2} + 1 \)

16. Let \( y = \frac{1}{\sqrt{x^3+1}} \). If \( x \) increases from 2 to 2.09, which of the following most closely approximates the change in \( y \)?

(A) 0.08  
(B) 0.04  
(C) -0.02  
(D) -0.06  
(E) -0.09

17. If \( f(1) = 1 \) and \( f'(1) = -1 \), then the value of \( \frac{d}{dx} \left[ \frac{f(x^2)}{xf(x^3)} \right] \) at \( x = 1 \) is equal to

(A) 1  
(B) 0  
(C) -1  
(D) -2  
(E) -3
18. If \( n \) is a positive integer, what is the value of the \( n^{th} \) derivative of \( f(x) = \frac{1}{1 - 2x} \) at \( x = \frac{1}{2} \)?

(A) \( \frac{1}{2} (n^n) \)  
(B) \( \frac{1}{2} (n!) \)  
(C) \( \frac{1}{2} n \)  
(D) \( n \)  
(E) \( \frac{n^n}{n!} \)

19. Let \( f(x) \) be continuous on a bounded interval, \([a, b]\), where \( a \neq b \), such that \( f(a) = 1 \) and \( f(b) = 3 \), and \( f'(x) \) exists for every \( x \) in \((a, b)\). What does the Mean-Value theorem say about \( f \)?

(A) There exists a number \( c \) in the interval \((a, b)\) such that \( f'(c) = 0 \).
(B) There exists a number \( c \) in the interval \((a, b)\) such that \( f(c) = 0 \).
(C) There exists a number \( c \) in the interval \((a, b)\) such that \( f'(c) = 2 \).
(D) There exists a number \( c \) in the interval \((a, b)\) such that \( f'(c) = 2(b - a) \).
(E) There exists a number \( c \) in the interval \((a, b)\) such that \( (b - a) f'(c) = 2 \).

20. What is the maximum area of a rectangle inscribed in a semicircle of radius \( a \)?

(A) \( \frac{\sqrt{2}}{2} a^2 \)  
(B) \( \frac{\sqrt{3}}{2} a^2 \)  
(C) \( a^2 \)  
(D) \( \frac{\pi}{2\sqrt{2}} a^2 \)  
(E) \( a^2 \sqrt{2} \)

21. The following function is defined for all positive \( x \):

\[
 f(x) = \int_{x}^{x+2\pi} \frac{\sin t}{t} dt
\]

At what value of \( x \) on the interval \((0, \frac{3\pi}{2})\) does this function attain a local maximum?

(A) \( \frac{\pi}{6} \)  
(B) \( \frac{\pi}{3} \)  
(C) \( \frac{\pi}{2} \)  
(D) \( \pi \)  
(E) \( \frac{2\pi}{3} \)

22. Let \( f(x) = x^k e^{-x} \), where \( k \) is a positive constant. For \( x > 0 \), what is the maximum value attained by \( f \)?

(A) \( \left( \frac{e}{k} \right)^k \)  
(B) \( \frac{e}{\sqrt{k} e^k} \)  
(C) \( \frac{(k \log k)^k}{k} \)  
(D) \( \left( \frac{e}{\log k} \right)^k \)  
(E) \( \left( \frac{k}{e} \right)^k \)

23. The radius of a circle is decreasing at a rate of 0.5 cm per second. At what rate, in \( \text{cm}^2/\text{sec} \), is the circle’s area decreasing when the radius is 4 cm?

(A) \( 4\pi \)  
(B) \( 2\pi \)  
(C) \( \pi \)  
(D) \( \frac{1}{2\pi} \)  
(E) \( \frac{1}{4\pi} \)

24. The function \( f(x) = \int_{x}^{x+\pi} t \log t \, dt \) has an absolute minimum at \( x = 0 \), and a local maximum at \( x = \)

(A) \( -\log 4 \)  
(B) \( -\log 2 \)  
(C) \( \log 2 \)  
(D) \( 1 \)  
(E) \( \log 4 \)
25. Evaluate the following integral:
\[ \int_{-1}^{0} x^2(x+1)^3 \, dx \]
\[ \begin{align*}
(A) & \quad -\frac{7}{20} & (B) & \quad -\frac{1}{60} & (C) & \quad \frac{2}{15} & (D) & \quad \frac{1}{60} & (E) & \quad \frac{7}{20} \\
\end{align*} \]

26. If \([x]\) denotes the greatest integer \(\leq x\), then \(\int_{0}^{\frac{7}{2}} [x] \, dx =\)
\[ \begin{align*}
(A) & \quad \frac{5}{2} & (B) & \quad \frac{7}{2} & (C) & \quad \frac{9}{2} & (D) & \quad \frac{17}{2} & (E) & \quad \frac{37}{2} \\
\end{align*} \]

27. If \[ f(x) = \begin{cases} 
-2(x+1) & \text{if } x \leq 0 \\
(k(1-x^2)) & \text{if } x > 0 
\end{cases} \]
then the value of \(k\) for which \(\int_{-1}^{1} f(x) \, dx = 1\) is
\[ \begin{align*}
(A) & \quad -1 & (B) & \quad 0 & (C) & \quad 1 & (D) & \quad 2 & (E) & \quad 3 \\
\end{align*} \]

28. Integrate \( \int \frac{x^2 \, dx}{\sqrt{1-x^4}} \).
\[ \begin{align*}
(A) & \quad \frac{1}{2} \left( \arcsin x - \sqrt{1-x^2} \right) + c & (B) & \quad \frac{1}{2} \left( \arcsin x + x\sqrt{1-x^2} \right) + c & (C) & \quad \frac{1}{2} \left( x\arcsin x - \sqrt{1-x^2} \right) + c \\
(D) & \quad \frac{1}{2} \left( \arcsin x - x\sqrt{1-x^2} \right) + c & (E) & \quad \frac{1}{2} \left( x\arcsin x + \sqrt{1-x^2} \right) + c \\
\end{align*} \]

29. What is the area of the region in the first quadrant bounded by the curve \(y = x \arctan x\) and the line \(x = 1\)?
\[ \begin{align*}
(A) & \quad \frac{\pi - 4}{4} & (B) & \quad \frac{\pi - 2}{4} & (C) & \quad \frac{\pi}{4} & (D) & \quad \frac{\pi + 2}{4} & (E) & \quad \frac{\pi + 4}{4} \\
\end{align*} \]

30. Simplify the following:
\[ \exp \int_{3}^{5} \frac{dx}{x^2 - 3x + 2} \]
[Note: Recall that \(\exp x\) is a standard, alternate notation for \(e^x\).]
\[ \begin{align*}
(A) & \quad \frac{3}{8} & (B) & \quad \frac{2}{3} & (C) & \quad \frac{4}{3} & (D) & \quad \frac{3}{2} & (E) & \quad \frac{5}{3} \\
\end{align*} \]
31. Calculate the area of the region in the first quadrant bounded by the graphs of \( y = 8x \), \( y = x^2 \), and \( y = 8 \).

(A) 12  (B) 8  (C) 6  (D) \( \frac{16}{3} \)  (E) 4

32. Which of the following expressions gives the area of the region bounded by the two circles pictured below?

\[
\int_0^\frac{\pi}{6} \frac{1}{2} (\sqrt{3} \sin \theta)^2 \, d\theta + \int_0^{\frac{\pi}{3}} \frac{1}{2} (\sqrt{3} \sin \theta)^2 \, d\theta
\]

(A) \( \int_0^\frac{\pi}{6} \left[ (\sqrt{3} \sin \theta)^2 - (3 \cos \theta)^2 \right] \, d\theta \)

(B) \( \int_0^\frac{\pi}{2} (3 \cos \theta)^2 \, d\theta + \int_0^{\frac{\pi}{6}} \frac{1}{2} (\sqrt{3} \sin \theta)^2 \, d\theta \)

(C) \( \int_0^\frac{\pi}{6} \frac{1}{2} (\sqrt{3} \sin \theta)^2 \, d\theta + \int_0^\frac{\pi}{3} \frac{1}{2} (3 \cos \theta)^2 \, d\theta \)

(D) \( \int_0^\frac{\pi}{6} (3 \cos \theta)^2 \, d\theta + \int_0^\frac{\pi}{3} \frac{1}{2} (\sqrt{3} \sin \theta)^2 \, d\theta \)

(E) \( \int_0^\frac{\pi}{6} \frac{1}{2} (\sqrt{3} \sin \theta)^2 \, d\theta + \int_0^{\frac{\pi}{3}} \frac{1}{2} (3 \cos \theta)^2 \, d\theta \)

33. Let \( a \) and \( b \) be positive numbers. The region in the second quadrant bounded by the graphs of \( y = ax^2 \) and \( y = -bx \) is revolved around the x-axis. Which of the following relationships between \( a \) and \( b \) would imply that the volume of this solid of revolution is a constant, independent of \( a \) and \( b \)?

(A) \( b^4 = 2a^5 \)  (B) \( b^3 = 2a^5 \)  (C) \( b^5 = 2a^3 \)  (D) \( b^4 = 2a^2 \)  (E) \( b^2 = 2a^3 \)

34. The region bounded by the graphs of \( y = x^2 \) and \( y = 6 - |x| \) is revolved around the y-axis. What is the volume of the generated solid?

(A) \( \frac{32 \pi}{3} \)  (B) 9\pi  (C) 8\pi  (D) \( \frac{20 \pi}{3} \)  (E) \( \frac{16 \pi}{3} \)
35. Calculate the length of the portion of the hypocycloid \( x^{2/3} + y^{2/3} = 1 \) in the first quadrant from the point \( \left( \frac{1}{8}, \frac{3\sqrt{3}}{8} \right) \), to the point (1, 0).

   (A) \( \frac{9}{8} \)  
   (B) \( \frac{3\sqrt{2}}{4} \)  
   (C) 1  
   (D) \( \frac{5\sqrt{2}}{8} \)  
   (E) \( \frac{\sqrt{3}}{2} \)

36. What positive value of \( a \) satisfies the following equation?

   \[
   \int_e^{e^a} \frac{dx}{x} \int_o^{e^y} \frac{dy}{y} = 1
   \]

   (A) \( \frac{1}{e} \)  
   (B) \( \sqrt{e} \)  
   (C) \( \sqrt{e} \)  
   (D) \( e \)  
   (E) \( e^2 \)

37. Evaluate the following limit:

   \[ \lim_{x \to 0} (\cos x)^{e^x} \]

   (A) \( \frac{1}{2} \)  
   (B) \( \frac{1}{\sqrt{e}} \)  
   (C) \( \frac{\sqrt{e}}{2} \)  
   (D) 1  
   (E) \( \sqrt{e} \)

38. Let \( n \) be a number for which the improper integral

   \[
   \int_{e}^{\infty} \frac{dx}{x(\log x)^n}
   \]

   converges. Determine the value of the integral.

   (A) \( \frac{1}{n+1} \)  
   (B) \( \frac{1}{n} \)  
   (C) \( \frac{1}{n-1} \)  
   (D) \( \frac{\log n}{n+1} \)  
   (E) \( \frac{\log n}{n-1} \)

39. Find the positive value of \( a \) that satisfies the equation:

   \[
   \int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \int_0^a \frac{x dx}{\sqrt{a^2 - x^2}}
   \]

   (A) \( \frac{2\sqrt{2}}{\pi} \)  
   (B) 1  
   (C) \( \frac{\pi}{2\sqrt{2}} \)  
   (D) \( \sqrt{2} \)  
   (E) \( \frac{\pi}{2} \)
40. Which of the following improper integrals converge?
   I. \( \int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^2} \)
   II. \( \int_{1}^{\infty} xe^{-x} \, dx \)
   III. \( \int_{0}^{2} \frac{dx}{(2-x)^2} \)
   (A) I only  (B) I and II only  (C) II only
   (D) I and III only  (E) II and III only

41. Which of the following infinite series converge?
   I. \( \sum_{n=1}^{\infty} \frac{\cos^4(\arctan n)}{n\sqrt{n}} \)
   II. \( \sum_{n=2}^{\infty} \frac{1}{n \log n} \)
   III. \( \sum_{n=3}^{\infty} \frac{(n+1)^3}{5(n+2)(n+3)(n+4)} \)
   (A) I only  (B) I and II only  (C) II only
   (D) I and III only  (E) II and III only

42. Find the smallest value of \( b \) that makes the following statement true:
   If \( 0 \leq a < b \), then the series \( \sum_{n=1}^{\infty} \frac{(n!)^2 a^n}{(2n)!} \) converges.
   (A) 1  (B) 2 log 2  (C) 2  (D) \( \sqrt{2} \)  (E) 4

43. Evaluate the following limit:
   \[ \lim_{n \to \infty} \sum_{k=1}^{n} \left[ \frac{k}{n^2} - \frac{k^2}{n^3} \right] \]
   (A) \( \frac{2}{3} \)  (B) \( \frac{1}{2} \)  (C) \( \frac{1}{3} \)  (D) \( \frac{1}{6} \)  (E) \( \frac{1}{12} \)
44. Which of the following statements are true?

I. If \( a_n \geq 0 \) for every \( n \), then \( \sum_{n=1}^{\infty} a_n \) converges \( \Rightarrow \) \( \sum_{n=1}^{\infty} \sqrt{a_n} \) converges.

II. If \( a_n \geq 0 \) for every \( n \), then \( \sum_{n=1}^{\infty} na_n \) converges \( \Rightarrow \) \( \sum_{n=1}^{\infty} a_n \) converges.

III. If \( a_n \geq 0 \) and \( a_{n+1} \leq a_n \) for every \( n \), then \( \sum_{n=1}^{\infty} a_n^2 \) converges \( \Rightarrow \) \( \sum_{n=1}^{\infty} (-1)^n a_n \) converges.

(A) I and II only  (B) I and III only  (C) II only
(D) II and III only  (E) III only

45. If \(-1 < x < 1\), then \( \sum_{n=1}^{\infty} nx^n = \)

(A) \( \frac{x^3}{(1-x)^2} \)  (B) \( \frac{x^2}{(1-x^2)^2} \)  (C) \( \frac{x}{(1+x^2)^2} \)

(D) \( \frac{x^3}{(1+x)^2} \)  (E) \( \frac{x^2}{(1+x^2)^2} \)

46. The smallest positive integer \( x \) for which the power series \( \sum_{n=1}^{\infty} \frac{n!(2n)!}{(3n)!} x^n \) does not converge is

(A) 4  (B) 6  (C) 7  (D) 8  (E) 9

47. In the Taylor series expansion (in powers of \( x \)) of the function \( f(x) = e^{x^2-3} \), what is the coefficient of \( x^2 \)?

(A) \(-7\)  (B) \(\frac{3}{2}\)  (C) \(\frac{7}{6}\)  (D) \(\frac{7}{6}\)  (E) \(\frac{3}{2}\)

48. If \( k_i \) (\( i = 0, 1, 2, 3, 4 \)) are constants such that \( x^4 = k_0 + k_1 (x+1) + k_2 (x+1)^2 + k_3 (x+1)^3 + k_4 (x+1)^4 \) is an identity in \( x \), what is the value of \( k_3 \)?

(A) \(-4\)  (B) \(-3\)  (C) \(-2\)  (D) 3  (E) 4