Math 130A Homework #4.

Due Wednesday, July 16 at the beginning of discussion.

1. Do Problem #5 in chapter 5.

2. Given a positive integer \( N \), define \( f(x) = \frac{2^x}{N^{N+1}} \) for \( x = 1, 2, \ldots, N \), and \( f(x) = 0 \) elsewhere. Show that \( f \) is a p.m.f.
   Also, given \( X \) is a random variable with p.m.f. \( f \), compute \( \mathbb{E}X \).

3. If \( X \sim \text{Binomial}(4, p) \), compute \( \mathbb{E}\sin(\pi X/2) \).

4. Given a random variable \( X \) that takes finitely many values \( x_1, \ldots, x_N \), show that \( \mathbb{E}(X - c)^2 \) is minimized when \( c = \mathbb{E}X \). (Calculus!)

5. Compute the variance of a \( \text{Poisson}(\lambda) \) random variable.

6. Use the definition of conditional probability to show that a geometric random variable \( X \) has the memoryless property:
   
   \[
   \text{For any } j, k > 0, \quad P(X = j + k \mid X > k) = P(X = j).
   \]

7. Define the function \( g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R} \) by
   
   \[
   g(x, y) = \begin{cases} 
   1 & \text{if } x \geq y \\
   0 & \text{if } x < y 
   \end{cases}
   \]

   (a) Show for any \( x \in \mathbb{N} \) that \( x = \sum_{y=1}^{\infty} g(x, y) \).

   (b) Let \( X \) be a random variable with \( P(X = x) > 0 \) only if \( x \in \{0, 1, 2, \ldots\} \). Prove that
   
   \[
   \sum_{x=1}^{\infty} xP(X = x) = \sum_{y=1}^{\infty} P(X \geq y).
   \]
   
   Hint: use part (a) and interchange the order of summation (this is justified because all terms in the infinite sum are non-negative – for a proof you can see Theorem 0.0.2 in [http://terrytao.files.wordpress.com/2012/12/gsm-126-tao5-measure-book.pdf]).

   (c) Use part (b) to prove that if \( X \sim \text{Geometric}(p) \) then \( \mathbb{E}X = \frac{1}{p} \). You’ll need to sum a geometric series.