CHAPTER 6 REVIEW QUESTIONS

Complete the following review questions using the techniques outlined in this chapter. Then, see Chapter 8 for answers and explanations.

1. There is only one integer, $x$, between 100 and 200 such that integer pair $(x, y)$ satisfies the equation $42x + 55y = 1$. What's the value of $x$ in this integer pair?

   (A) 127  (B) 148  (C) 158  (D) 167  (E) 183

2. Let $L$ be the least common multiple of 1001 and 10101. What's the sum of the digits of $L$? (All numbers are written in their usual decimal representation.)

   (A) 6  (B) 11  (C) 17  (D) 22  (E) 33

3. Let $x_1$ and $x_2$ be the two smallest positive integers for which the following statement is true: “$85x - 12$ is a multiple of 19.” Then $x_1 + x_2 =$

   (A) 19  (B) 27  (C) 31  (D) 38  (E) 47

4. If $x, y$, and $z$ are positive integers such that $4x - 5y + 2z$ is divisible by 13, then which one of the following must also be divisible by 13?

   (A) $x + 13y - z$  (B) $6x - 10y - z$  (C) $x - y - 2z$  (D) $-7x + 12y + 3z$  (E) $-5x + 3y - 4z$

5. When expressed in its usual decimal notation, the number 100! (that is, 100 factorial) ends in how many consecutive zeros?

   (A) 20  (B) 24  (C) 30  (D) 32  (E) 50

6. How many generators does the group $(\mathbb{Z}_{2^m} +)$ have?

   (A) 2  (B) 6  (C) 8  (D) 10  (E) 12

7. Which one of the following groups is cyclic?

   (A) $\mathbb{Z}_2 \times \mathbb{Z}_4$  (B) $\mathbb{Z}_2 \times \mathbb{Z}_6$  (C) $\mathbb{Z}_2 \times \mathbb{Z}_8$  (D) $\mathbb{Z}_3 \times \mathbb{Z}_9$  (E) $\mathbb{Z}_3 \times \mathbb{Z}_5$

8. If $G$ is a group of order 12, then $G$ must have a subgroup of all of the following orders EXCEPT

   (A) 2  (B) 3  (C) 4  (D) 6  (E) 12
9. How many subgroups does the group \( Z_3 \oplus Z_{10} \) have?

(A) 6  (B) 10  (C) 12  (D) 20  (E) 24

10. If \( S = \{ a \in \mathbb{R}^*: a \neq 1 \} \), with the binary operation \( \cdot \) defined by the equation \( a \cdot b = e^{\log a \cdot b} \) (where \( \log b = \log e \cdot b \)), then \((S, \cdot)\) is a group. What is the inverse of \( a \in S\)?

(A) \( \frac{1}{a \log a} \)  (B) \( \frac{e}{\log a} \)  (C) \( e^{-\log a} \)  (D) \( e^{\log(1/a)} \)  (E) \( e^{1/\log a} \)

11. Which of the following are subgroups of \( GL(2, \mathbb{R}) \), the group of invertible 2 by 2 matrices (with real entries) under matrix multiplication?

I. \( T = \{ A \in GL(2, \mathbb{R}) : \det A = 2 \} \)
II. \( U = \{ A \in GL(2, \mathbb{R}) : A \text{ is upper triangular} \} \)
III. \( V = \{ A \in GL(2, \mathbb{R}) : \text{tr}(A) = 0 \} \)

Note: \( \text{tr}(A) \) denotes the trace of \( A \), which is the sum of the entries on the main diagonal.

(A) I and II only  (B) II only  (C) II and III only  (D) III only  (E) I and III only

12. Let \( p \) and \( q \) be distinct primes. How many (mutually nonisomorphic) Abelian groups are there of order \( p^2 q^3 \)?

(A) 6  (B) 8  (C) 10  (D) 12  (E) 16

13. Let \( G \) be the group generated by the elements \( x \) and \( y \) and subject to the following relations:
\( x^2 = y^3, y^6 = 1, \) and \( x^{-1}yx = y^{-1} \). Express in simplest form the inverse of the element \( z = x^2yxy^3y^3 \).

(A) \( y^2x^{-1} \)  (B) \( xy^2 \)  (C) \( xy \)  (D) \( yx \)  (E) \( y^2x \)

14. Let \( H \) be the set of all group homomorphisms \( \phi: Z_3 \rightarrow Z_v \). How many functions does \( H \) contain?

(A) 1  (B) 2  (C) 3  (D) 4  (E) 6

15. Let \( G \) be a group of order 9, and let \( e \) denote the identity of \( G \). Which one of the following statements about \( G \) CANNOT be true?

(A) There exists an element \( x \) in \( G \) such that \( x \neq e \) and \( x^{-1} = x \).
(B) There exists an element \( x \) in \( G \) such that \( x \neq e \) and \( x^2 = x^6 \).
(C) There exists an element \( x \) in \( G \) such that \( \langle x \rangle \) has order 3.
(D) \( G \) is cyclic.
(E) \( G \) is Abelian.
16. Let $R$ be a ring; an element $x$ in $R$ is said to be idempotent if $x^2 = x$. How many idempotent elements does the ring $\mathbb{Z}_{20}$ contain?

(A) 2  (B) 4  (C) 5  (D) 8  (E) 10

17. Which of the following rings are integral domains?
   I. $\mathbb{Z} \oplus \mathbb{Z}$
   II. $\mathbb{Z}_p'$ where $p$ is a prime
   III. $\mathbb{Z}_p''$ where $p$ is a prime

(A) I and II only  (B) II only  (C) II and III only  (D) III only  (E) I and III only

18. Which one of the following rings does NOT have the same number of units as the other four?

(A) $\mathbb{Z} \oplus \mathbb{Z}$  (B) $\mathbb{Z} \oplus \mathbb{Z}_3$  (C) $\mathbb{Z} \oplus \mathbb{Z}_5$  (D) $\mathbb{Z} \oplus \mathbb{Z}_6$  (E) $\mathbb{Z}_3 \oplus \mathbb{Z}_3$

19. How many elements $x$ in the field $\mathbb{Z}_{11}$ satisfy the equation $x^{12} - x^{10} = 2$?

(A) 1  (B) 2  (C) 3  (D) 4  (E) 5

20. Which of the following are subfields of $\mathbb{C}$?

I. $K_1 = \left\{ a + b\sqrt{2} : a, b \in \mathbb{Q} \right\}$

II. $K_2 = \left\{ a + b\sqrt{2} : a, b \in \mathbb{Q} \text{ and } ab < \sqrt{2} \right\}$

III. $K_3 = \left\{ a + bi : a, b \in \mathbb{Z} \text{ and } i = \sqrt{-1} \right\}$

(A) I only  (B) I and II only  (C) III only  
(D) I and III only  (E) None of the $K_i$ are subfields of $\mathbb{C}$.