Due Monday, July 21 at the beginning of discussion.

1. Do Problem #3 in chapter 7, and Problems #1 and #5 in chapter 8.

2. Let $X \sim \text{Geometric}(p)$. Compute the p.m.f. of $X^2$ and $X + 2$.

3. Write down an explicit sample space $\Omega$, the probabilities $P(\omega)$ for each outcome $\omega \in \Omega$, and the rule for the functions $X : \Omega \to \mathbb{R}$, $Y : \Omega \to \mathbb{R}$, such that $X \sim \text{Uniform}([1, 2, \ldots, N])$ and $Y \sim \text{Geometric}(p)$ are independent random variables.

   Be sure to check that $X$ and $Y$ actually satisfy the definition of independence.

   Show that $E(XY) = E(X)E(Y)$ by computing the sum $\sum_{x=1}^{N} \sum_{y=1}^{\infty} xyP(X = x, Y = y)$.

4. Let $X$ and $Y$ be independent $\text{Geometric}(p_1)$ and $\text{Geometric}(p_2)$ random variables. Compute the probability $P(X = Y)$. (You’ll have to sum a geometric series). In particular, if I roll a die until I see a 6, and you toss a coin until you obtain Heads, what is the chance that my number of rolls equals your number of tosses?

5. In the context of the previous problem, compute $P(X \geq Y)$. Hint: decompose the event $\{X \geq Y\}$ into a disjoint union of events $\{X = x, Y \geq X\}$.

6. In a population of $n$ individuals, assume that each person’s birthday is equally likely to fall on any one of the 365 days in the year. Let $X_i$ be the of people born on day $i$.

   (a) What is the distribution of $X_i$? Give an approximation to $P(X_i \geq 2)$.

   (b) If two or more people are born on the same day, then a large party is thrown on that day. Compute the expected number of large parties that occur during the year. Hint: Indicator trick.

   (c) How large does $n$ need to be to ensure that the above expectation is at least 1?

7. Review Example 3.7 in the notes (the Matching problem). In the context of that problem, let $X$ be the number of employees who end up with their own gift. Our goal is to find the approximate p.m.f. of $X$.

   (a) Show that the probability that employees $1, 2, \ldots, k$ end up with their own gift is $\frac{1}{(n)_k}$.

   (b) Show that the probability that only employees $1, 2, \ldots, k$ end up with their own gift is

   $$\frac{1}{(n)_k} \left[ 1 - \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{n-k} \frac{1}{(n-k)!} \right].$$

   (Apply the result of Example 3.7 to the remaining $n - k$ employees).

   (c) Show that $P(X = k) \approx e^{-1} \frac{1}{k!}$ for $n$ large. Thus $X$ is approximately a Poisson(1) random variable.