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# Solving Polynomials After Klein: The Theory of Resolvent Degree

Thursday, May $16^{th}$, 2019

## Solving Polynomials - Classical Viewpoint

### Classical Question

Given a polynomial

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

find and understand the roots of $P(z)$ in terms of $a_1, \ldots, a_n$.

$1 \leq n \leq 4$ - Have formulas using radicals and $+, -, \times, \div$

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But polynomials still have roots!

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Understand formulas via algebraic geometry and topology.

Start with the quadratic formula.

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## Quadratic Formula - Classic

$$P(z) = z^2 + bz + c$$

$$z = \frac{-b + \sqrt{b^2 - 4c}}{2}$$

Remark: View radicals in the classical sense, i.e. as a *multi-valued* function

$$\sqrt[d]{w} := \left\{ z \mid z^d - w = 0 \right\}.$$

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# Quadratic Formula - Topology

$\mathbb{C}^2_{(b,c)}$ - space parametrizing the polynomials $z^2 + bz + c$

Key component is the square root of the discriminant

$$\begin{array}{ccc} & & \mathbb{C}^1 \\ & & \Big\downarrow{\scriptstyle z \mapsto z^2} \\ \mathbb{C}^2_{(b,c)} & \xrightarrow[b^2-4c]{} & \mathbb{C}^1 \end{array}$$

# Quadratic Formula - Topology

Complete this to a pullback square

$$\begin{array}{ccc} E_1 & \longrightarrow & \mathbb{C}^1 \\ \downarrow & & \downarrow {\scriptstyle z \mapsto z^2} \\ \mathbb{C}^2_{(b,c)} & \xrightarrow[b^2-4c]{} & \mathbb{C}^1 \end{array}$$

$$E_1 = \left\{(b, c, \delta) \in \mathbb{C}^3 \mid \delta^2 = b^2 - 4c\right\}$$

## Quadratic Formula - Topology

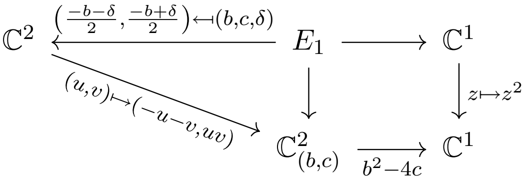
From here, we can get the roots...

$$\begin{array}{ccccc}
\mathbb{C}^2 & \xleftarrow{\left(\frac{-b-\delta}{2},\frac{-b+\delta}{2}\right) \leftarrow\!\shortmid (b,c,\delta)} & E_1 & \longrightarrow & \mathbb{C}^1 \\
 & & \big\downarrow & & \big\downarrow {\scriptstyle z \mapsto z^2} \\
 & & \mathbb{C}^2_{(b,c)} & \xrightarrow[b^2-4c]{} & \mathbb{C}^1
\end{array}$$

$$E_1 = \left\{(b, c, \delta) \in \mathbb{C}^3 \mid \delta^2 = b^2 - 4c\right\}$$

# Quadratic Formula - Topology

... and then get back to the original polynomial.

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## Quadratic Formula - Topology

Focus on the map $E_1 \to \mathbb{C}^2_{(b,c)}$

- Comes from pullback square
- Top - 2-sheeted branched cover
- Alg Geom - generically finite, dominant, rational map

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## Quadratic Formula - Topology

### Definition

A **covering space** (or simply, a **cover**) is a continuous surjection $p : Y \to X$ that can be locally trivialized around every point.

More explicitly, we can find a neighborhood $U_x$ of every point $x$ such that

$$p^{-1}(U_x) \cong \bigsqcup_{i \in I} U_x \, .$$

We say $p : Y \to X$ is $n$-**sheeted** if $|I| = n$ for all $x$.

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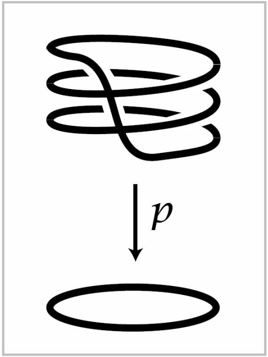


Figure 1: A $3$-sheeted cover of $S^1$

*Image from Allen Hatcher's Algebraic Topology, p.56*

## Quadratic Formula - Topology

### Definition

A **branched covering space (branched cover)** of complex varieties is a map $p : Y \to X$ such that

$$p|_{X \setminus \mathcal{B}} : p^{-1}(X \setminus \mathcal{B}) \to X \setminus \mathcal{B}$$

is a cover (in classical topology) for some Zariski closed subvariety $\mathcal{B}$ of $X$.

We refer to the minimal such $\mathcal{B}$ as the **branch locus** of $p$.

# Quadratic Formula - Topology

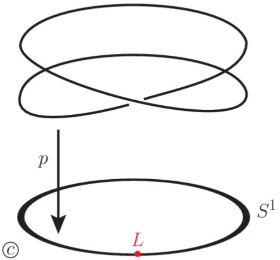


Figure 1: A $2$-sheeted branched cover of $S^1$

*Image from Christoper Dustin's blog,*
*"Representing Spacetime as a Branched Covering Space", (Link)*

# Quadratic Formula - Topology

$$\begin{array}{ccc} E_1 & \longrightarrow & \mathbb{C}^1 \\ \downarrow & & \downarrow{\scriptstyle z \mapsto z^2} \\ \mathbb{C}^2_{(b,c)} & \xrightarrow[b^2-4c]{} & \mathbb{C}^1 \end{array}$$

Why is $E_1 \to \mathbb{C}^2_{(b,c)}$ a *branched* cover?

When $b^2 - 4c = 0$, the fiber collapses to a point
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## Complex Varieties

### Definition

A **complex variety (variety over** $\mathbb{C}$) is a reduced scheme of finite type over $\text{Spec}(\mathbb{C})$.

Varieties are reduced, but may not be irreducible.

# Categories

We define two categories:

- **IrrVars/$\mathbb{C}$** - objects are irreducible complex varieties, morphisms are dominant rational maps
- **Fields/$\mathbb{C}$** - objects are field extensions of $\mathbb{C}$ with finite transcendence degree, morphisms are field embeddings

## Equivalences of Categories

### *Lemma*

*The functor induced by*

$$\begin{aligned} \mathbb{C} : \textbf{IrrVars}/\mathbb{C}^{\textbf{op}} &\to \textbf{Fields}/\mathbb{C} \\ X &\mapsto \mathbb{C}(X) \end{aligned}$$

*is an equivalence of categories.*

## Equivalences of Categories

### *Corollary*

*The induced functor on arrow categories*

$$\mathbf{Ar}(\mathbb{C}) : \mathbf{Ar}(\mathbf{IrrVars}/\mathbb{C}^{\mathbf{op}}) \to \mathbf{Ar}(\mathbf{Fields}/\mathbb{C})$$
$$(Y \to X) \mapsto (\mathbb{C}(X) \hookrightarrow \mathbb{C}(Y))$$

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**Takeaway:** Today - branched covers of complex varieties. Can also tell the same story in terms of field extensions

## Equivalences of Categories

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**Takeaway:** Today - branched covers of complex varieties. Can also tell the same story in terms of field extensions

# Algebraic Functions

## Definition

Let $X$ be a complex variety. An **algebraic function** on $X$ is an $n$-valued function

$$\phi : X \to \mathbb{C}$$
$$x \mapsto \left\{z \mid z^n + a_1(x)z^{n-1} + \cdots + a_n(x) = 0\right\}$$

where each $a_i$ is a rational function on $X$

## Algebraic Functions

### Example

Let $X$ be the complex variety $\mathbb{C}^n$ and define $a_i$ to be the $i^{\text{th}}$ coordinate function

$$\begin{aligned} a_i : X &\to \mathbb{C} \\ x &\mapsto x_i \end{aligned}$$

Define the algebraic function $\Phi_n$ as follows:

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Re-state classical question in this language:

## Classical Question (Re-stated)

Give a formula for $\Phi_n$.

What is a formula for an algebraic function?

Generalization of the topological version of quadratic formula

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# Formula for a Branched Cover

Given a branched cover of complex varieties $Y \to X$, a **formula in functions of** $d$ **variables** for $Y \to X$ of length $r$ is a finite tower of branched covers of complex varieties

$$X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 \subseteq X$$

such that

- $X_0 \subseteq X$ is a dense Zariski open,
- $X_r \to X$ factors through a branched cover $X_r \to Y$,
- each map $X_i \to X_{i-1}$ comes from a pullback square of complex varieties of dimension at most $d$.

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## Formula for an Algebraic Function

How does this help us define formulas for algebraic functions?

Given an algebraic function $\phi$, we construct a canonical branched cover

## Construction of a Branched Cover Associated to an Algebraic Function

Let $X$ be a complex variety and $\phi$ an algebraic function on $X$ given by

$$x \mapsto \left\{z \mid z^n + a_1(x)z^{n-1} + \cdots + a_n(x) = 0\right\}.$$

Explicitly write $a_i(x) = \frac{f_i(x)}{g_i(x)}$ and set $U = X \setminus Z(g_1, \ldots, g_n)$. Construct

$$E_\phi = \overline{\{(x,z) \in U \times \mathbb{P}^1 \mid z^n + a_1(x)z^{n-1} + \cdots + a_n(x) = 0\}} \subseteq X \times \mathbb{P}^1.$$

Get branched cover $E_\phi \to X$ given by $(x,z) \mapsto x$.

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Want a formula for $\Phi_n$. Moreover, want the formula to be as simple as possible. Need to make this precise.

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## Resolvent Degree and Essential Dimension

### Definition

An $n$-sheeted cover $Y \to X$ is **defined** over a variety $X_0$ if there is an $n$-sheeted cover $Y_0 \to X_0$ such that

$$Y \cong Y_0 \times_{X_0} X$$

for some map $X \to X_0$.

The **essential dimension** of $Y \to X$ is

ed$(Y \to X) = \min \{\dim(X_0) \mid Y \to X$ is defined over $X_0\}$.

# Resolvent Degree and Essential Dimension

Equivalently, the **essential dimension** of $Y \to X$ is

$$\text{ed}(Y \to X) = \min \{d \mid \exists \text{ a formula of length 1 in } d \text{ variables}\}.$$

.

# Resolvent Degree and Essential Dimension

## Definition

The **resolvent degree** of $Y \to X$ is

$$\text{RD}(Y \to X) = \min \{d \mid \exists \text{ a formula in } d \text{ variables}\}$$

## Resolvent Degree and Essential Dimension

Given an algebraic function $\phi$ on $X$, the **essential dimension** / **resolvent degree** of $\phi$ is the essential dimension / resolvent degree of $E_\phi \to X$.

- ed($\phi$) - how simply we can write $\phi$
- RD($\phi$) - how simply we can write a formula for $\phi$

$$\text{ed}(n) := \text{ed}(\Phi_n) \qquad \text{RD}(n) := \text{RD}(\Phi_n)$$

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## Examples of RD and ed

What do we know?

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| ed$(n)$ | | | | | |
| RD$(n)$ | | | | | |

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"Kronecker's Theorem" - Felix Klein

Solving quintic in one step requires functions of two variables

Using longer towers, only need functions of one variable

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## Upper Bounds on RD

Essential dimension $\neq$ resolvent degree

Focus on resolvent degree

**Upper bounds on resolvent degree:**

$\mathrm{RD}(5) = 1$ (Bring, Klein)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
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## Upper Bounds on RD

Essential dimension $\neq$ resolvent degree

Focus on resolvent degree

**Upper bounds on resolvent degree:**

RD$(6) \leq 2$ (Hamilton, Klein)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| RD$(n)$ | 1 | 1 | 1 | 1 | 1 | $\leq 2$ | | | |

# Upper Bounds on RD

Essential dimension $\neq$ resolvent degree

Focus on resolvent degree

**Upper bounds on resolvent degree:**

RD$(7) \leq 3$ (Hamilton, Klein)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
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**Upper bounds on resolvent degree:**

RD$(8) \leq 4$ (Hamilton)

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|---|---|---|---|---|---|---|---|---|---|
| RD$(n)$ | 1 | 1 | 1 | 1 | 1 | $\leq 2$ | $\leq 3$ | $\leq 4$ | |

# Upper Bounds on RD

Essential dimension $\neq$ resolvent degree

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**Upper bounds on resolvent degree:**

RD$(9) \leq 4$ (Hilbert)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| RD$(n)$ | 1 | 1 | 1 | 1 | 1 | $\leq 2$ | $\leq 3$ | $\leq 4$ | $\leq 4$ |

# Hilbert's Conjectures

Translate Hilbert's conjectures into modern language:

- Hilbert's Sextic Conjecture: $\text{RD}(6) = 2$
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## Bounds on Resolvent Degree

Hamilton (1836), Sylvester (1887), Brauer (1975), and Wolfson - Upper bounds on RD$(n)$

Bounds not expected to be sharp (for large $n$)

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## Quote by Dixmier

Conclusion to Dixmier's summary on Hilbert's 13th problem [1]

> *"Let's end on a dramatic note, which proves our incredible ignorance. Although this seems unlikely, it is not impossible that RD$(n) = 1$ for all $n$!*
> *. . . Any reduction of RD$(n)$ would be serious progress. In particular, it is time to know whether RD$(6) = 1$ or RD$(6) = 2$."*

[1] J. Dixmier, "Histoire du $13^e$ problème de Hilbert," in: Analyse diophantienne et géom'etrie algébrique, Cahiers Sém. Hist. Math., Sér 2, vol. 3, Univ. Paris VI, Paris, 1993, p85-94.

## Remaining Goals for the Talk

Formulas for Sextic - Hamilton, detailed sketch by Klein

Research Goal - Understand precise geometric relationship between solutions of Hamilton and Klein

Hopefully gives insight to resolvent degree, as well.

Start with Klein's solution of the quintic.

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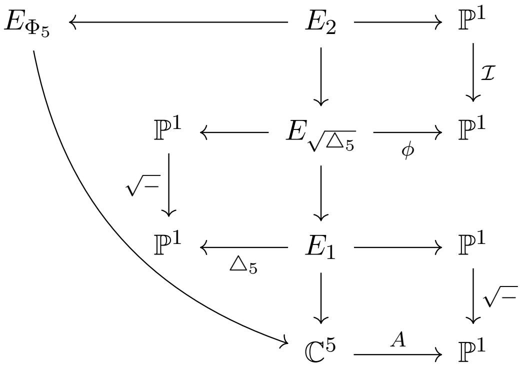
# Formula for the Quintic

**Theorem** (Klein)




is a formula for the quintic (in one variable functions).

## Components of The Tower

- $E_1 \to \mathbb{C}^5$ - reduction of quintic to the normal form $z^5 + az^2 + bz + c$
- $E_{\sqrt{\triangle_5}} \to E_1$ - adjoin square root of discriminant
- Icosahedral cover

$$\mathcal{I} : \mathbb{P}^1 \to \mathbb{P}^1 \cong \mathbb{P}^1 / A_5$$
$$[z_1 : z_2] \mapsto \left[H(z_1, z_2)^3 : 1728 f(z_1, z_2)^5\right]$$

$f, H$ - polynomials invariant under action of $A_5$ (correspond to vertices, faces)

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## The Analytic Part

Klein - complete algebraic solution of the quintic

Further, use analytic functions to solve polynomials.

Example:

$$z^n = w \quad \Leftrightarrow \quad z = e^{\frac{1}{n}\log(w)}$$

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## The Bring Curve

Bring (1786) also gave a solution to the quintic

Bring reduced generic quintic to $z^5 + az + b$

If $z_1, \dots, z_5$ are roots of a polynomial of the form $z^5 + az + b$, then

$$\sum_{k=1}^{5} z_k = \sum_{k=1}^{5} z_k^2 = \sum_{k=1}^{5} z_k^3 = 0$$

Equations define a subvariety $C_B \subseteq \mathbb{P}^4$ - **Bring curve**

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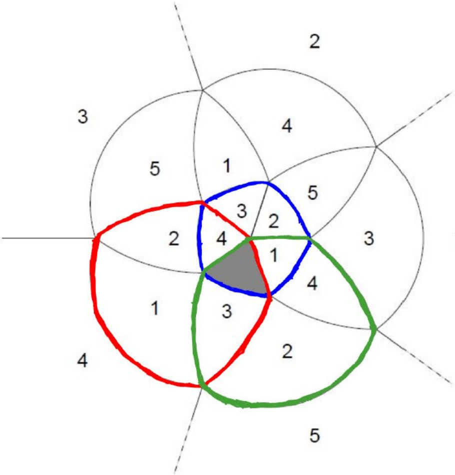

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(Green) Natural 3-sheeted branched covering $C_B \to \mathbb{P}^1$

<u>Remark:</u> Analogues of Bring curve for degrees 2,3,4 are rational.

$C_B$ is not a rational curve

# Total Solution of the Quintic

Total solution of the quintic (both the algebraic and analytic parts) comes down to understanding:

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Last stage of Klein's solution for quintic comes from icosahedron, $A_5 \curvearrowright \mathbb{P}^1$

$A_6$ does not act on $\mathbb{P}^1$, but does act on $\mathbb{P}^2$

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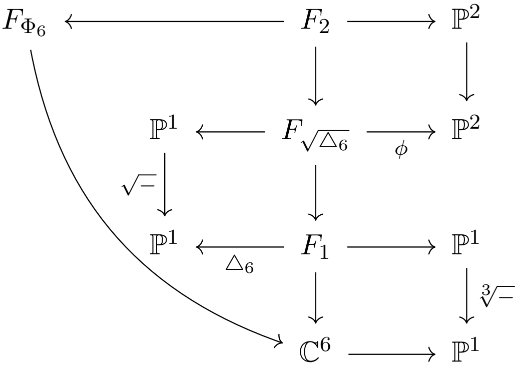
# Ideal Formula for the Sextic



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Hamilton's reduction to normal form for sextic defines analogue of Bring curve - a surface $S_H$

Expect that $S_H$ is a $K3$ surface

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$$A_6 \curvearrowright S_H \overset{2:1}{\dashrightarrow} \mathbb{P}^2 \curvearrowleft A_6$$

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# Thank You!

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