Quiz 2

**Instruction:** There are three problems (see back side). No calculators. Show all work clearly and in order.

1. (4pts) You roll a die 5 times. What is the chance of seeing exactly two 5's or exactly two 6's (or both)?

Let \( A = \{ \text{exactly two 5's} \} \), \( B = \{ \text{exactly two 6's} \} \).

Want \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \).

\[
P(A) = \frac{\binom{5}{2} \cdot 5^3}{6^5} \quad \text{(choose locations of 5's, choose 3 other die values)}
\]

\[
P(A \cap B) = \frac{1}{6^5} \left( \binom{5}{2} \cdot \binom{3}{2} \right) \cdot 4 \quad \text{values of 5's, loc. of 6's, 3 remaining spots}
\]

So \( P(A \cup B) = \frac{1}{6^5} \left[ 2 \left( \binom{5}{2} \right) 5^3 - \binom{5}{2} \binom{3}{2} \cdot 4 \right] \)

2. (4pts) An urn contains exactly \( r \) red balls and \( b \) black balls. You will draw a ball. If it's red, you throw it away. If it's black, you return it to the urn with another black ball.

Then draw a second ball. What is the probability that the second ball is black?

On each draw all balls in the urn are equally likely to be chosen.

Let \( B_1 = \{ 1^{st} \text{ ball is black} \} \), \( B_2 = \{ 2^{nd} \text{ ball is black} \} \).

So Bayes' theorem:

\[
P(B_2) = P(B_2 | B_1) \cdot P(B_1) + P(B_2 | B_1^c) \cdot P(B_1^c)
\]

\[
= \frac{b+1}{r+b+1} \cdot \frac{b}{r+b} + \frac{b}{r-1+b} \cdot \frac{r}{r+b}
\]

For the second ball, gained black ball, threw red ball away.
3. (+2pts) In the context of the previous problem, for which values of \( r \) and \( b \) are the events \{first draw is black\} and \{second draw is black\} independent?

If \( r = 0 \) or \( b = 0 \), then \( \Pr(B_1) = \Pr(B_2) = 0 \) or 1

Else: Need

\[
\Pr(B_2 | B_1) = \Pr(B_2)
\]

for independence to hold.

\[
\frac{b+1}{r+b+1} = \frac{b+1}{r+b} \cdot \frac{b}{r+b} + \frac{b}{r-1+b} \cdot \frac{r}{r+b} \quad \text{by Q2}
\]

\[
\Rightarrow \quad 1 - \frac{b}{r+b} = \frac{r}{r+b}
\]

\[
\frac{b+1}{r+b+1} \cdot \frac{r}{r+b} = \frac{b}{r-1+b} \cdot \frac{r}{r+b}
\]

So

\[
\frac{b+1}{r+b+1} = \frac{b}{r-1+b} \quad \Rightarrow \quad b(r+b+1) + 1(r-1+b) = b(r+b+1)
\]

\[
\Rightarrow \quad r-1+b = 2b
\]

So \( r = b+1 \)