Solving Polynomials After Klein:  
The Theory of Resolvent Degree  
Alexander J. Sutherland  
May 16th, 2019

Abstract
Solving polynomials is a classical problem. Here, we take a geometric/topological approach that combines the classical viewpoints pioneered by Klein and Hilbert (among others) and the modern frameworks of algebraic geometry, topology, and differential topology. In Section 1, we examine the quadratic formula and introduce the covering space theory. Section 2 describes algebraic functions and their relation to solving polynomials, while also introducing monodromy. In Section 3, we provide a modern description of formulas for branched covers and algebraic functions, as well as introducing irrationalities and Galois theory for covers. Essential dimension and resolvent degree - notions of simplicity regarding branched covers/algebraic functions, as well as their formulas - are given in Section 4. Moreover, we state the problems of Kronecker and Klein, Hilbert’s conjectures, and the current state of the theory of resolvent degree. Section 5 is concerned with properties of essential dimension and resolvent degree. In Section 6, we discuss Bring and Klein’s algebraic solutions of the quintic, along with Green’s analytic solution of the quintic. Finally, Section 7 describes the analogues of Bring and Klein given by Hamilton and Klein for the sextic and the work that remains to describe a complete solution of the sextic.
## Contents

1. **Introduction** 3
2. **Algebraic Functions** 8
3. **Formulas for Algebraic Functions** 13
4. **Essential Dimension and Resolvent Degree** 17
5. **Properties of Essential Dimension and Resolvent Degree** 22
6. **Solving the Quintic** 23
7. **Solving the Sextic** 28
8. **Appendix A: Category Theory** 31
9. **Appendix B: The Theory of Resolvent Degree for Field Extensions** 32
10. **Appendix C: Proof of Lemma 5.6** 35
11. **Bibliography** 36
1 Introduction

We start with the following classical problem.

Problem 1.1. (Classical Problem)
Given a polynomial
\[ P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \]
find and understand the roots of \( P(z) \).

While many of the definitions and results discussed in this paper work in a more general context, we will restrict ourselves to working over \( \mathbb{C} \).

Definition 1.2. (Generic Polynomials)
A polynomial
\[ P_n(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n \]
is generic over \( \mathbb{C} \) if the set \( \{a_1, \ldots, a_n\} \) is algebraically independent over \( \mathbb{C} \).

Abel proved that \( P_n \), the generic polynomial of degree \( n \), is solvable in radicals if and only if \( n = 1, 2, 3, \) or \( 4 \). However, this does not mean that we cannot understand polynomials of higher degree. Rather, it requires that we expand the context of what we understand a formula for a generic polynomial to be. We will address this question precisely in Section 3. For now, we look to the generic quadratic for our motivation.

Example 1.3. (The Quadratic Formula)
Consider the generic polynomial \( z^2 + bz + c \). We are used to thinking of the quadratic formula as the equation
\[ z = \frac{-b + \sqrt{b^2 - 4c}}{2}. \]
Note that we are using "\( \sqrt{b^2 - 4c} \)" here instead of "\( \pm \sqrt{b^2 - 4c} \)", which is not well-defined for \( b, c \in \mathbb{C} \). The approach that is most fruitful going forward is to view \( \sqrt{\_} \) as an algebraic function that is multi-valued, i.e.
\[ \sqrt{d} = \{ z \mid z^2 - d = 0 \}. \]
This will become more clear once we formally introduce algebraic functions in Section 2 and their formulas in Section 3.

We will now give another perspective on the quadratic formula that is more geometric and topological. Our goal is to give a geometric construction that describes the roots of \( z^2 + bz + c \). We start with \( \mathbb{C}^2(b,c) \), which parametrizes the possible coefficients of \( z^2 + bz + c \). As in equation (1), the square root of
the discriminant is the crucial component and it now takes form as a pullback square:

\[
\begin{array}{ccc}
E_1 & \longrightarrow & C \\
\downarrow & & \downarrow \\
C^2_{(b,c)} & \dashrightarrow & b^2 - 4c
\end{array}
\]

where \( E_1 = \{(b,c,\delta) \in \mathbb{C}^3 \mid \delta^2 = b^2 - 4c\} \). From \( E_1 \), we can describe the roots of \( z^2 + bz + c \) in \( \mathbb{C}^2 \):

\[
\begin{array}{ccc}
C^2_{(u,v)} & \xleftarrow{(u,v) \mapsto (-u-v,uv)} & E_1 \\
\downarrow & & \downarrow \\
C^2_{(b,c)} & \longrightarrow & C
\end{array}
\]

Finally, from the roots \( (u,v) \), we can obtain the polynomial \( z^2 + bz + c \) via elementary symmetric polynomials:

\[
\begin{array}{ccc}
C^2_{(u,v)} & \xrightarrow{(u,v) \mapsto (-u-v,uv)} & E_1 \\
\downarrow & & \downarrow \\
C^2_{(b,c)} & \longrightarrow & C
\end{array}
\]

Before we examine these maps further, we explicitly state our definition of a variety to avoid any ambiguity.

**Definition 1.4. (Complex Varieties)**

A **complex variety** (or a **variety over** \( \mathbb{C} \)) is a reduced scheme of finite type over \( \text{Spec}(\mathbb{C}) \).

In particular, varieties are reduced, but not necessarily irreducible.

We remark that \( E_1 \) is indeed a pullback in the category of varieties and further examine the map \( \pi : E_1 \rightarrow C^2_{(b,c)} \). Note that \( \sqrt{d} \) is a 2-valued function for all \( d \in \mathbb{C} \setminus \{0\} \) and is 1-valued at \( d = 0 \). Define

\[
\mathcal{B} = \{(b,c) \mid b^2 - 4c = 0\} \subseteq C^2_{(b,c)}
\]

One can check that the map

\[
\pi|_{C^2_{(b,c)} \setminus \mathcal{B}} : \pi^{-1} \left( C^2_{(b,c)} \setminus \mathcal{B} \right) \rightarrow C^2_{(b,c)} \setminus \mathcal{B}
\]
is a covering space, when we view these spaces with their classical topology. For the unfamiliar reader, we provide the formal definition and the intuition behind covering spaces.

**Definition 1.5. (Covering Spaces)**
Let $X$ be a topological space. A **covering space** (or simply, a **cover**) of $X$ is a topological space $Y$ and a continuous surjection $p : Y \to X$ such that for any $x \in X$, there is an open neighborhood $O$ of $x$ such that there exists a discrete space $F$ and an isomorphism $f : p^{-1}(O) \cong O \times F$ such that

$$
p^{-1}(O) \xrightarrow{f} O \times F \xrightarrow{\text{proj}} O
$$

commutes, where the map $O \times F \to O$ is given by projection onto $O$. In such a situation, we refer to $X$ as the **base space**, $Y$ as the **total space**, and $f$ as a **local trivialization** of $p$ at $x$.

Finally, we say that $p$ is **finitely-sheeted** (respectively, **$n$-sheeted** if the cardinality of every fiber is finite (respectively, has cardinality $n$). When $p$ is $n$-sheeted, we also refer to $n$ as the **degree** of the map.

The key idea about covering spaces is that the total space locally looks like disjoint copies of the base space, but may have more complicated global structure. We can extend the notion of a covering space to include our map $\pi$ from above.

**Remark 1.6.** As above and from here on out, whenever we speak of a (branched) cover of complex varieties $Y \to X$, we mean that it is a (branched) cover when we view $X$ and $Y$ with their classical topologies. This allows us to sidestep notions of generically étale maps.

**Definition 1.7. (Branched Covering Spaces)**
Let $p : Y \to X$ be a map of complex varieties. We say that $p$ is a **branched covering space** (or simply, a **branched cover**) if $p|_{X \setminus B} : p^{-1}(X \setminus B) \to X \setminus B$

is a cover, where $B$ is a Zariski closed subvariety of $X$. In such a case, we say the minimal such $B$ is the **branch locus** of $p$. Finally, we say that $p$ is **finite-sheeted** (respectively, **$n$-sheeted**) if the cover $p|_{X \setminus B}$ is.

**Remark 1.8.** A map of varieties $p : Y \to X$ is a (finite-sheeted) branched cover if and only if it is a (generically finite) dominant, rational map.
Indeed, the map $\pi : E_1 \to C_{(b,c)}$ is exactly a 2-sheeted branch cover and $n$-sheeted covers will be pervasive from here on out. Before moving on to describing algebraic functions in detail, it is worth delving into the relationship between covering spaces and field extensions.

Suppose that we have an affine complex variety $X$. Then, we have a corresponding $\mathbb{C}$-algebra of regular functions on $X$, which we denote by $A(X)$. Moreover, when $X$ is irreducible, $A(X)$ is an integral domain. Thus, we have a field of rational functions on $X$, which is $\mathbb{C}(X) := \text{Frac}(A(X))$. To make precise the relationship between $X$ and $\mathbb{C}(X)$, we must introduce additional language and notation.

**Definition 1.9. (Category of Irreducible Complex Varieties)**

We denote by $\text{IrrVars}/\mathbb{C}$ the category whose objects are irreducible complex varieties and where morphisms are dominant rational maps.

**Definition 1.10. (Category of Fields over $\mathbb{C}$)**

We denote by $\text{Fields}/\mathbb{C}$ the category whose objects are field extensions $\mathbb{C} \hookrightarrow K$ of finite transcendence degree and where morphisms are field embeddings $K \hookrightarrow L$ over $\mathbb{C}$.

We leave it to the reader to verify that the above are indeed categories.

**Lemma 1.11. (Equivalence of Categories)**

The functor induced by

$$C : \text{IrrVars}/\mathbb{C}^{\text{op}} \to \text{Fields}/\mathbb{C}$$

$$X \mapsto \mathbb{C}(X)$$

is an equivalence of categories.

**Proof.** First, we remark that any irreducible complex variety has a dense open subvariety that is affine, so the map $X \mapsto \mathbb{C}(X)$ is well-defined for general irreducible complex varieties. From algebraic geometry, we have the well-known bijection:

$$\left\{ \text{Maps } Y \to X \text{ of affine complex varieties} \right\} \leftrightarrow \left\{ \text{Maps of coordinate rings } A(X) \to A(Y) \right\}.$$

When $X$ and $Y$ are irreducible, $A(X)$ and $A(Y)$ are integral domains. Consequently, we have a bijection:

$$\left\{ \text{Dominant, rational maps } Y \to X \text{ irreducible affine complex varieties} \right\} \leftrightarrow \left\{ \text{Field extensions } \text{Frac}(R) \to \text{Frac}(S) \right\}.$$

Consequently, $C$ is a fully-faithful functor. It remains to show is that $C$ is essentially surjective.

Suppose that $\mathbb{C} \hookrightarrow L$ is a field extension of transcendence degree $n$. Let $\{e_1, \ldots, e_n\}$ be a transcendence basis for $L$ over $\mathbb{C}$. Then, $L/\mathbb{C}(e_1, \ldots, e_n)$ is
a finite extension of fields. As $C$ is characteristic 0, the extension is separable. Consequently, the primitive element theorem yields that there is $\alpha \in L$ such that $L \cong C(e_1, \ldots, e_n) (\alpha)$. Let $m(x)$ be the minimal polynomial for $\alpha$ over $C(e_1, \ldots, e_n)$ and take $S = C[e_1, \ldots, e_n]$. By clearing denominators of $m(x)$, we get a polynomial $p(x) \in S[x]$. Set $R = S[x]/p(x)$ and note that $p(x)$ irreducible yields $R$ is an integral domain. Thus, $X = \text{Spec}(R)$ is an irreducible complex variety with $C(X) \cong L$.

The above equivalence is useful in its own right, but we will actually use the equivalence of Corollary 1.13 more often. However, we must first introduce the notion of an arrow category.

**Definition 1.12. (Arrow Categories)**

Let $C$ be a category. We define the arrow category of $C$, denoted by $\text{Ar}(C)$ (or $C \triangleleft_1$) to be the category whose objects are the morphisms of $C$ and where a morphism of

$$f : X \rightarrow Y, \ g : Z \rightarrow W \in \text{Ar}(C)$$

is a commuting diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Z \\
\downarrow f & & \downarrow g \\
Y & \xrightarrow{\beta} & W
\end{array}
$$

Again, we leave it to the reader to verify that if $C$ is a category, then $\text{Ar}(C)$ is indeed a category. As a consequence of Lemma 1.11 and Lemma 8.1 (in Appendix A), we can conclude

**Corollary 1.13. (Induced Equivalence of Arrow Categories)**

The equivalence of categories $C$ induces an equivalence of arrow categories

$$\text{Ar}(C) : \text{Ar}(\text{IrrVars}/C^{\text{op}}) \rightarrow \text{Ar}(\text{Fields}/C).$$

In particular, it induces an equivalence between the full subcategories of finite-sheeted branched covers of irreducible, affine complex varieties and finite extensions of fields $K \hookrightarrow L$ over $C$. 

7
2 Algebraic Functions

In the previous section, we mentioned that \( \sqrt{d} = \{ z \mid z^2 - d = 0 \} \) was an algebraic function. We now make precise what this means.

**Definition 2.1. (Algebraic Functions)**

Let \( X \) be a complex variety. An algebraic function \( \phi \) on \( X \) is an \( n \)-valued function

\[
\phi : X \rightarrow \mathbb{C}
\]

\[
x \mapsto \{ z \mid z^n + a_1(x)z^{n-1} + \cdots + a_n(x) = 0 \}
\]

for some rational functions \( a_1, \ldots, a_n : X \rightarrow \mathbb{C} \).

Our goal is to better understand generic polynomials through the lens of algebraic functions. In particular, we will soon provide a construction that takes an algebraic function \( \phi \) on a complex variety \( X \) and produces a branched cover of \( X \). We can then study the monodromy group of this branched cover to gain insight into \( \phi \).

With this in mind, we first show how to think about generic polynomials as algebraic functions. We emphasize here that algebraic functions are *multi-valued* instead of necessarily single-valued. While this is in contrast with much of the mathematics of the 20th century, the functions of classical interest that arise are multi-valued.

**Example 2.2. (Generic Polynomials as Algebraic Functions)**

Let \( X \) be the complex variety \( \mathbb{C}^n \) and define the \( a_i \) to be the coordinate functions

\[
a_i : X \rightarrow \mathbb{C}
\]

\[
(x_1, \ldots, x_n) \mapsto x_i.
\]

Then, the polynomial \( P(z) = z^n + a_1 z^{n-1} + \cdots + a_n \) is generic and we have an algebraic function

\[
\phi : X \rightarrow \mathbb{C}
\]

\[
x \mapsto \{ z \mid z^n + a_1(x)z^{n-1} + \cdots + a_n(x) = 0 \}
\]

taking a specified polynomial to its roots. Note that \( \phi \) is generically \( n \)-valued.

We now proceed to construct a branched cover from an algebraic function.

**Construction 2.3.** Let \( X \) be a complex variety and \( \phi \) an algebraic function given by

\[
x \mapsto \{ z \mid z^n + a_1(x)z^{n-1} + \cdots + a_n(x) = 0 \}.
\]

for some rational functions \( a_1, \ldots, a_n \). For each \( 1 \leq i \leq n \), write

\[
a_i(x) = \frac{f_i(x)}{g_i(x)}
\]
By setting $Z = Z(g_1, \ldots, g_n)$, we observe that all of the $a_i$ are well-defined on $U = X \setminus Z$. Thus,

$$\{(x, z) \in U \times \mathbb{P}^1 \mid z^n + a_1(x)z^{n-1} + \cdots + a_n(x) = 0\} \subseteq X \times \mathbb{P}^1.$$ 

is well-defined. From here, we construct the complex variety

$$E_\phi = \{(x, z) \in U \times \mathbb{P}^1 \mid z^n + a_1(x)z^{n-1} + \cdots + a_n(x) = 0\} \subseteq X \times \mathbb{P}^1,$$

which has canonical projections

$$\begin{array}{ccc}
E_\phi & \xrightarrow{\pi_1} & \mathbb{P}^1 \\
\pi_0 & \downarrow & \\
X & & 
\end{array}$$

Definition 2.4. (Riemann Varieties)

Given an algebraic function $\phi$ on a complex variety $X$, we refer to $E_\phi$ (as in Construction 2.3) as the Riemann variety of $\phi$.

Proposition 2.5. (Branched Covers of Algebraic Functions)

Given an algebraic function $\phi$ on a complex variety $X$, the map

$$\pi_\phi : E_\phi \rightarrow X$$

is a branched covering space.

We refer to $\pi_\phi : E_\phi \rightarrow X$ as the branched cover associated to $\phi$.

Proof. As $\phi$ is an algebraic function on $X$, it has the form

$$x \mapsto \{z \mid z^n + a_1(x)z^{n-1} + \cdots + a_n(x) = 0\}.$$

for some rational functions $a_1, \ldots, a_n$ on $X$. Thus, given $x \in X$, define $p_x(z)$ to be the polynomial

$$z^n + a_1(x)z^{n-1} + \cdots + a_n(x).$$

We can thus describe the branch locus of $\pi_\phi$ as

$$\mathcal{B} = \{x \in X \mid p_x(z) \text{ has a repeated root}\},$$

since this exactly characterizes when the fiber of $x$ has fewer than $n$ points. We can equivalently describe this set using the vanishing of the discriminant; i.e.

$$\mathcal{B} = \{x \in X \mid \Delta_n(p_x) = 0\}.$$
which shows that $B$ is indeed a Zariski closed subvariety. Now, to show that
\[
\pi_{\phi}\mid_{X\setminus B} : \pi_{\phi}^{-1}(X \setminus B) \to X \setminus B
\]
is a cover, it suffices to construct local trivializations. Indeed, the roots of a polynomial vary smoothly as the coefficients vary, so it is easy to see that for any sufficiently small open $O \subseteq X \setminus B$ (in $X \setminus B$’s classical topology), we have
\[
\pi_{\phi}^{-1}(O) \cong \bigsqcup_{i=1}^{n} O.
\]
Equivalently, $\pi_{\phi}\mid_{X\setminus B}$ is a generically finite, dominant map by construction, which also yields that $\pi_{\phi}$ is a finite-sheeted cover.

We now define the monodromy group of a cover.

**Proposition 2.6. (Uniqueness of Path-Lifting for Covers)**
Suppose that $p : Y \to X$ is a cover of topological spaces with $Y$ connected and locally path connected. For any $x \in X$, $y \in p^{-1}(x)$, and a loop $\gamma : [0, 1] \to X$ at $x$, there is a unique lift of $\gamma$ in $Y$ starting at $y$, which we denote by $\widetilde{\gamma}$.

This result is well-known in covering space theory, so we do not repeat it here. However, we do note that the end point of $\widetilde{\gamma}$ (i.e. $\widetilde{\gamma}(1)$) again must be in the fiber of $x$ and only depends on the class of $\gamma$ in $\pi_1(X, x)$. Consequently, this defines a right group action $\pi_1(x) \acts \pi_1(X, x)$.

**Definition 2.7. (Monodromy of a Cover)**
Let $p : Y \to X$ be a cover of topological spaces with $Y$ connected and locally path connected. Then, we refer to the right group action $\pi_1(x) \acts \pi_1(X, x)$ as the **monodromy action** of $p$.

As $X$ is connected, we have a permutation representation
\[
\rho : \pi_1(X, x) \to \text{Perm}(p^{-1}(x))
\]
for each $x \in X$. Moreover, the connectedness of $X$ yields that the monodromy action of $x$ is conjugate to the monodromy action of any $x' \in X$. The **monodromy group** of $p$ is the image of the permutation representation, and is well-defined up to conjugacy.

**Definition 2.8. (Monodromy of a Branched Cover)**
Let $p : Y \to X$ be a branched cover of topological spaces with $Y$ connected and locally path-connected. Denote the branch locus of $p$ by $B$. The **monodromy action** (respectively, **group**) of $p$ is given the monodromy action (respectively, group) of the cover
\[
p\mid_{X\setminus B} : p^{-1}(X \setminus B) \to X \setminus B.
\]
Note that any complex variety (with its classical topology) is locally path connected and any irreducible complex variety is also connected. Consequently, the monodromy of any cover of irreducible complex varieties is well-defined up to conjugacy. We finally extend the definition of monodromy to algebraic functions by their corresponding branched covers.

**Definition 2.9. (Monodromy of an Algebraic Function)**
Given an algebraic function \( \phi \) on an irreducible complex variety \( X \), we define the monodromy action (respectively, group) of \( \phi \) to be the monodromy action (respectively, group) of the branched cover \( \pi_\phi : E_\phi \to X \).

We now revisit Example 2.2 to determine its monodromy.

**Proposition 2.10. (Monodromy of Generic Polynomials)**
Let \( X = \mathbb{C}^n \) and take \( a_1, \ldots, a_n \) to be the coordinate functions on \( X \). Define \( \phi \) as in Example 2.2. Then, the monodromy group of \( \phi \) is \( S_n \).

**Proof.** To avoid confusion with the fundamental group, we will refer to the branched cover \( \pi_\phi \) as \( p \) for this example.

Note that it suffices to compute the monodromy group for any point in \( X \setminus B \). As a result, we compute the monodromy at the point \( z^n - 1 \), i.e.

\[
x = (a_1(z^n - 1), \ldots, a_n(z^n - 1)) = (0, \ldots, 0, -1).
\]

To show \( \text{Mon}(\phi) = S_n \), it suffices to show each transposition of the form \( (i \ i+1) \) is in the monodromy group.

Observe that \( p^{-1}(x) \) is the set of \( n^{th} \) roots of unity \( \{1, \zeta, \ldots, \zeta^{n-1}\} \), where \( \zeta = e^{2\pi i/n} \). We label the roots \( z_i = \zeta^{i-1} \) so that \( z_1 = 1, \ldots, z_n = \zeta^{n-1} \). For the transposition \( (i \ i+1) \) to be in the monodromy group, it suffices to show that there is a path lifting a loop at \( x \) taking \( \zeta^{i-1} \) to \( \zeta^i \).

We start by constructing the path \( \gamma_1 \) from 1 to \( \zeta \) given by traversing the circle including 1 and \( \zeta \) that is centered at \( \frac{1+i\zeta}{2} \), as is depicted in Figure 1.
We set the following notation:

\[ M = \frac{1 + \zeta}{2}, \]
\[ r = \cos^{-1}(\theta), \]
\[ \theta = \sin^{-1}(|M|), \]
\[ \delta = \pi - \left( \frac{\pi}{n} + \theta \right) = \frac{(n-1)\pi}{n} - \theta. \]

Then, \( \gamma_1 \) is given by

\[ \gamma_1 : [0,1] \to \mathbb{C} \]
\[ t \mapsto M + re^{(\pi - \delta)it}. \]

By rotating around the circle from \( \zeta \) to 1, we obtain another path \( \gamma_2 \):

\[ \gamma_2 : [0,1] \to \mathbb{C} \]
\[ t \mapsto M + re^{(\pi + \delta)it}. \]

However, in \( p^{-1}(X \setminus \mathcal{B}) \) - the space of square free polynomials parametrized by the roots - \( \gamma_1 \) and \( \gamma_2 \) induce a path \( \Gamma \)

\[ \Gamma : [0,1] \to p^{-1}(X \setminus \mathcal{B}) \]
\[ t \mapsto (z - \gamma_1(t))(z - \gamma_2(t))(z - \zeta^2) \cdots (z - \zeta^{n-1}). \]

By the uniqueness of path-lifting for covers, \( \Gamma \) projects to a loop in \( X \setminus \mathcal{B} \) which yields that \((1 2)\) is in \( \operatorname{Mon}(\phi) \). Moreover, for each \( 1 \leq i \leq n - 1 \), we can define
paths

\[ \gamma_1^i : [0, 1] \to \mathbb{C} \]
\[ t \mapsto \zeta^i \left( M + re^{(\pi t - \delta)i} \right) \]
\[ \gamma_2^i : [0, 1] \to \mathbb{C} \]
\[ t \mapsto \zeta^i \left( M + re^{(\pi (t+1) - \delta)i} \right) \]

which induce a path

\[ \Gamma^i : [0, 1] \to p^{-1}(X \setminus \mathcal{B}) \]
\[ t \mapsto (z - 1) \cdots (z - \gamma_1^i(t))(z - \gamma_2^i(t)) \cdots (z - \zeta^{n-1}) \]

in \( p^{-1}(X \setminus \mathcal{B}) \). Again, uniqueness of path-lifting for covers yields that \( \Gamma^i \) projects to a loop in \( X \setminus \mathcal{B} \) yielding each transposition \( (i \ i+1) \) is in \( \text{Mon}(\phi) \).

\[ \square \]

3 Formulas for Algebraic Functions

Recall that, ultimately, our goal is to be able to determine a root of degree \( n \) polynomials given their coefficients. In the previous section, we have seen how to view generic polynomials as algebraic functions and we can use this language to give a rigorous definition for a formula of a generic polynomial, which is the appropriate generalization of Example 1.3.

From here on out, we will focus on the geometry and work in terms of (branched) covering spaces. Corollary 1.13 allows us to state everything for field extensions as well. We collect all of the definitions and statements for fields in Appendix B.

Definition 3.1. (Formulas for Branched Covers)
Let \( Y \to X \) be a branched cover of complex varieties. A formula in functions of \( d \) variables for \( Y \to X \) is a finite tower of branched covers of complex varieties

\[ X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 \subseteq X \]

such that

- \( X_0 \subseteq X \) is a dense, Zariski open subvariety,
- \( X_r \to X \) factors through a branched cover \( X_r \to Y \),
- and for each \( 1 \leq i \leq r \), there are complex varieties \( Z_i \) and \( \tilde{Z}_i \) of dimension at most \( d \) and a branched cover \( \tilde{Z}_i \to Z_i \) so that the square

\[
\begin{array}{ccc}
n & \to & Z_i \\
\downarrow & & \downarrow \\
X_{i-1} & \to & Z_i
\end{array}
\]
is a pullback diagram and \( d = \max \left\{ \dim \left( \tilde{Z}_i \right) \mid 1 \leq i \leq r \right\} \).

We refer to \( r \) as the **length** of the formula.

**Definition 3.2. (Formulas for Algebraic Functions)**

Let \( \phi \) be an algebraic function on a complex variety \( X \), with associated branched cover \( \pi_X : E_{\phi} \to X \). A **formula in functions of \( d \) variables** for \( \phi \) is a formula in functions of \( d \) variables for the associated branched cover \( \pi_{\phi} : E_{\phi} \to X \).

The **length** of a formula for \( \phi \) is the length of the formula for \( \pi_{\phi} : E_{\phi} \to X \).

Not only do we want to write down formulas for generic polynomials, we ultimately want to write down formulas that are as simple as possible (in the fullest sense of "simple"). Notions of simplicity for formulas will be the subject of the next section. We will understand these notions of simplicity, and formulas in general, in terms of "irrationalities."

Before giving the definitions we will use, we recall some historical context. In solving a generic polynomial, Klein \([13]\) would refer to \( \mathbb{C}(a_1, \ldots, a_n) \) as the "field of rationality" \([1]\) for the generic polynomial of degree \( n \). With this in mind, an "irrationality" is an element:

\[
\alpha \in \overline{\mathbb{C}(a_1, \ldots, a_n)} \setminus \mathbb{C}(a_1, \ldots, a_n).
\]

Such an \( \alpha \) generates a finite extension of fields

\[
\mathbb{C}(a_1, \ldots, a_n) \hookrightarrow \mathbb{C}(a_1, \ldots, a_n)(\alpha)
\]

with \( \alpha \) as a primitive element. This motivates the following definitions.

**Definition 3.3. (Primitive Element of a Cover)**

Let \( p : Y \to X \) be a cover of topological spaces. A **primitive element** for \( p \) is a map \( \alpha : Y \to \mathbb{C} \) such that the map

\[
Y \hookrightarrow X \times \mathbb{C}
\]

\[
y \mapsto (p(y), \alpha(y))
\]

is an embedding.

**Definition 3.4. (Irrationalities of Complex Varieties)**

Let \( X \) be a complex variety. An **irrationality** of \( X \) is a branched cover \( p : Y \to X \) along with a primitive element \( \alpha \) of the cover

\[
p|_{X \setminus \mathcal{B}} : p^{-1}(X \setminus \mathcal{B}) \to X \setminus \mathcal{B}.
\]

\[\text{1}\] For more on this, see "Development of Mathematics in the 19th Century," Chapter VII: Deeper Insight Into the Nature of Algebraic Varieties and Structures, Section 3: The Theory of Algebraic Integers and Its Interactions with the Theory of Algebraic Functions - in particular, p.312-314.
It is worth remarking that we often refer to an irrationality without a choice of primitive element. Given a branched cover \( Y \to X \), we wish to describe certain types of irrationalities relative to \( Y \to X \). For this, we need to introduce notions of Galois theory for covers.

**Definition 3.5. (Deck Transformations and the Deck Group)**

Let \( p : Y \to X \) be a cover of topological spaces. A **deck transformation** (or **automorphism**) of \( p \) is a homeomorphism \( f : Y \to Y \) such that the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & Y \\
\downarrow p & & \downarrow p \\
X & \xleftarrow{p} & X
\end{array}
\]

commutes. The set of all deck transformations of \( p \) forms a group under composition and is called the **deck group** (or **automorphism group**) of \( p \); it is denoted by \( \text{Aut}(p) \).

**Remark 3.6.** Let \( p : Y \to X \) be a cover of topological spaces and \( f \) a deck transformation of \( p \). From the definition of a deck transformation, it follows that \( f \) takes a fiber of \( p \) to itself, but possibly permutes the elements in the fiber. Consequently, for each \( x \in X \), this defines a left group action \( \text{Aut}(p) \circ p^{-1}(x) \).

**Definition 3.7. (Galois Theory Terminology for Covers)**

Let \( Y \to X \) be a branched cover of complex varieties.

1. \( Y \to X \) is **Galois** if \( |\text{Aut}(Y \to X)| = |\text{Mon}(Y \to X)| \).
2. A branched cover of varieties \( Y' \to X \) is a **Galois closure** of \( Y \to X \) if it factors as \( Y' \to Y \to X \), \( Y' \to X \) is Galois, and no proper subcover of \( Y' \to X \) is Galois.
3. Given a pair of branched covers of varieties \( Z \to Y \to X \) with \( Z \to X \) Galois, a **Galois closure** of \( Y \to X \) in \( Z \to X \) is a Galois closure \( Y' \to X \) so that \( Z \to X \) factors as

\[
Z \to Y' \to Y \to X.
\]

We can now define natural and accessory irrationalities.

**Definition 3.8. (Natural and Accessory Irrationalities)**

Let \( Z \to X \) be a branched cover of complex varieties.

- An irrationality \( Y \to X \) is a **natural irrationality** of \( Z \to X \) if any Galois closure \( \tilde{Z} \to X \) of \( Z \to X \) factors as

\[
\tilde{Z} \to Y \to X.
\]
An irrationality $Y \rightarrow X$ is an **accessory irrationality of** $Z \rightarrow X$ if the map
\[ \pi_1(Y) \rightarrow \pi_1(X) \rightarrow \text{Mon}(Z \rightarrow X) \]

is surjective.

In particular, we describe all relative irrationalities in terms of natural and accessory irrationalities.

**Lemma 3.9. (Decomposition of Irrationalities)**

Let $Z \rightarrow X$ be a branched cover of complex varieties with an irrationality $Y \rightarrow X$. Then, there exists a maximal branched subcover $W \rightarrow X$ such that

- $W \rightarrow X$ is a natural irrationality for both $Z \rightarrow X$ and $Y \rightarrow X$, and
- $Y \rightarrow W$ is an accessory irrationality for $\tilde{Z} \rightarrow W$.

for any Galois closure $\tilde{Z} \rightarrow X$ of $Z \rightarrow X$.

We summarize the above lemma with the following diagram

\[
\begin{array}{ccc}
Z & \rightarrow & W \\
\downarrow & & \downarrow \text{natural} \\
Y & \rightarrow & X
\end{array}
\]

**Proof.** Since $\tilde{Z} \rightarrow X$ is Galois,
\[ \text{Mon}(\tilde{Z} \rightarrow X) \cong \text{Aut}(\tilde{Z} \rightarrow X) \cong \pi_1(X)/\pi_1(\tilde{Z}). \]

Any natural irrationality of $Z \rightarrow X$ must be fixed by $\pi_1(\tilde{Z})$. As a result, the maximal subcover of $Y \rightarrow X$ that is a natural irrationality for $Z \rightarrow X$ is the subcover fixed by
\[ H = \left\langle \pi_1(\tilde{Z}), \pi_1(Y) \right\rangle \subseteq \pi_1(X). \]

We denote this subcover by $W \rightarrow X$. However, from this construction, it is clear that $\pi_1(Y)$ surjects onto $\text{Mon}(\tilde{Z} \rightarrow W)$. 

We now extend the notion of an irrationality to algebraic functions by considering its corresponding cover.

**Definition 3.10. (Irrationalities of Algebraic Functions)**

Let $\phi$ be an algebraic function on a complex variety $X$. An algebraic function $\psi$ is an **irrationality of** $X$ if $E_\psi \rightarrow X$ is a non-trivial cover. Similarly, $\psi$ is a **natural** (respectively, **accessory**) irrationality for $\phi$ if $E_\psi \rightarrow X$ is a natural (respectively, accessory) irrationality of $E_\phi \rightarrow X$. 

16
4 Essential Dimension and Resolvent Degree

We can now make precise two notions of simplicity. Essential dimension is a measure of how simply one can write an algebraic function. Resolvent degree is a measure of how simply one can write a formula for an algebraic function. We will use these notions of simplicity to formulate the classical problems stated by Kronecker (as in [14]) and Klein (as in [12]).

Definition 4.1. (Essential Dimension of a Branched Cover)
Let $Y \rightarrow X$ be an $n$-sheeted branched cover of complex varieties. We say $Y \rightarrow X$ is defined over a complex variety $X_0$ if there is an $n$-sheeted branched cover $Y_0 \rightarrow X_0$ and a map $X \rightarrow X_0$ such that

$$Y \rightarrow X \cong X \times_{X_0} Y_0 \rightarrow X.$$ 

The essential dimension of $Y \rightarrow X$, denoted $\text{ed}(Y \rightarrow X)$, is

$$\text{ed}(Y \rightarrow X) = \min \left\{ \dim(X_0) \mid Y \rightarrow X \text{ is defined over } X_0 \right\}.$$

Definition 4.2. (Resolvent Degree of a Branched Cover)
Let $Y \rightarrow X$ be a branched cover of complex varieties. The resolvent degree of $Y \rightarrow X$ is the minimal $d$ for which there exists a formula for $Y \rightarrow X$ of in functions of $d$ variables. We denote the resolvent degree of $Y \rightarrow X$ by $\text{RD}(Y \rightarrow X)$.

As we have seen before, our definitions for branched covers yield definitions for algebraic functions via their associated branched covers.

Definition 4.3. (Essential Dimension of an Algebraic Function)
Let $\phi$ be an algebraic function on a complex variety $X$. The essential dimension of $\phi$ is the essential dimension of $E_\phi \rightarrow X$. We denote the essential dimension of $\phi$ by $\text{ed}(\phi)$.

Definition 4.4. (Resolvent Degree of an Algebraic Function)
Let $\phi$ be an algebraic function. The resolvent degree of $\phi$ is the minimal $d$ for which there exists a formula for $\phi$ of in functions of $d$ variables. We denote the resolvent degree of $\phi$ by $\text{RD}(\phi)$.

The difference between essential dimension and resolvent degree is the restricting to formulas of length 1 or allowing towers of arbitrary length, as is made precise by the following lemma.

Lemma 4.5. (Re-statement of Essential Dimension for Covers)
Let $Y \rightarrow X$ be a branched cover of complex varieties. The essential dimension of $Y \rightarrow X$ is the minimal $d$ for which there exists a formula for $Y \rightarrow X$ of length 1 in a function of $d$ variables.

\[\text{ed}(Y \rightarrow X) = \min \{ \dim(X_0) \mid Y \rightarrow X \text{ is defined over } X_0 \} \]

\[\text{RD}(Y \rightarrow X) = \min \{ d \mid \text{formula for } Y \rightarrow X \text{ of length } d \} \]

\[\text{ed}(\phi) = \min \{ \dim(X_0) \mid \phi \text{ is defined over } X_0 \} \]

\[\text{RD}(\phi) = \min \{ d \mid \text{formula for } \phi \text{ of length } d \} \]

\[2\text{Note that Klein’s problem is also given by Hilbert and can be found in [11].} \]
Proof. For the purpose of this proof, we will denote the definition of essential dimension as in Definition 4.1 by \( ed_1 \) and we will denote the invariant described in Lemma 4.5 by \( ed_2 \).

Suppose that \( Y \to X \) is defined over a complex variety \( X_0 \) of dimension \( d \). Then, we can construct the formula

\[
\begin{array}{ccc}
Y & \cong & X \times_{X_0} Y_0 \\
\downarrow & & \downarrow \\
X & \to & X_0
\end{array}
\]

with a single function in \( d \) variables. By taking a minimum over the dimensions of all such \( X_0 \), we conclude

\[
ed_2(Y \to X) \leq ed_1(Y \to X).
\]

Now, suppose that we have a length 1 formula for \( Y \to X \) in \( d \) variables:

\[
\begin{array}{ccc}
Y & \to & Z \\
\downarrow & & \downarrow \\
X & \to & Z_0
\end{array}
\]

However, from the Galois correspondence for covers, we get a corresponding diagram of groups

\[
\begin{array}{ccc}
\pi_1(Z) & \to & \pi_1(Z_2) \\
\downarrow & & \downarrow \\
\pi_1(Y) & \to & \pi_1(Y_0) \\
\downarrow & & \downarrow \\
\pi_1(X) & \to & \pi_1(Z_0)
\end{array}
\]

where the diagram without \( \pi_1(Y) \) is a pullback square. Denote the image of \( \pi_1(Y) \) in \( \pi_1(Z_0) \) by \( G \) and then consider

\[
H = \langle \pi_1(Z_2), G \rangle < \pi_1(Z_0).
\]

We then have an updated diagram

\[
\begin{array}{ccc}
\pi_1(Z) & \to & \pi_1(Z_2) \\
\downarrow & & \downarrow \\
\pi_1(Y) & \to & H \\
\downarrow & & \downarrow \\
\pi_1(X) & \to & \pi_1(Z_0)
\end{array}
\]

18
From the Galois correspondence, there is a complex variety $Z_1$ such that
$$Z_2 \twoheadrightarrow Z_1 \twoheadrightarrow Z_0$$
corresponds to
$$\pi_1(Z_2) \hookrightarrow H = \pi_1(Z_1) \hookrightarrow \pi_1(Z_0).$$
Moreover, the diagram
$$\begin{array}{ccc}
Z & \rightarrow & Z_2 \\
\downarrow & & \downarrow \\
Y & \rightarrow & Z_1 \\
\downarrow & & \downarrow \\
X & \rightarrow & Z_0
\end{array}$$
is comprised of two pullback diagrams by construction. Since
$$Y = Z_1 \times_{Z_0} X,$$
taking a minimum over the dimensions of all length 1 formulas of $E \rightarrow X$ allows us to conclude
$$\text{ed}_1(Y \rightarrow X) \leq \text{ed}_2(Y \rightarrow X).$$

When dealing with a branched cover of complex varieties whose monodromy group is simple, the difference between essential dimension and resolvent degree is whether or not we allow accessory irrationalities. This will become more clear in the next section, once we give some properties of essential dimension and resolvent degree. From their definitions, it is clear that the essential dimension of an object is always greater than or equal to its resolvent degree. It is worth verifying, however, that these invariants are distinct.

**Example 4.6. (Essential Dimension $\neq$ Resolvent Degree)**
Consider the complex variety $X = \mathbb{C}^n$ and let $\{a_1(x), \ldots, a_n(x)\}$ be the coordinate functions on $X$. We define the algebraic function
$$\psi_n : X \rightarrow \mathbb{C}$$
$$x \mapsto \{z \mid (z^2 - a_1(x)) \cdots (z^2 - a_n(x)) = 0\}.$$Then, $\text{ed}(\psi_n) = n$ and $\text{RD}(\psi_n) = 1$. It is clear that $\text{ed}(\psi_n) \leq n$ by checking that
$$\begin{array}{ccc}
E_{\psi_n} & \leftarrow & X \times_{\mathbb{C}^n} \mathbb{C}^n_{\text{roots}} \\
\downarrow & & \downarrow \\
X & \leftarrow & \mathbb{C}^n
\end{array}$$

$$\begin{array}{ccc}
E_{\psi_n} & \rightarrow & \mathbb{C}^n_{\text{roots}} \\
\downarrow & \leftarrow & \downarrow \\
X & \rightarrow & \mathbb{C}^n
\end{array}$$
is a formula for $\psi_n$. The most expeditious way to show that $\text{ed}(\psi_n) = n$ uses
the notion of the essential dimension of a group, which we do not wish to get
into here. However, we note that all of the necessary tools can be found in [8]
(in particular, see sections 2, 3, and 6).

To prove that $RD(\psi_n) = 1$, we note that a formula of $\psi$ in 1 variable of length
$n$ can be constructed by appropriately combining pullback diagrams of the fol-
lowing form:

$$
\begin{array}{ccc}
X \times \mathbb{C}^1 & \rightarrow & \mathbb{C}^1 \\
\downarrow & & \downarrow \gamma \mapsto z^2 \\
X & \rightarrow & \mathbb{C}^1 \\
\end{array}
$$

We will proceed to state the problems of Kronecker and Klein/Hilbert. To
do this, however, we must introduce additional notation. Denote the algebraic
function associated to the generic degree $n$ polynomial by $\Phi_n$. The branched
cover corresponding to $\Phi_n$, 

$$
\pi_{\Phi_n} : E_{\Phi_n} \rightarrow \mathbb{C}^n,
$$

has a canonical natural irrationality for all $n \geq 2$ given by adjoining the square
root of the discriminant. We denote this natural irrationality by $E_{\sqrt{\Delta_n}} \rightarrow \mathbb{C}^n$.
Moreover, providing a formula to solve the generic degree $n$ polynomial given the
square root of the discriminant is exactly to provide a formula for the branched
cover $E_{\Phi_n} \rightarrow E_{\sqrt{\Delta_n}}$.

**Problem 4.7. (Kronecker’s Problem)**

Compute the essential dimension of the generic degree $n$ polynomial after ad-
joining the square root of the discriminant; i.e. compute $\text{ed} \left( E_{\Phi_n} \rightarrow E_{\sqrt{\Delta_n}} \right)$.

**Problem 4.8. (Klein’s Problem)**

Compute the resolvent degree of solving the generic degree $n$ polynomial; i.e.
compute $\text{RD} \left( E_{\Phi_n} \rightarrow \mathbb{C}^n \right)$.

Klein himself describes the difference between his perspective and the per-
spective of Kronecker in his lectures on the icosahedron [13]:

"True, we shall have to investigate ... how far we can get with
the use of natural irrationalities only. But, over and above this,
the question arises, what is the state of affairs with regard to the
accessory irrationalities which aid us in further reduction; what are
the simplest results which we can attain to with their help?"

From here on out, when we speak of solving $\Phi_n$ as simply as possible, we
mean providing a formula for $\Phi_n$ that realizes $\text{RD}(\Phi_n)$. Our next consideration
of simplicity is length - i.e. of all formulas realizing $\text{RD}(\Phi_n)$, the simplest is the
one of minimal length.
It is appropriate now to take stock of what is currently known about Klein’s problem. To simplify notation, we denote $\text{RD}(\Phi_n)$ by $\text{RD}(n)$. From classical Galois theory, the solvability of $\Phi_n$ by radicals yields that

$$\text{RD}(n) = 1, \ n = 1, 2, 3, 4$$

The work of Bring [1] and Klein [13] provide constructions for solving the quintic using algebraic functions of one variable and we can thus conclude $\text{RD}(5) = 1$. Moreover, the work of Hamilton [2], Hilbert [11], and Klein [13] provide upper bounds on $\text{RD}(n)$. Here we list the best known upper bounds for $n \leq 9$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{RD}(n)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>≤2</td>
<td>≤3</td>
<td>≤4</td>
<td>≤4</td>
<td></td>
</tr>
</tbody>
</table>

**Figure 2:** $\text{RD}(n)$ for $1 \leq n \leq 9$

In [11], Hilbert sketches a reduction of Hamilton’s estimate of $\text{RD}(9) \leq 5$ to $\text{RD}(9) \leq 4$. Moreover, he provides his Sextic and Octic conjectures, recalls his 13th Problem, and makes explicit his conjecture on $\text{RD}(9)$. We summarize them as simply as possible here.

**Conjecture 4.9. (Hilbert’s Conjectures)**

1. $\text{RD}(6) = 2$
2. $\text{RD}(7) = 3$
3. $\text{RD}(8) = 4$
4. $\text{RD}(9) = 4$

The work of Hamilton [2], Sylvester-Hammond [5], and Brauer [7] have provided general upper bounds on $\text{RD}(n)$ for all $n$. The best known upper bounds are in upcoming work of Wolfson. While the growth rate of $\text{RD}(n)$ is not currently known, it is reasonable to expect that

$$\lim_{n \to \infty} \text{RD}(n) = +\infty.$$  

However, there are no known non-trivial lower bounds on $\text{RD}(n)$ - it is, in theory, possible that $\text{RD}(n) \equiv 1$. In fact, in 1993, Dixmier concluded his summary on Hilbert’s 13th problem [4] by saying:

”Let’s end on a dramatic note, which proves our incredible ignorance. Although this seems unlikely, it is not impossible that $\text{RD}(n) = 1$ [4] for all $n$! ... Any reduction of $\text{RD}(n)$ would be serious progress. In particular, it is time to know whether $\text{RD}(6) = 1$ or $\text{RD}(6) = 2."$

---


[2] What we denote by $\text{RD}(n)$, Dixmier denotes by $s(n)$.
5 Properties of Essential Dimension and Resolvent Degree

Having stated our classical problem (Problem 1.1) within the framework of resolvent degree (as in Problem 4.8), it is worth making note of some of the properties of essential dimension and resolvent degree.

Remark 5.1. First and foremost, consider a branched cover $Y \rightarrow X$ of complex varieties. Then, a formula of $Y \rightarrow X$ in functions of $d$ variables

$$X_r \rightarrow X_{r-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \subseteq X$$

is exactly a tower of branched covers of essential dimension $\leq d$.

We will use the above remark to prove statements about essential dimension and then pass them on to statements about resolvent degree. We start by providing upper bounds on resolvent degree.

Proposition 5.2. (Easy Upper Bounds on ed/RD)
Let $Y \rightarrow X$ be a branched cover of complex varieties. Then,

1. $\text{RD}(Y \rightarrow X) \leq \text{ed}(Y \rightarrow X) \leq \text{dim}(X)$
2. Given a dominant map $Z \rightarrow X$ of complex varieties,
   - $\text{ed}(Y \times_X Z \rightarrow Z) \leq \text{ed}(Y \rightarrow X)$, and
   - $\text{RD}(Y \times_X Z \rightarrow Z) \leq \text{RD}(Y \rightarrow X)$.
3. If the branched covers of complex varieties $Y \rightarrow X$ and $Y' \rightarrow X'$ are birationally equivalent, then
   - $\text{ed}(Y \rightarrow X) = \text{ed}(Y' \rightarrow X')$, and
   - $\text{RD}(Y \rightarrow X) = \text{RD}(Y' \rightarrow X')$.

Proof. The first statement follows from the definitions of a formula, of essential dimension, and of resolvent degree. As in the previous proof, the second statement follows from pulling back any formula for $Y \rightarrow X$ along $Z \rightarrow X$. The third statement follows from pulling back formulas solving $Y \rightarrow X$ and formulas solving $Y' \rightarrow X'$ along a birational equivalence and its inverse.

We remark on the following properties of essential dimension and resolvent degree. These results are useful in the theory of resolvent degree, but the proofs would take us far afield from solving polynomials. Thus, we remark that the proofs of these results can be found in [10] (see Lemma 2.8, Lemma 2.9, and Lemma 2.13).

Proposition 5.3. (ed/RD and Irreducible Components)
Consider a branched cover of complex varieties $Y \rightarrow X$. Denote by $\{X_i\}$ the set of irreducible components of $X$ and let $\{Y_{i,j}\}$ denote the set of irreducible components of $Y|_{X_i} \rightarrow X_i$. Then,
• \( \text{ed}(Y \to X) \geq \max_{i,j} \{ \text{ed}(Y_{i,j} \to X_i) \} \),

• \( \text{ed}(Y \to X) \leq \max_i \left\{ \sum_j \text{ed}(Y_{i,j} \to X_i) \right\} \), and

• \( \text{RD}(Y \to X) = \max_{i,j} \{ \text{RD}(Y_{i,j} \to X_i) \} \).

**Lemma 5.4. (Resolvent Degree of Compositions of Covers)**

Let \( Z \to Y \to X \) be a pair of branched covers of complex varieties. Then,

\[
\text{RD}(Z \to X) = \max \{ \text{RD}(Z \to Y), \text{RD}(Y \to X) \}.
\]

**Lemma 5.5. (Resolvent Degree Invariant Under Galois Closure)**

Let \( Y \to X \) be a branched cover of complex varieties. Let \( \tilde{Y} \to X \) be a Galois closure of \( Y \to X \). Then,

\[
\text{RD}(\tilde{Y} \to X) = \text{RD}(Y \to X).
\]

While solving generic polynomials is of interest in its own right, it also plays a universal role in the theory of resolvent degree as a whole in a precise sense.\(^5\)

**Lemma 5.6. (Universality of Solving \( \Phi_n \))**

Let \( Y \to X \) be an \( n \)-sheeted branched cover of complex varieties. Then,

\[
\text{RD}(Y \to X) \to \text{RD}(n).
\]

The proof of this lemma is not difficult, but is most accessible via field theory and thus can be found in Appendix C.

### 6 Solving the Quintic

We now have a sufficient description of the theory of resolvent degree and solving generic polynomials to state the first modern result of the theory - Kronecker’s Theorem, as Klein names it in [13]. In fact, here is the statement Klein gives:

"We have now all the requisite materials for completing the proof of the oft-mentioned theorem of Kronecker. Our object is to prove that it is impossible, in the case of any proposed equation of the fifth degree, even after the adjunction of the square root of the discriminant, to construct a rational resolvent which contains only one parameter."

- Felix Klein, p.286.

\(^5\)This lemma is also from [11] - see Lemma 2.11.
We now restate the theorem in modern language and using the notation describing Problems 4.7 and 4.8.

**Theorem 6.1. (Kronecker’s Theorem)**

\[
ed(E_{\Phi_5} \rightarrow \mathbb{C}^5) = \ed(E_{\Phi_5} \rightarrow E_{\sqrt{-5}}) = 2
\]

As was mentioned in Section 4, however, Klein went on to show in [13] that

**Theorem 6.2. (Klein, Bring)**

\[
\RD(E_{\Phi_5} \rightarrow \mathbb{C}^5) = 1.
\]

We refer to this jointly as a result of Bring and Klein as they provided independent proofs and took different approaches to the problem; Bring’s proof is the subject of [1]. We will address some of the components of his proof after we outline Klein’s formula for the quintic. All of the original content of his construction and more can be found in Klein’s lectures ([13]), including a proof of Kronecker’s theorem. Here, however, we follow the complete, but more expedient version given by Nash in [15].

If the essence of Klein’s insight about solving the quintic had to be distilled to one remark, it would be that everything follows from the symmetries of the icosahedron. Let us take a regular icosahedron in \(\mathbb{R}^3\) centered at the origin whose vertices have magnitude 1. We can then radially project the faces of the icosahedron onto its circumsphere, which is \(S^2\). We also make use of the identification of \(S^2\) with complex projective space \(\mathbb{P}^1\), and thus also with the extended complex plane. Let \(\Gamma\) denote the symmetry group of the icosahedron (i.e. automorphisms of \(\mathbb{P}^1\) fixing the icosahedron). Then, we have an isomorphism \(\Gamma \cong A_5\). In [15], Nash provides the following diagram (labeled as Figure 3), which uses stereographic projection to split the extended complex plane into regions corresponding to faces of the icosahedron. The labeling of these faces yields the isomorphism of \(\Gamma\) with \(A_5\):
As $\Gamma$ is the symmetry group of the icosahedron, it naturally gives the branched cover $\mathbb{P}^1 \rightarrow \mathbb{P}^1/\Gamma$. Understanding the branched cover $\mathbb{P}^1 \rightarrow \mathbb{P}^1/\Gamma$ is the crux of Klein's formula.

It is first worth remarking that $\mathbb{P}^1/\Gamma \cong \mathbb{P}^1$. To do this, we provide an isomorphism of graded rings between $\mathbb{C}[z_1, z_2]$ and a subring of $\mathbb{C}[z_1, z_2]^\Gamma$ that is regraded so that it is generated by elements in degree 1. The action of $\Gamma$ is free on $\mathbb{P}^1$, except for on the orbits of the vertices, midpoints of the edges, and face centers of the icosahedron. The stabilizer subgroups of these orbits have orders 5, 2, and 3, respectively. Thus, it suffices to construct the corresponding invariant homogeneous polynomials of degrees 12, 20, and 30 - we will denote them respectively by $f$, $H$, and $\phi$. Using the original embedding of the icosahedron into $\mathbb{R}^3$ and the appropriate isomorphisms, it is possible to construct the polynomials directly, as the roots are given by the vertices, edge midpoints, and face centers. By doing this for $f$, one obtains that

$$f(z_1, z_2) = z_1z_2 (z_1^{10} + 14z_1^5z_2^5 - z_2^{10}).$$

Instead of computing them directly, we note that $H$ and $\phi$ can be recovered from $f$ by using the Hessian and Jacobian covariants of $f$. Indeed,

$$H = \frac{1}{121} \mathcal{H}(f) \quad \phi = \frac{1}{20} \mathcal{J}(f, H)$$
where
\[ \mathcal{H}(g(z_1, z_2)) = \begin{vmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{vmatrix} \quad \mathcal{J}(g(z_1, z_2), h(z_1, z_2)) = \begin{vmatrix} g_1 & g_2 \\ h_1 & h_2 \end{vmatrix} \]

\[ g_{i,j} = \frac{\partial^2}{\partial_i \partial_j} g(z_1, z_2) \quad g_i = \frac{\partial}{\partial_i} g(z_1, z_2) \quad h_i = \frac{\partial}{\partial_i} h(z_1, z_2). \]

Explicitly,
\[ H(z_1, z_2) = -\left( z_1^{30} + z_2^{30} \right) + 288 \left( z_1^{15} z_2^5 - z_1^5 z_2^{15} \right) - 494 z_1^{10} z_2^{10} \]
\[ \phi(z_1, z_2) = \left( z_1^{30} + z_2^{30} \right) + 522 \left( z_1^{25} z_2^5 - z_1^5 z_2^{25} \right) - 10,005 \left( z_1^{20} z_2^{10} + z_1^{10} z_2^{30} \right). \]

One can check that the relation
\[ H^3 + \phi^2 = 1728 f^5 \]
holds. In fact, we have an isomorphism
\[ A \cong \mathbb{C}[z_1, z_2]^\Gamma = \mathbb{C}[f, H, \phi]. \]

where
\[ A = \mathbb{C}[x, y, z]/(1728x^3 - y^2 - z^3). \]

Let \( A = \bigoplus_{n \geq 0} A_n \) be the graded decomposition of \( A \). Since \( f, H, \) and \( \phi \) have degrees 12, 20, and 30, respectively, and \( H^3 + \phi^2 = 1728 f^5 \) is a minimal relation between these generators, any homogeneous element of \( A \) must have degree divisible by 60. Consequently, we re-grade by setting
\[ A^{(60)} = \bigoplus_{n \geq 0} A_{60n}. \]

\( A^{(60)} \) is isomorphic to the homogeneous coordinate ring of \( \mathbb{P}^1/\Gamma \) and is isomorphic to \( \mathbb{C}[x^5, y^3] \), which yields \( \mathbb{P}^1/\Gamma \cong \mathbb{P}^1 \). In fact, it yields that the map
\[ I : \mathbb{P}^1 \to \mathbb{P}^1/\Gamma \] is isomorphic to
\[ \mathbb{P}^1 \to \mathbb{P}^1 \]
\[ [z_1 : z_2] \mapsto [H(z_1, z_2)^3 : 1728 f(z_1, z_2)^5] \]

Having discussed the branched cover \( \mathbb{P}^1 \to \mathbb{P}^1/\Gamma \), we take a brief aside to mention the reduction of the generic quintic to Klein’s normal form. By first solving an auxiliary square root (i.e. solving an accessory irrationality), we can reduce the generic quintic to the normal form
\[ y^5 + 5\alpha y^2 + 5\beta y + \gamma = 0. \]
In the case of the non-trivial quintic (i.e. $\alpha, \beta, \gamma$ not all 0), the roots $z_0, \ldots, z_4$ determine a point in $\mathbb{P}^4$ (once ordered). When a square root of the discriminant of the quintic is given along with the $\alpha, \beta, \gamma$, the roots $z_0, \ldots, z_4$ are ordered up to the action of $A_5$. Thus, they determine an $A_5$ orbit in $\mathbb{P}^4$. Moreover, by Klein’s normal form of the quintic, this orbit lies in the smooth quadric

$$Q = \left\{ [z_0 : \cdots : z_4] \in \mathbb{P}^4 \mid \sum_{i=0}^{4} z_i = \sum_{i=0}^{4} z_i^2 = 0 \right\}. $$

that is $S_5$-invariant. Note that $Q$ is doubly-ruled - i.e. $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. By projecting onto a factor of $\mathbb{P}^1$ and taking a quotient by $A_5$ leads to the construction of an invariant known as the icosahedral invariant, $Z$. 

It is now possible to construct Klein’s formula for the quintic. Using the notation as in establishing Problems 4.7 and 4.8 his formula is

$$E_2 \rightarrow E_{\sqrt{\Delta_5}} \rightarrow E_1 \rightarrow X$$

where

- $E_1 \rightarrow X$ comes from the square root required to reduce to the normal form,
- $E_{\sqrt{\Delta_5}} \rightarrow E_1$ comes from adjoining the square root of the discriminant, and
- $E_2 \rightarrow E_{\sqrt{\Delta_5}}$ comes from pulling back by the icosahedral cover $\mathbb{P}^1 \rightarrow \mathbb{P}^1/\Gamma$.

The explicit factorization of the map $E_2 \rightarrow E_{\Phi_5} \rightarrow X$ is left to [15].

The work of Bring [1] (and later, Jerrard) proves Klein’s theorem by showing that the quintic can be reduced to the Bring-Jerrard form

$$y^5 + y + \gamma = 0$$

after adjoining the square root of the discriminant. Note that this normal form corresponds to a curve

$$C_B = \left\{ [z_0 : \cdots : z_4] \in \mathbb{P}^4 \mid \sum_{i=0}^{4} z_i = \sum_{i=0}^{4} z_i^2 = \sum_{i=0}^{4} z_i^3 = 0 \right\}. $$

that we refer to as the Bring curve. Bring further extends this construction to a full solution of the quintic.
So far, we have focused entirely on the algebraic component of solving generic polynomials. There is also an analytic component. Let us first describe the simplest scenario. Given \( w \in \mathbb{C} \), how can we interpret an \( n \text{th} \) root analytically?

Observe:

\[
    z^n = w \iff \log(z^n) = \log(w) \iff n \log(z) = \log(w) \\
    \iff \log(z) = \frac{1}{n} \log(w) \\
    \iff z = e^{\frac{1}{n} \log(w)}
\]

All we remark here is that there is a similar story for solving the generic quintic. To do this, we must generalize from the exponential and logarithm to elliptic modular functions on the upper-half plane \( \mathcal{H} \), the uniformization of \( \mathbb{P}^1 \setminus \{\infty\} \). However, there is a natural 3-sheeted branched cover \( C_B \to \mathbb{P}^1 \) coming from Bring’s solution to the quintic and whose uniformization is also \( \mathcal{H} \). With all of this in mind, the complete solution of the quintic distills to understanding the following picture:

\[
\begin{array}{c}
G \curvearrowright \mathcal{H} \rightarrow \mathcal{H} \curvearrowright SL_2(\mathbb{Z}, 5) \\
A_5 \curvearrowright C_B \rightarrow \mathbb{P}^1 \cap A_5
\end{array}
\]

Diagram 1: Solution of the Quintic

The maps \( C_B \to \mathbb{P}^1 \) and \( \mathcal{H} \to \mathbb{P}^1 \) constitute the algebraic and analytic components, respectively, of the solution of the quintic. The work of Green ([19]) explicitly handles the analytic component of the solution and completes the above diagram.

7 Solving the Sextic

In this last section, we outline the future work to be done. Our goal is to fully understand the total solution of the generic sextic by extending the methods of Klein, Bring, and Green. Hamilton [2] extends the methods of Bring to give a complete solution of the sextic. In particular, this construction yields a surface \( S_H \) that is analogous to the Bring curve \( C_B \).

Klein extends his methods for solving the quintic to a detailed sketch of a solution for the sextic in [12]. First, Klein reduces to a normal form for the sextic via an accessory cubic irrationality. Then, Klein adjoins the square root.

\[\text{Diagram 1: Solution of the Quintic}\]
of the discriminant of the normal form, just as in the case of the quintic. The last stage comes from the analogue of Klein’s icosahedral cover of $\mathbb{P}^1$. Recall that in the case of the quintic, this cover comes from the realization of $A_5$ as the symmetry group of the icosahedron and the topological identification of the icosahedron with $\mathbb{P}^1$.

There is no non-trivial action of $A_6$ on $\mathbb{P}^1$. However, there is an action of $A_6$ on $\mathbb{P}^2$ arising from the icosahedron. Consider an embedding of the icosahedron based at the origin in $\mathbb{R}^3$. Then, the automorphisms of the icosahedron give a 3-dimensional real representation of $A_5$, which we then complexify. This gives a copy of $A_5$ in $PGL(3, \mathbb{C})$. We can then enlarge this group by an order 4 transformation (coming from a tetrahedral subgroup) to get a copy of $A_6$ in $PGL(3, \mathbb{C})$ (see [9]).

There is a lift of $A_5 < PGL(3, \mathbb{C})$ to a subgroup $2A_5 < GL(3, \mathbb{C})$. Indeed, the lift comes from an extension of $A_5$ by $\mathbb{Z}/2\mathbb{Z}$ (hence the notation $2A_5$). The accessory square root in Klein’s solution of the quintic [that does not come from the square root of the discriminant] comes from this extension. Similarly, there is a lift of $A_6 < PGL(3, \mathbb{C})$ to a subgroup $3A_6 < GL(3, \mathbb{C})$ and this lift is an extension of $A_6$ by $\mathbb{Z}/3\mathbb{Z}$. This is where the cubic accessory irrationality $F_1 \rightarrow \mathbb{C}^6$ comes from. $3A_6$ is known as the Valentiner group, after the work of Valentiner [16]. It was studied further by Wiman [17].

Understanding the action of $A_6 \curvearrowright \mathbb{P}^2$ allows Klein to identify a minimal collection of $A_6$-invariant polynomials that generate the homogeneous coordinate ring of $\mathbb{P}^2/A_6$. From this, Klein can construct an isomorphism of graded rings which yields $\mathbb{P}^2/A_6 \cong \mathbb{P}^2$. Moreover, from these polynomials, we can explicitly describe the branched cover

$$\mathbb{P}^2 \rightarrow \mathbb{P}^2/A_6 \cong \mathbb{P}^2,$$

in analogy with the icosahedral cover for the quintic. This yields the branched cover $F_2 \rightarrow F_{\sqrt{\Delta_6}}$ and thus we have determined a formula

$$F_2 \rightarrow F_{\sqrt{\Delta_6}} \rightarrow F_1 \rightarrow \mathbb{C}^6$$

for the sextic.

What remains is to generalize the work of Green for the sextic. In other words, we wish to create an analogous version of Diagram 1 for the sextic. First, we seek to construct a branched cover of $S_H \rightarrow \mathbb{P}^2$, in analogy with $C_B \rightarrow \mathbb{P}^1$. Next, we wish to uniformize $\mathbb{P}^2$ by Hilbert modular functions (generalizing the uniformization of $\mathbb{P}^1$ by elliptic modular functions). We expect that the uniformization is $\mathcal{H} \times \mathcal{H}$ and that it comes with an action of

$$SL_2(\mathbb{Z}\langle \sqrt{2}; 3 \rangle) = \ker \left( SL_2(\mathbb{Z}\langle \sqrt{2} \rangle) \rightarrow PSL_2(\mathbb{F}_9) \right).$$

29
Finally, we want to show that $S_H$ is a K3 surface and determine its uniformization. We summarize our current understanding of the total solution of the sextic in the following diagram:

$$
?? \circlearrowleft ??? \longrightarrow \mathcal{H} \times \mathcal{H} \circlearrowright \widetilde{SL}_2(\mathbb{Z}(\sqrt{2}; 3))
$$

$$
\downarrow \quad \downarrow
$$

$$
A_6 \circlearrowright S_H \longrightarrow \mathbb{P}^2 \circlearrowleft A_6
$$

*Diagram 2: Solution of the Sextic*
Lemma 8.1. Let $\mathcal{C}$ and $\mathcal{D}$ be small categories (i.e. their collections of objects are sets) and $F : \mathcal{C} \to \mathcal{D}$ an equivalence of categories. Then, for any category $\mathcal{E}$, we have an induced equivalence of categories

$$F_{\mathcal{E}} : \text{Fun}(\mathcal{E}, \mathcal{C}) \to \text{Fun}(\mathcal{E}, \mathcal{D})$$

$$(G : \mathcal{E} \to \mathcal{C}) \mapsto (F \circ G : \mathcal{E} \to \mathcal{D}).$$

In particular, $F$ induces an equivalence of categories $\text{Ar}(\mathcal{C}) \to \text{Ar}(\mathcal{D})$.

Proof. We first show that $F_{\mathcal{E}}$ is essentially surjective. Suppose that $H : \mathcal{E} \to \mathcal{D}$ is a functor. For each $e \in \mathcal{E}$, the essential surjectivity of $F$ yields that there is $e_c \in \mathcal{C}$ such that $F(e_c) \cong H(e)$. Moreover, the assignment $e \mapsto e_c$ induces a functor $G : \mathcal{E} \to \mathcal{C}$, where morphisms

$$\text{Hom}_\mathcal{E}(e, e') \to \text{Hom}_\mathcal{C}(e_c, e'_c)$$

$$\phi \mapsto \phi_c$$

where $\phi_c$ is determined by the isomorphisms $F(e_c) \cong H(e)$ and $F(e'_c) \cong H(e')$, as

$$F(e_c \to e'_c) \cong H(e \to e').$$

Thus, $F_{\mathcal{E}}(G) \cong H$. Now, observe that the map

$$\text{Hom}_{\text{Fun}(\mathcal{E}, \mathcal{C})}(G, G') \to \text{Hom}_{\text{Fun}(\mathcal{E}, \mathcal{D})}(F \circ G, F \circ G')$$

$$\{\alpha_e \mid e \in \mathcal{E}\} \mapsto \{F \circ \alpha_e \mid e \in \mathcal{E}\}$$

is bijective, as $F$ is fully faithful. Thus, $F_{\mathcal{E}}$ is indeed an equivalence of categories.

Now, consider the specific case where $\mathcal{E}$ is the category with two objects $x, y$ and exactly one non-identity morphism, $x \to y$. Then, we have an equivalence of categories

$$\text{Ar}(\mathcal{C}) \to \text{Fun}(\mathcal{E}, \mathcal{C}) \to \text{Fun}(\mathcal{E}, \mathcal{D}) \to \text{Ar}(\mathcal{D}).$$

\[ \Box \]
Appendix B: The Theory of Resolvent Degree for Field Extensions

As was mentioned in Section 3, formulas, essential dimension, resolvent degree and everything related to them that was given in the context of branched covers of varieties makes sense for field extensions (in light of Corollary 1.13). As the story has been told for branched covers and follows from the mentioned equivalence, we provide the relevant definitions and statements, but omit their proofs.

Definition 9.1. (Formulas for Field Extensions)
Analogue of Definition 3.1

Let \( K \hookrightarrow L \) be a finite extension of fields over \( \mathbb{C} \). A formula in extensions of \( d \) variables for \( L/K \) is a finite sequence of finite extensions of fields

\[
K = L_0 \hookrightarrow L_1 \hookrightarrow \cdots \hookrightarrow L_r
\]

where \( L_r/L \) is a field extension over \( K \) and for all \( 1 \leq i \leq r \),

\[
L_i = L_{i-1} \otimes_{F_i} \tilde{F}_i
\]

for some subfield \( F_i \hookrightarrow L_{i-1} \) with \( \text{tr.deg}_\mathbb{C}(F_i) \leq d \) and \( \tilde{F}_i/F_i \) is a finite extension.

We refer to \( r \) as the length of the formula.

Definition 9.2. (Irrationalities of Complex Field Extensions)
Analogue of Definition 3.4

Let \( \mathbb{C} \hookrightarrow K \) be an extension of fields. An irrationality of \( K \) is a finite extension of fields

\[
K \hookrightarrow L
\]

along with a choice of primitive element \( \alpha \in L \).

Definition 9.3. (Natural and Accessory Irrationalities)
Analogue of Definition 3.8

Let \( K \hookrightarrow L \) be a finite extension of fields over \( \mathbb{C} \).

- An irrationality \( K \hookrightarrow F \) is a natural irrationality of \( K \hookrightarrow L \) if any Galois closure \( K \hookrightarrow \bar{L} \) of \( K \hookrightarrow L \) factors as

\[
K \hookrightarrow F \hookrightarrow \bar{L}.
\]

- An irrationality \( K \hookrightarrow F \) is an accessory irrationality of \( K \hookrightarrow L \) if \( K \) is the maximal common subfield of \( F \) and \( L \) over \( K \).
Lemma 9.4. (Decomposition of Irrationalities)
Analogue of Lemma 3.9

Let $K \hookrightarrow L$ be a finite extension of fields over $\mathbb{C}$ with an irrationality $K \hookrightarrow F$. Then, there exists a maximal subextension $K \hookrightarrow E$ such that

- $K \hookrightarrow E$ is a natural accessory irrationality for both $K \hookrightarrow L$ and $K \hookrightarrow F$;
- $E \hookrightarrow F$ is an accessory irrationality for $E \hookrightarrow \tilde{L}$.

for any Galois closure $K \hookrightarrow \tilde{L}$. We summarize the above lemma with the following diagram:

\[ K \twoheadrightarrow E \twoheadrightarrow \tilde{L} \quad \text{natural} \quad \text{accessory} \quad \quad L \]

Definition 9.5. (Essential Dimension of Field Extensions)
Analogue of Definition 4.1

Let $K \hookrightarrow L$ be an extension of fields over $\mathbb{C}$ of degree $n$. Given $K_0 \subseteq K$, we say that $F$ is defined over $K_0$ if there is an extension $K_0 \hookrightarrow L_0$ of degree $n$, such that $L_0K = L$. The essential dimension of $K \hookrightarrow L$ is

\[ \text{ed}(L/K) = \min \{ \text{tr.deg}(K_0) \mid L/K \text{ is defined over } K_0 \} \].

Definition 9.6. (Resolvent Degree of Field Extensions)
Analogue of Definition 4.2

Let $K \hookrightarrow L$ be a finite extension of fields over $\mathbb{C}$. The resolvent degree of $K \hookrightarrow L$ is the minimal $d$ for which there exists a formula for $K \hookrightarrow L$ of extensions in $d$ variables. We denote the resolvent degree of $K \hookrightarrow L$ by $\text{RD}(K \hookrightarrow L)$.

Lemma 9.7. (Re-statement of Essential Dimension for Fields)
Analogue of Lemma 4.5

Let $K \hookrightarrow L$ be a finite extension of fields over $\mathbb{C}$. The essential dimension of $K \hookrightarrow L$ is the minimal $d$ for which there exists a formula of length 1 in functions of $d$ variables.

Lemma 9.8. (Equivalence of Resolvent Degree)

Let $Y \rightarrow X$ be a branched cover of irreducible complex varieties. Then,

\[ \text{ed}(Y \rightarrow X) = \text{ed}(\mathbb{C}(Y)/\mathbb{C}(X)) \quad \text{and} \quad \text{RD}(Y \rightarrow X) = \text{RD}(\mathbb{C}(Y)/\mathbb{C}(X)). \]
**Proposition 9.9. (Easy Upper Bounds on ed/RD)**  
*Analogue of Proposition 5.2*

Let $K \hookrightarrow L$ be a branched cover of complex varieties. Then,

1. $\text{RD}(K \hookrightarrow L) \leq \text{ed}(K \hookrightarrow L) \leq \text{tr.deg}(K)$
2. Given a finite extension of fields $K \hookrightarrow F$,
   - $\text{ed}(K \hookrightarrow L \otimes_K F) \leq \text{ed}(K \hookrightarrow L)$, and
   - $\text{RD}(K \hookrightarrow L \otimes_K F) \leq \text{RD}(K \hookrightarrow L)$.

**Lemma 9.10. (Resolvent Degree of Compositions of Covers)**  
*Analogue of Lemma 5.4*

Let $K \hookrightarrow L \hookrightarrow F$ be a pair of finite extensions of fields. Then,

$$\text{RD}(K \hookrightarrow F) = \max \{\text{RD}(K \hookrightarrow L), \text{RD}(L \hookrightarrow F)\}.$$ 

**Lemma 9.11. (Resolvent Degree Invariant Under Galois Closure)**  
*Analogue of Lemma 5.5*

Let $K \hookrightarrow L$ be a finite extension of fields over $\mathbb{C}$. Let $K \hookrightarrow L'$ be a Galois closure of $K \hookrightarrow L$. Then,

$$\text{RD}(K \hookrightarrow L') = \text{RD}(K \hookrightarrow L).$$
10 Appendix C: Proof of Lemma 5.6

Lemma 5.6 (Universality of Solving $\Phi_n$)
Let $Z \to Y$ be an $n$-sheeted branched cover of complex varieties. Then,
\[ \text{RD}(Z \to Y) \to \text{RD}(n). \]

Proof. Observe that $\mathbb{C}(Y) \hookrightarrow \mathbb{C}(Z)$ is a degree $n$, separable field extension. By the primitive element theorem, there exists $\alpha \in \mathbb{C}(Z)$ such that
\[ \mathbb{C}(Z) \cong \mathbb{C}(Y)(\alpha) \cong \mathbb{C}(Y)[z]/m(z), \]
where
\[ m(z) = z^n + b_1 z^{n-1} + \cdots + b_n \]
is a minimal polynomial for $\alpha$. Let $\{Y_\alpha\}$ be an affine open cover of $Y$. As the $b_i$ are rational functions on $Y$, we can write
\[ b_{i,\alpha} = \frac{f_{i,\alpha}}{g_{i,\alpha}} \]
on each $Y_\alpha$. We then define $U_\alpha = Y_\alpha \setminus Z(g_{1,\alpha}, \ldots, g_{n,\alpha})$ and these glue together to give a dense, Zariski open $U \subseteq Y$ on which all of the $b_i$ are well-defined. Recalling $X = \mathbb{C}^n$ as in our definition of $\Phi_n$, we have a well defined map
\[ U \xrightarrow{m} X \]
\[ u \mapsto (b_1(u), \ldots, b_n(u)) \]
which itself defines a pullback square
\[ Z|_U \xrightarrow{m} E_{\Phi_n} \]
\[ Y \xrightarrow{m} X \]
Thus, by Proposition 5.2 we have
\[ \text{RD}(Z \to Y) = \text{RD}(Z|_U \to U) \leq \text{RD}(E_{\Phi_n} \to X) = \text{RD}(n). \]
\[ \square \]

35
11 Bibliography

We break up the references into groups:

- Classic - [1], [2], [3], [4], [5], [6]
- Modern - [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17]
- Future - [18], [19], [20], [21], [22], [23]

References - Classic


References - Modern


References - For Future Work


