DE GIORGI'S CONJECTURE FOR THE ALLEN-CAHN EQUATION AND RELATED PROBLEMS FOR CLASSICAL AND FRACTIONAL LAPLACIANS

by

BRYAN DIMLER, B.S.

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COMMITTEE MEMBERS:
Mostafa Fazly, Ph.D., Co-Chair
Changfeng Gui, Ph.D., Co-Chair
Gelu Popescu, Ph.D.
Fengxin Chen, Ph.D.

THE UNIVERSITY OF TEXAS AT SAN ANTONIO
College of Sciences
Department of Mathematics
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DEDICATION

I dedicate this thesis to my loving family and friends who have been so supportive throughout the duration of this project.

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Bryan Dimler, M.S.
The University of Texas at San Antonio, 2020

Supervising Professors: Mostafa Fazly, Ph.D. and Changfeng Gui, Ph.D.

We present results related to a famous conjecture of Enrico De Giorgi for a special class of bounded monotone solutions, called layer solutions, to nonlinear equations of the form

$$-\Delta u = f(u) \text{ in } \mathbb{R}^n$$

and

$$(-\Delta)^s u = f(u) \text{ in } \mathbb{R}^n,$$

where $(-\Delta)^s$ denotes the fractional Laplace operator with fractional exponent $s \in (0,1)$. Here, we assume that $f : \mathbb{R} \to \mathbb{R}$ is at least of class C^1. We begin by defining the fractional Laplace operator, and prove many of its fundamental properties. We then present the extension problem in $\mathbb{R}^{n+1}_+ := \{(x,y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n,\ y > 0\}$ for the operator $(-\Delta)^s$ introduced by Caffarelli and Silvestre in [13] and develop a fundamental solution and Poisson kernel for $(-\Delta)^s$ in $\mathbb{R}^{n+1}_+$. Subsequently, we prove De Giorgi's conjecture for layer solutions to the first equation above in dimensions $n \leq 3$. Precisely, we show that layer solutions are necessarily one-dimensional. We then turn our attention to the fractional De Giorgi conjecture, and present the fractional versions of the results obtained in the classical case (i.e. similar results for the second equation above). To supplement, we discuss some Pohozaev-type monotonicity formulae for the operators Δ and $(-\Delta)^s$, along with some closely related problems. We close with a brief discussion of some open problems and topics of further interest to the author.

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CHAPTER 1: INTRODUCTION

Perhaps the most studied operator in partial differential equations (PDE) is the Laplacian operator, given by

$$\Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}. \tag{1.1}$$

The Laplacian is the prototype representative of a class of partial differential operators, termed (uniformly) elliptic operators[1], which are characterized by their tendency to revert the value of a function at a point to its mean value over a region. Equations involving elliptic operators (i.e. elliptic PDE) commonly arise as steady state (time independent) counterparts of hyperbolic and parabolic PDE; their solutions tending to possess favorable regularity properties. Of particular interest in both the theory and application of PDE are solutions to Laplace's equation

$$-\Delta u = 0. \tag{1.2}$$

Laplace's equation is a second-order elliptic PDE, first studied by Pierre-Simon Laplace in the 18th century. It occurs frequently in the applied sciences, specifically in the study of steady state phenomena. For example, Laplace's equation may be viewed as the steady-state counterpart to the heat equation

$$u_t - \Delta u = 0. \tag{1.3}$$

That is to say, after setting $u_t = 0$ in the heat equation we obtain Laplace's equation. Physically, this corresponds to the heat distribution in, say, a rod at infinite time, $u = u(x, +\infty)$. Assuming no heat is being supplied to the rod, we should expect the heat distribution in the rod to reach an equilibrium or steady-state. Symbolically, we may then write

$$u_t(x,t) \to 0 \text{ as } t \to +\infty$$

[1]For the definition, see the appendix, Chapter B.

and obtain (1.2) in the limit.

Solutions to Laplace's equation are called harmonic functions, and their study is important to branches of physics such as electrostatics, gravitation, and fluid dynamics. Harmonic functions are also of interest in pure mathematics, not only in the study of PDE, but also fields ranging from complex analysis, harmonic analysis, and differential geometry. More generally, one may consider *Poisson's equation*

$$-\Delta u = f, \tag{1.4}$$

which plays a role in conservative fields (i.e. electrical, magnetic, gravitational, ...), where the vector field is derived from the gradient of a potential. We briefly note that Laplace's equation is merely the Poisson equation when we set $f \equiv 0$.

Recall from introductory analysis that, for a twice continuously differentiable function $u \in C^2(a, b)$, a second-order Taylor expansion gives

$$-u''(x) = \lim_{y \to 0} \frac{2u(x) - u(x+y) - u(x-y)}{y^2} \tag{1.5}$$

for each $x \in (a, b) \subseteq \mathbb{R}$. The right-hand side of equation (1.5) is known as a symmetric difference quotient of order two. Consider now the *spherical surface operator* and *solid averaging operator* defined by

$$\mathcal{M}_y u(x) = \frac{u(x+y) + u(x-y)}{2} \tag{1.6}$$

$$\mathcal{A}_y u(x) = \frac{1}{2y} \int_{x-y}^{x+y} u(t)\, dt, \tag{1.7}$$

respectively. We have

$$-u''(x) = 2 \lim_{y \to 0} \frac{u(x) - \mathcal{M}_y u(x)}{y^2} \tag{1.8}$$

$$= 6 \lim_{y \to 0} \frac{u(x) - \mathcal{A}_y u(x)}{y^2}, \tag{1.9}$$

where the second equality is obtained from the first after applying L'Hospital's rule. After making the necessary adjustments in higher dimensions, we obtain the *Blaschke-Privalov Laplacian*.

Proposition 1.0.1 (Blaschke-Privalov Laplacian)**.** *Let $\Omega \subset \mathbb{R}^n$ be open. For any $u \in C^2(\Omega)$ and $x \in \Omega$*

$$-\Delta u(x) = 2n \lim_{R\to 0} \frac{u(x) - \mathcal{M}_R u(x)}{R^2} \tag{1.10}$$
$$= 2(n+2) \lim_{R\to 0} \frac{u(x) - \mathcal{A}_R u(x)}{R^2} \tag{1.11}$$

where the Laplacian operator Δ is defined by (1.1)*, and the spherical surface and solid averaging operators in $\mathbb{R}^n$ are defined by*

$$\mathcal{M}_R u(x) = \frac{1}{\sigma_{n-1} R^{n-1}} \int_{\partial B_R(x)} u(y)\, dS^{n-1}(y) \tag{1.12}$$
$$\mathcal{A}_R u(x) = \frac{1}{\omega_n R^n} \int_{B_R(x)} u(y)\, dy, \tag{1.13}$$

where $B_R(x)$ denotes the ball of radius R centered at $x \in \mathbb{R}^n$ and $dS^{n-1}(y)$ denotes the $n-1$-dimensional spherical surface measure[2] *on $\mathbb{R}^n$.*

Remark 1.0.1. In the above proposition, σ_{n-1} and ω_n denote the Lebesgue measure[3] of the unit sphere and unit ball in $\mathbb{R}^n$, respectively. Moreover, setting $n = 1$ yields (1.8) and (1.9). We also observe, as a bit of foreshadowing, that we can write (1.10) more suggestively as

$$-\Delta u(x) = \frac{(n+2)\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}}} \lim_{R\to 0^+} \int_{\mathbb{R}^n} [2u(x) - u(x+y) - u(x-y)] \frac{1}{R^{n+2}} \chi_{B(0,R)}(y)\, dy, \tag{1.14}$$

where, as usual, χ_E denotes the characteristic function of a set $E \subseteq \mathbb{R}^n$.

In view of Proposition 1.0.1, we see that a function being harmonic is deeply related to the action of comparing the function value at a point with the surrounding values and reverting to the

[2]We often times will write dS^{n-1} in place of $dS^{n-1}(y)$ whenever there is no confusion regarding the variable of integration. Furthermore, on a general domain $\Omega \subset \mathbb{R}^n$, we write dS to represent the measure on $\partial\Omega$.

[3]Throughout this project, we will work almost exclusively with the Lebesgue measure on $\mathbb{R}^n$. In accordance, we will henceforth say "measure" when referring to the Lebesgue measure unless the distinction is necessary.

averaged values in a neighborhood of that point. That is, the idea behind the integral representations (1.10) and (1.11) is that the Laplacian tries to model an elastic reaction: it aims to level out differences in function values in order to make the function as uniform as possible. Precisely, we see that, for a harmonic function u defined on a domain Ω,

$$u(x) = \mathcal{M}_R u(x) = \mathcal{A}_R u(x) \tag{1.15}$$

for any ball of radius $R > 0$ such that $B_R(x) \subset\subset \Omega$. That is to say that the value of a harmonic function at a point in its domain is identically equal to its average over any ball (or boundary of any ball) compactly contained in its domain. This tendency to revert to the surrounding mean suggests that harmonic equations or, more generally, equations involving elliptic operators, possess some kind of regularity properties that prevent the solutions from oscillating too wildly.

From Proposition 1.0.1, we also see that the Laplacian is the infinitesimal limit of integral operators. In what follows, we discuss another integral operator, the fractional Laplacian, from which we recover the Laplacian in an appropriate limit sharing a similar property of averaging function values. Unlike the case of the Laplacian, such averaging procedure will not be limited to a small neighborhood of a given point in the domain, but will encompass all the possible values of the function by assigning weights to points corresponding to their proximity to the point of interest via an integral kernel. Due to this global nature, we say that the fractional Laplacian is a nonlocal operator.

We will first acquaint the reader with some basic facts and definitions necessary to study the fractional Laplacian. Subsequently, we will provide two definitions of the fractional Laplacian and prove their equivalence. This will be followed by a brief discussion of some fundamental facts regarding the fractional Laplacian. A lot of attention will be given to the subsequent section, namely, the fractional Laplacian as the solution to the harmonic extension problem in dimension $\mathbb{R}^{n+1+a}$, first recognized by Caffarelli and Silvestre in their pioneering paper (see [13]). We will then focus on a famous conjecture of Enrico De Giorgi regarding a particular class of solutions called layer solutions to the nonlinear problem

$$-\Delta u = f(u) \text{ in } \mathbb{R}^n,$$

where $f \in C^1(\mathbb{R})$, which is resolved aside from a mild limit assumption imposed in dimensions $4 \leq n \leq 8$ by Savin in [49]. The focus of this section will be to provide the reader with an understanding of the work of Ambrosio and Cabré in [2] for dimension $n = 3$, inspired by that of Gui and Ghoussoub in [31] for dimension $n = 2$. We further aim to develop the necessary tools for studying the problem in the nonlocal case using the extension problem of Caffarelli and Silvestre. Due to the deep connection between De Giorgi's conjecture, the theory of phase transitions, and minimal surfaces, we provide a brief discussion of minimal surface problems in the framework of De Giorgi, as well as motivate with simple examples from the theory of phase transitions. As supplementary material, we provide a short section on an estimate and monotonicity formula of Modica, proved in [41] and [43], respectively.

In the nonlocal case, we prove nonlocal analogues to the local results by following the work of Cabré and Sire ([9], [10]), as well the work of Cabré and Cinti ([7], [8]) and Dipierro et al. ([18]). We also discuss the difficulties that arise when transitioning from local to nonlocal problems via the extension problem. We conclude the study of the nonlocal case by considering a nonlocal Modica-type estimate and nonlocal monotonicity formula, along with a brief discussion of recent results, open problems, and subjects of interest to the author for future study.

1.1 Fractional Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a general, possibly non-smooth, open set in $\mathbb{R}^n$. Just as the Sobolev spaces $W^{k,p}(\Omega)$ are needed to establish appropriate regularity of solutions u for a given PDE, our forthcoming considerations of the fractional Laplacian $(-\Delta)^s$ require us to define a fractional counterpart, that is, the *fractional Sobolev spaces* $W^{s,p}(\Omega)$ for $s \in (0,1)$. For any $s \in (0,1)$ and any $p \in [1,\infty)$, we define the fractional Sobolev Space $W^{s,p}(\Omega)$ as follows:

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \ : \ \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}. \tag{1.16}$$

The space $W^{s,p}$ is an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}}, \tag{1.17}$$

where the term

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}}$$

is called the *Gagliardo seminorm* of u.

We present some fundamental properties of the fractional Sobolev spaces $W^{s,p}(\Omega)$. The results will be stated without proof, though proofs can be found in [21], [46], and [48]. For $s \in (0,1)$, the space $W^{s',p}(\Omega)$ is continuously embedded[4] in $W^{s,p}(\Omega)$ when $s \leq s'$.

Proposition 1.1.1. *Let* $p \in [1, \infty)$ *and* $0 < s \leq s' < 1$. *Let* Ω *be an open set in* $\mathbb{R}^n$ *and* $u : \Omega \to \mathbb{R}$ *be a measurable function. Then*

$$\|u\|_{W^{s,p}(\Omega)} \leq C \|u\|_{W^{s',p}(\Omega)}$$

for some suitable positive constant $C = C(n,s,p) \geq 1$. *In particular,*

$$W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega).$$

With some additional regularity assumptions on the $\partial\Omega$ (namely, that $\Omega \subset \mathbb{R}^n$ is open of class $C^{0,1}$ with bounded boundary), one can prove that $W^{1,p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$.

[4] Recall that a Banach space U is continuously embedded in a Banach space V if $U \subseteq V$ and $\|v\|_V \leq C\|u\|_U$ for all $u \in U$ and $v \in V$ and some positive constant C.

Proposition 1.1.2. *Let $p \in [0, \infty)$ and $s \in (0, 1)$. Let Ω be an open set in $\mathbb{R}^n$ of class $C^{0,1}$ with bounded boundary and $u : \Omega \to \mathbb{R}$ be a measurable function. Then*

$$\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

for some suitable constant $C = C(n, s, p) \geq 1$. In particular,

$$W^{1,p}(\Omega) \subseteq W^{s,p}(\Omega).$$

Remark 1.1.1. It should be pointed out that the definition of the fractional Sobolev space $W^{s,p}(\Omega)$ can be extended to the case $s \geq 1$ with some care (see [46]), however, this is not our focus.

We thereby see that there is a direct relationship between the fractional and classical Sobolev spaces via continuous embedding. In line with the classic case of s being an integer, we obtain the subsequent approximation result.

Proposition 1.1.3. *For any $s \in (0, 1)$, the space $C_0^\infty(\mathbb{R}^n)$ of smooth functions with compact support is dense in $W^{s,p}(\mathbb{R}^n)$.*

Remark 1.1.2. More generally, Proposition 1.1.3 holds for all $s > 0$.

Let $W_0^{s,p}(\mathbb{R}^n)$ denote the closure of $C_0^\infty(\mathbb{R}^n)$ in the norm $\|\cdot\|_{W^{s,p}(\Omega)}$ defined by (1.17). As a consequence of Proposition 1.1.3, we have

$$W_0^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n). \tag{1.18}$$

However, for a general domain $\Omega \subset \mathbb{R}^n$ these spaces do not coincide. That is to say that, in general, $C_0^\infty(\Omega)$ is not dense in $W^{s,p}(\Omega)$. Furthermore, the same conclusions stated in Proposition 1.1.1 and Proposition 1.1.2 hold for the spaces $W_0^{s,p}(\Omega)$.

1.2 Defining the Fractional Laplacian

We are finally ready to introduce the fractional Laplace operator $(-\Delta)^s$. We will first present the reader with a couple equivalent definitions of the fractional Laplacian. This is by no means exhaustive, and a more complete list of definitions can be found in [1] and [38]. The definitions to be considered are as follows:

1. The fractional Laplacian as a singular integral operator.

2. The fractional Laplacian via the Fourier transform on $\mathbb{R}^n$.

The equivalence of the above characterizations will be proven in full detail, however, we first motivate the material that follows.

In the applied sciences, it is often helpful to consider fractional derivatives of functions. Perhaps the most prominent way to define such fractional derivatives is founded on the notion of Marcel Riesz' potential of a function. Assume $n \geq 3$. In potential theory, the *Newtonian potential* of a function $f \in \mathscr{S}(\mathbb{R}^n)$ is given by

$$I_2(f)(x) = \frac{1}{4\pi^{\frac{n}{2}}}\Gamma\Big(\frac{n-2}{2}\Big)\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-2}}\,dy.$$

Recognizing the convolution kernel $\frac{1}{4\pi^{\frac{n}{2}}}\Gamma\big(\frac{n-2}{2}\big)\frac{1}{|x|^{n-2}}$ is simply the fundamental solution for Laplace's equation

$$\Psi(x) = \frac{1}{(n-2)\sigma_{n-1}}\frac{1}{|x|^{n-2}},$$

and recalling the identity of Gauss-Green, we observe that for any $f \in \mathscr{S}(\mathbb{R}^n)$ one has

$$I_2(-\Delta f) = f.$$

That is, the Newtonian potential is the inverse of the Laplacian operator: $I_2 = (-\Delta)^{-1}$. From this observation, we introduce a generalization of the Newtonian potential.

Definition 1.2.1 (Riesz' Potentials)**.** For any $n \in \mathbb{N}$, let $0 < \alpha < n.$ The *Riesz potential of order* α is the operator whose action on a function $f \in \mathscr{S}(\mathbb{R}^n)$ is given by:

$$I_\alpha(f)(x) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

It can be shown that $I_\alpha \in C^\infty(\mathbb{R}^n)$ for any $f \in \mathscr{S}(\mathbb{R}^n)$. Moreover, the operator I_α has been defined so that the normalization constant reduces to that of I_2 when setting $\alpha = 2$ in order to guarantee the validity of the following "fundamental theorem of fractional calculus."

Theorem 1.2.1. *For any $f \in \mathscr{S}(\mathbb{R}^n)$, one has in $\mathscr{S}'(\mathbb{R}^n)$*

$$I_\alpha(-\Delta)^{\frac{\alpha}{2}} f = (-\Delta)^{\frac{\alpha}{2}} I_\alpha f = f.$$

Of course, we must specify what is meant by the fractional operator $(-\Delta)^{\frac{\alpha}{2}}$. This is done most naturally by defining the action of $(-\Delta)^{\frac{\alpha}{2}}$ on the Fourier transform:

$$\mathscr{F}((-\Delta)^{\frac{\alpha}{2}} u) = (2\pi|\cdot|)^\alpha \mathscr{F}(u) \text{ for } u \in \mathscr{S}'(\mathbb{R}^n).$$

In other words, the above equation shows that the operator I_α inverts the fractional powers $0 < \alpha < n$ of the Laplacian: $I_\alpha = (-\Delta)^{-\frac{\alpha}{2}}$.

Since we are focused on the fractional Laplacian $(-\Delta)^s$ with s residing in $(0,1)$ we let $s = \frac{\alpha}{2}$ with $0 < \alpha < 2$ in the above expressions and the remainder of this project. Though such an operator has already been formally introduced, the prior definition has a major drawback for our purposes. Namely, it is not easy to analyze a given function by prescribing its Fourier transform. For this reason, we introduce an alternative definition of the fractional Laplacian, equivalent to the previous. This definition is helpful since it is more directly connected to the symmetric difference quotient of order two presented in the introduction, thereby allowing its probabilistic interpretation to be highlighted (see [30] for more on this). We will, however, return to the definition of the fractional Laplacian as determined by its action on the Fourier transform to tie up loose ends and introduce useful results.

1.2.1 The Fractional Laplacian as a Singular Integral Operator

Let us first specify what is meant by a singular integral operator. A *singular integral operator* is an integral operator with a singular kernel, each having its only singularities at a finite point, called *the origin*, and at infinity. An excellent resource on this topic is Elias M. Stein's text [58].

Having the above definition at hand, we may define the fractional Laplacian as a singular integral operator.

Definition 1.2.2. Let $u \in \mathscr{S}(\mathbb{R}^n)$ and $s \in (0,1)$. The fractional Laplacian operator is defined as a singular integral operator by

$$(-\Delta)^s u(x) := C(n,s) P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy. \tag{1.19}$$

Remark 1.2.1. The notation "$P.V.$" in (1.19) stands for "in the Principal Value sense," that is

$$\begin{aligned} (-\Delta)^s u(x) &= C(n,s) P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy \\ &= C(n,s) \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy. \end{aligned} \tag{1.20}$$

The constant $C(n,s)$ is a dimensional constant that depends only on n and s, and is given precisely by

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1-\cos \zeta_1}{|\zeta|^{n+2s}}\, d\zeta \right)^{-1}. \tag{1.21}$$

In fact, for $0 < s < 1$, we have

$$C(n,s) = \frac{s 2^{2s} \Gamma(\frac{n+2s}{2})}{\pi^{\frac{n}{2}} \Gamma(1-s)} \text{ (see (5.10) in [30])}.$$

We further note the constant $C(n,s)$ plays plays no essential role for fixed $s \in (0,1)$. As such, unless stated otherwise, we will take $C(n,s) \equiv 1$. Furthermore, we will drop the symbol $P.V.$

in computations with the convention that integrals are being taken in the $P.V.$ sense unless this distinction is necessary.

In the spirit of the opening remarks, we show that the singular integral in (1.19) can be written as a weighted second order differential quotient much like in the classical case. The proofs of the following facts have been adapted from [46].

Lemma 1.2.1. *Let $s \in (0,1)$ and let $(-\Delta)^s$ be the fractional Laplacian operator defined by* (1.19). *Then, for any $u \in \mathscr{S}(\mathbb{R}^n)$,*

$$(-\Delta)^s u(x) = -\frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y) - u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy \tag{1.22}$$

for each $x \in \mathbb{R}^n$.

Remark 1.2.2. The integral in (1.22) *is not* in the $P.V.$ sense.

Proof. The equivalence of definitions (1.19) and (1.22) is the result of a standard change of variables argument. Set $z = y - x$ and observe that

$$\begin{aligned}(-\Delta)^s u(x) &= -\int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|x-y|^{n+2s}}\, dy \\ &= -\int_{\mathbb{R}^n} \frac{u(x+z) - u(x)}{|z|^{n+2s}}\, dz\end{aligned}$$

Substituting $z' = -z$ above, we have

$$\int_{\mathbb{R}^n} \frac{u(x+z) - u(x)}{|z|^{n+2s}}\, dz = \int_{\mathbb{R}^n} \frac{u(x-z') - u(x)}{|z'|^{n+2s}}\, dz'$$

and so after relabeling z' as z

$$\begin{aligned}
&2\int_{\mathbb{R}^n} \frac{u(x+z)-u(x)}{|z|^{n+2s}}\,dz \\
&= \int_{\mathbb{R}^n} \frac{u(x+z)-u(x)}{|z|^{n+2s}}\,dz + \int_{\mathbb{R}^n} \frac{u(x-z)-u(x)}{|z|^{n+2s}}\,dz \\
&= \int_{\mathbb{R}^n} \frac{u(x+z)-u(x-z)-2u(x)}{|z|^{n+2s}}\,dz.
\end{aligned}$$

Therefore, if we rename z as y in the above expression, we can write the fractional Laplacian operator in (1.19) as

$$(-\Delta)^s u(x) = -\frac{1}{2} P.V. \int_{\mathbb{R}^n} \frac{u(x+y)-u(x-y)-2u(x)}{|y|^{n+2s}}\,dy,$$

where the integral is being taken in the $P.V.$ sense due to the singularity at the origin. By the above representation, we may remove the singularity at the origin. Indeed, for any smooth function u, a second order Taylor expansion yields

$$\frac{u(x+y)-u(x-y)-2u(x)}{|y|^{n+2s}} \le \frac{\|D^2u\|_{L^\infty}}{|y|^{n+2s-2}},$$

which is integrable near zero for any fixed $s \in (0,1)$. Therefore, since $u \in \mathscr{S}(\mathbb{R}^n)$ we may get rid of the $P.V.$ and write (1.22). □

Remark 1.2.3. As mentioned earlier, a primary advantage of the expression (1.22) is in the fact that the probabilistic interpretation of the fractional Laplacian (1.19) becomes apparent. Moreover, (1.22) may sometimes be used to simplify computations, allowing one to disregard the $P.V.$ symbol in (1.19).

We point out that the fractional Laplacian is well-defined for every $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, which can be shown by estimating the integral (1.19) over $B_\epsilon(x)$, for $\epsilon > 0$ small enough, and its complement in $\mathbb{R}^n$ and applying Lemma 1.2.1. Similarly, one may show $(-\Delta)^s u \in L^2(\mathbb{R}^n)$ whenever $u \in \mathscr{S}(\mathbb{R}^n)$.

Let $0 < s < 1$. We denote by $\mathscr{L}_s(\mathbb{R}^n)$ the space of measurable functions $u : \mathbb{R}^n \to \overline{\mathbb{R}}$ for which the norm

$$\|u\|_{\mathscr{L}_s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} \, dx$$

is finite. In particular, we have the inclusion $\mathscr{S}(\mathbb{R}^n) \hookrightarrow \mathscr{L}_s(\mathbb{R}^n)$[5], where the symbol "$\hookrightarrow$" denotes continuous embedding. Furthermore, if $u \in C^2(\mathbb{R}^n) \cap \mathscr{L}_s(\mathbb{R}^n)$, we can define $(-\Delta)^s u(x)$ as in (1.19) for every $x \in \mathbb{R}^n$.

1.2.2 The Fractional Laplacian Via the Fourier Transform

We now introduce an alternative definition of the space $H^s(\mathbb{R}^n) := W^{s,2}(\mathbb{R}^n)$ by way of the Fourier transform[6]. The proofs in this section have been adapted from [46]. Define

$$\widehat{H}^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\mathscr{F}u(\xi)|^2 \, d\xi < +\infty\}. \tag{1.23}$$

We show the space $\widehat{H}^s(\mathbb{R}^n)$ is equivalent to the space defined earlier in (1.16). Before we do so, we will prove an important result. Returning to our initial definition of the fractional Laplacian given in the opening of this chapter, we observe that the fractional Laplacian $(-\Delta)^s$ can be viewed as a *pseudo-differential operator of symbol* $(2\pi|\xi|)^{2s}$. In short, pseudo-differential operators are defined by their action on functions specified by the following formula:

$$\mathscr{P}u(x) = \int_{\mathbb{R}^n} e^{2\pi i \xi \cdot x} p(x, \xi) \hat{u}(\xi) \, d\xi.$$

Here, the function $p(x, \xi)$ is known as a symbol. In the case of the fractional Laplacian, we see that $p(x, \xi) = (2\pi|\xi|)^{2s}$.

[5]Even more, we have the inclusion $L^p(\mathbb{R}^n) \hookrightarrow \mathscr{L}_s(\mathbb{R}^n)$ for each $1 \leq p \leq \infty$.

[6]We take

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \, dx$$

as our definition of the Fourier transform. Here, we assume $f \in L^1(\mathbb{R}^n)$, of course.

Proposition 1.2.1. *Let* $s \in (0,1)$ *and let* $(-\Delta)^s : \mathscr{S}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *be the fractional Laplacian operator defined by* (1.19)*. Then, for any* $u \in \mathscr{S}(\mathbb{R}^n)$*,*

$$(-\Delta)^s u = \mathscr{F}^{-1}((2\pi|\xi|)^{2s}(\mathscr{F}u)) \tag{1.24}$$

for all $\xi \in \mathbb{R}^n$*.*

Proof. In view of Lemma 1.2.1, we may use the definition via the weighted second order differential quotient in (1.22). We denote by Lu the integral in (1.22), that is

$$Lu = -\frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y) - u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy.$$

We note that L is a linear operator and we are looking for its symbol, that is, a function $p : \mathbb{R}^n \to \mathbb{R}$ such that

$$Lu = \mathscr{F}^{-1}(p \cdot \mathscr{F}u). \tag{1.25}$$

We want to prove that

$$p(\xi) = (2\pi|\xi|)^{2s}, \tag{1.26}$$

where we denote by ξ the frequency variable. Applying a second order Taylor expansion, we observe that

$$\frac{|u(x+y) + u(x-y) - 2u(x)|}{|y|^{n+2s}} \leq$$
$$C\Big(\chi_{B_1}(y)|y|^{2-n-2s} \sup_{B_1(x)} |D^2u| + \chi_{\mathbb{R}^n \setminus B_1(y)}|y|^{-n-2s}|u(x+y) + u(x-y) - 2u(x)|\Big) \in L^1(\mathbb{R}^{2n})$$

for some constant C. Consequently, by Fubini's theorem, we can exchange the integral in y with the Fourier transform in x. Thus, we apply the Fourier transform in the variable x and obtain

$$
\begin{aligned}
p(\xi)(\mathscr{F}u)(\xi) &= \mathscr{F}(Lu) \\
&= -\frac{1}{2}\int_{\mathbb{R}^n} \frac{\mathscr{F}(u(x+y)+u(x-y)-2u(x))}{|y|^{n+2s}}\,dy \\
&= -\frac{1}{2}\int_{\mathbb{R}^n} \frac{e^{2\pi i\xi y}+e^{-2\pi i\xi y}-2}{|y|^{n+2s}}\,dy(\mathscr{F}u)(\xi) \\
&= \int_{\mathbb{R}^n} \frac{1-\cos(2\pi\xi\cdot y)}{|y|^{n+2s}}\,dy(\mathscr{F}u)(\xi).
\end{aligned}
\tag{1.27}
$$

Hence, in order to obtain (1.26), it suffices to show that

$$
\int_{\mathbb{R}^n} \frac{1-\cos(2\pi\xi\cdot y)}{|y|^{n+2s}}\,dy = (2\pi|\xi|)^{2s}. \tag{1.28}
$$

First, we observe that, if $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$, we have

$$
\frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}} \le \frac{|\zeta_1|^2}{|\zeta|^{n+2s}} \le \frac{1}{|\zeta|^{n-2+2s}}
$$

near $\zeta = 0$, which can be seen by applying a Taylor expansion about zero. Thus,

$$
\frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}} \in L^1(\mathbb{R}^n). \tag{1.29}
$$

Now, we consider the function $q : \mathbb{R}^n \to \mathbb{R}$ defined as follows:

$$
q(\xi) = \int_{\mathbb{R}^n} \frac{1-\cos(\xi\cdot y)}{|y|^{n+2s}}\,dy.
$$

From (1.29), we see that p is well-defined. Moreover, q is invariant under the rotation $\xi \mapsto |\xi|e_1$. That is

$$
q(\xi) = q(|\xi|e_1), \tag{1.30}
$$

where e_1 denotes the first direction in $\mathbb{R}^n$. Indeed, when $n = 1$, we may deduce (1.28) by the fact that $q(-\xi) = q(\xi)$. When $n \ge 2$, we consider a rotation R for which $R(|\xi|e_1) = \xi$ and we denote by R^T its transpose. Then, by substituting $y' = R^T y$, we obtain

$$\begin{aligned}
q(\xi) &= \int_{\mathbb{R}^n} \frac{1-\cos((R(|\xi|e_1))\cdot y)}{|y|^{n+2s}}\,dy \\
\text{(Since R is self adjoint)} \;\; &= \int_{\mathbb{R}^n} \frac{1-\cos\big((|\xi|e_1)\cdot(R^T y)\big)}{|y|^{n+2s}}\,dy \\
&= \int_{\mathbb{R}^n} \frac{1-\cos((|\xi|e_1)\cdot y')}{|y|^{n+2s}}\,dy' \\
&= q(|\xi|e_1),
\end{aligned}$$

proving (1.30).

As a consequence of (1.29) and (1.30), the substitution $\zeta = 2\pi|\xi|y$ gives that

$$\begin{aligned}
p(\xi) &= p(|\xi|e_1) \\
&= \int_{\mathbb{R}^n} \frac{1-\cos(2\pi|\xi|y_1)}{|y|^{n+2s}}\,dy \\
&= \frac{1}{(2\pi|\xi|)^n}\int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{\left|\frac{\zeta}{2\pi|\xi|}\right|^{n+2s}}\,d\zeta \\
&= C(n,s)^{-1}(2\pi|\xi|)^{2s} \\
&= (2\pi|\xi|)^{2s}
\end{aligned}$$

We thereby deduce (1.28) and the proof is complete. □

Specifically, Proposition 1.2.1 provides us with a direct relationship between the fractional Laplacian of a suitable function u and its Fourier transform given by (1.24). Moreover, we have the following:

Proposition 1.2.2. *Let $s \in (0,1)$. Then the fractional Sobolev space $H^s(\mathbb{R}^n)$ defined in* (1.16) *coincides with $\widehat{H}^s(\mathbb{R}^n)$ defined in* (1.23). *In particular, for any $u \in H^s(\mathbb{R}^n)$*

$$[u]^2_{H^s(\mathbb{R}^n)} = 2\int_{\mathbb{R}^n} |\xi|^{2s}|\mathscr{F}u(\xi)|^2\,d\xi.$$

Proof. For every fixed $y \in \mathbb{R}^n$, by applying the change of variables $x = x - y$, we get

$$
\begin{aligned}
\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\right)dy &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(z+y)-u(y)|^2}{|z|^{n+2s}}\,dz\,dy \\
&= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left|\frac{u(z+y)-u(y)}{|z|^{\frac{n}{2}+2}}\right|^2 dy\right)dz \\
&= \int_{\mathbb{R}^n}\left\|\frac{u(z+\cdot)-u(\cdot)}{|z|^{\frac{n}{2}+s}}\right\|^2_{L^2(\mathbb{R}^n)} dz \\
&= \int_{\mathbb{R}^n}\left\|\mathscr{F}\left(\frac{u(z+\cdot)-u(\cdot)}{|z|^{\frac{n}{2}+s}}\right)\right\|^2_{L^2(\mathbb{R}^n)} dz,
\end{aligned}
$$

where we have used Plancherel's formula. Using (1.28) we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^n}\left\|\mathscr{F}\left(\frac{u(z+\cdot)-u(\cdot)}{|z|^{\frac{n}{2}+s}}\right)\right\|^2_{L^2(\mathbb{R}^n)} dz &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|e^{i\xi\cdot z}-1|^2}{|z|^{n+2s}}|\mathscr{F}u(\xi)|^2\,d\xi\,dz \\
&= 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(1-\cos(\xi\cdot z))}{|z|^{n+2s}}|\mathscr{F}u(\xi)|^2\,dz\,d\xi \\
&= 2C(n,s)^{-1}\int_{\mathbb{R}^n}|\xi|^{2s}|\mathscr{F}u(\xi)|^2\,d\xi \\
&= 2\int_{\mathbb{R}^n}|\xi|^{2s}|\mathscr{F}u(\xi)|^2\,d\xi.
\end{aligned}
$$

This completes the proof. □

In other words, Proposition 1.2.2 establishes the equivalence of the spaces $H^s(\mathbb{R}^n)$ and $\hat{H}^s(\mathbb{R}^n)$, mirroring the case when s is an integer[7]. For more on the fractional Sobolev theory and trace theory for $(-\Delta)^s$, we encourage the reader to take a look at [46].

1.3 Fundamental Properties of the Fractional Laplacian

Before proceeding, we introduce some elementary yet fundamental properties of the operator $(-\Delta)^s$. Many of these facts will be helpful in later computations and the reader should notice the similarity between the following results and the local analogues for the Laplacian. We will

[7] See Appendix Chapter A.

conclude this section, as well as the chapter, by proving that the fractional Laplacian of a suitable function u converges to u as $s \downarrow 0$ and to $-\Delta u$ as $s \uparrow 1$. This will tie together the similarities between the fractional Laplacian and Laplacian operators which we have observed thus far. For more information on the following material, we recommend [30].

Appealing to Definition 1.2.2, one my easily verify that the fractional Laplacian (1.19) is linear and translation invariant. Furthermore, from above, $(-\Delta)^s$ is a homogeneous operator of order $2s$. We thereby have the following proposition.

Proposition 1.3.1. *For every function $u \in \mathscr{S}(\mathbb{R}^n)$, we have for every $h \in \mathbb{R}^n$*

$$(-\Delta)^s(\tau_h u) = \tau_h((-\Delta)^s u) \tag{1.31}$$

and every $\lambda > 0$

$$(-\Delta)^s(\delta_\lambda u) = \lambda^{2s}\delta_\lambda((-\Delta)^s u), \tag{1.32}$$

where τ_h and δ_λ are the translation and dilation operators by $h \in \mathbb{R}^n$ and $\lambda > 0$, respectively.

Much like the classical Laplacian, we also have invariance with respect to the action of the orthogonal group on $\mathbb{R}^n$.

Proposition 1.3.2. *Let $u := u(|x|)$ be a function with spherical symmetry in $C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then, $(-\Delta)^s u$ also has spherical symmetry.*

Proof. Let $O \in \mathbb{O}(n)$, where $\mathbb{O}(n)$ denotes the orthogonal group on $\mathbb{R}^n$. Compute $(-\Delta)^s u(Ox)$ directly to find

$$(-\Delta)^s u(Ox) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{2u(|x|) - u(|x+O^T y|) - u(|x - O^T y|)}{|y|^{n+2s}}\, dy.$$

Applying the change of variable $z = O^T y$, we have

$$\begin{aligned}(-\Delta)^s u(Ox) &= \frac{1}{2}\int_{\mathbb{R}^n} \frac{2u(|x|) - u(|x+z|) - u(|x-z|)}{|z|^{n+2s}} \\ &= (-\Delta)^s u(x)\end{aligned}$$

by Lemma 1.2.1. $\square$

Moreover, we have the following estimate:

Proposition 1.3.3. *Let $u \in \mathscr{S}(\mathbb{R}^n)$. Then, for every $x \in \mathbb{R}^n$ with $|x| > 1$, we have*

$$|(-\Delta)^s u(x)| \le C(u, n, s)|x|^{-(n+2s)}, \tag{1.33}$$

where $C(u, n, s)$ is a dimensional constant depending also on the function u and fractional exponent s.

From Proposition 1.3.3, we may obtain a nontrivial regularity result for $(-\Delta)^s$:

Corollary 1.3.1. *Let $u \in \mathscr{S}(\mathbb{R}^n)$. Then, $(-\Delta)^s u \in C^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.*

With the opening of this note in mind, we provide the fractional analogue of the averaging expression for $-\Delta u(x)$ in terms of a pointwise limit involving the spherical mean-value operator $\mathscr{M}_R$. Unlike the classical case, our new expression will be provided in terms of an integral. This integral representation is a result of the nonlocal nature of $(-\Delta)^s$.

Proposition 1.3.4. *Let $u \in \mathscr{S}(\mathbb{R}^n)$. For every $0 < s < 1$, one has*

$$(-\Delta)^s u(x) = -\sigma_{n-1} \int_0^\infty R^{-1-2s}[\mathscr{M}_R u(x) - u(x)]\, dR, \tag{1.34}$$

with $\mathscr{M}_R$ denoting the spherical mean-value operator (1.12).

Proof. By (1.10) in Proposition 1.0.1, we see that $\mathscr{M}_R u(x) - u(x) = O(R^2)$ as $R \to 0^+$. Thus, the integrand in right-hand side of the integral (1.34) behaves like R^{1-2s} as $R \to 0^+$. Moreover, at infinity the integrand behaves like R^{-1-2s}. We thereby conclude that the integral on the right-hand side of (1.34) is convergent. By Cavalieri's principle, we may write

$$\begin{aligned}(-\Delta)^s u(x) &= \int_0^\infty \int_{\partial B_R(x)} \frac{u(x)-u(y)}{R^{n+2s}}\, dS^{n-1}(y)\, dR \\ &= \int_0^\infty \frac{1}{R^{n+2s}} \int_{\partial B_R(x)} [u(x)-u(y)]\, dS^{n-1}(y)\, dR \\ &= -\sigma_{n-1} \int_0^\infty \frac{R^{n-1}}{R^{n+2s}} [\mathscr{M}_R u(x) - u(x)]\, dR,\end{aligned}$$

giving the desired conclusion. $\square$

The following symmetry property is also useful.

Proposition 1.3.5 (Symmetry Property). *Let* $0 < s \leq 1$. *Then, for any* $u, v \in \mathscr{S}(\mathbb{R}^n)$ *we have*

$$\int_{\mathbb{R}^n} u(x)(-\Delta)^s v(x)\, dx = \int_{\mathbb{R}^n} (-\Delta)^s u(x) v(x)\, dx. \tag{1.35}$$

Proof. We omit the case $s = 1$, as it is well known and follows immediately from the integration by parts formula. Thus, let us focus on the case $0 < s < 1$. By Corollary 1.3.1, we know that $\widehat{(-\Delta)^s} u, \widehat{(-\Delta)^s} v \in L^1(\mathbb{R}^n)$, so we may use the following formula, valid for any $f, g \in L^1(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} \hat{f}(\xi) g(\xi)\, d\xi = \int_{\mathbb{R}^n} f(\xi) \hat{g}(\xi)\, d\xi. \tag{1.36}$$

By (1.36) and Proposition 1.2.1, we see that

$$\begin{aligned}\int_{\mathbb{R}^n} (-\Delta)^s u(x) v(x)\, dx &= \int_{\mathbb{R}^n} (-\Delta)^s u(x) \mathscr{F}\big(\mathscr{F}^{-1} v\big)(x)\, dx \\ &= \int_{\mathbb{R}^n} \mathscr{F}\big((-\Delta)^s u(\xi)\big) \mathscr{F}^{-1} v(\xi)\, d\xi \\ &= \int_{\mathbb{R}^n} \hat{u}(\xi)(2\pi|\xi|)^{2s} \mathscr{F}^{-1} v(\xi)\, d\xi\end{aligned}$$

Applying Proposition 1.2.1 once more, we find

$$\mathscr{F}^{-1}\big((-\Delta)^s v\big)(\xi) = (2\pi|\xi|)^{2s}\mathscr{F}^{-1}v(\xi).$$

Using the above and applying (1.36) again, we see

$$\begin{aligned}\int_{\mathbb{R}^n}(-\Delta)^s u(x)v(x)\,dx &= \int_{\mathbb{R}^n}\hat{u}(\xi)\mathscr{F}^{-1}\Big((-\Delta)^s v\Big)(\xi)\,d\xi\\ &= \int_{\mathbb{R}^n}\mathscr{F}^{-1}(\hat{u})(x)(-\Delta)^s v(x)\,dx\\ &= \int_{\mathbb{R}^n}u(x)(-\Delta)^s v(x)\,dx,\end{aligned}$$

as desired. This concludes the proof. □

Before moving on, we remark that many of the classical results for the Laplacian can be extended, with appropriate adjustments, to the fractional Laplacian. In particular, this holds true for the maximum principles and other related theorems (e.g. Liouville theorem, Harnack inequality). For more on this, see [30].

1.3.1 The Limit as $s \downarrow 0$ and $s \uparrow 1$

In this opening chapter, we have introduced the fractional Laplacian $(-\Delta)^s$ for $s \in (0,1)$ and have highlighted its similarity to the more familiar operator, the Laplacian. However, after examining definition (1.19), the relationship between these two operators is not apparent. By viewing the fractional Laplacian as a pseudo-differential operator (see Proposition 1.2.1), the relationship becomes clear.

Theorem 1.3.1. *For any $u \in \mathscr{S}(\mathbb{R}^n)$, the following hold:*

$$\lim_{s\to 0^+}(-\Delta)^s u = u, \tag{1.37}$$

$$\lim_{s\to 1^-}(-\Delta)^s u = -\Delta u. \tag{1.38}$$

Proof. By Proposition 1.2.1, we see that

$$(-\Delta)^s u(x) = \mathscr{F}^{-1}((2\pi|\xi|)^{2s}\mathscr{F}(u))(x) \text{ for all } x \in \mathbb{R}^n. \tag{1.39}$$

Taking the limit in (1.39) as $s \to 0^+$ and $s \to 1^-$ and using the fact that $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ along with the dominated convergence theorem, we see that

$$\begin{aligned}
\lim_{s\to 0^+} (-\Delta)^s u(x) &= \lim_{s\to 0^+} \mathscr{F}^{-1}((2\pi|\xi|)^{2s}\mathscr{F}(u))(x) \\
&= \mathscr{F}^{-1}\Big(\lim_{s\to 0^+} (2\pi|\xi|)^{2s}\mathscr{F}(u)\Big)(x) \\
&= \mathscr{F}^{-1}(\mathscr{F}(u))(x) \\
&= u(x) \text{ for each } x \in \mathbb{R}^n
\end{aligned} \tag{1.40}$$

and

$$\begin{aligned}
\lim_{s\to 1^-} (-\Delta)^s u(x) &= \lim_{s\to 1^-} \mathscr{F}^{-1}(|\xi|^{2s}\mathscr{F}(u))(x) \\
&= \mathscr{F}^{-1}\Big(\lim_{s\to 1^-} (2\pi|\xi|)^{2s}\mathscr{F}(u)\Big)(x) \\
&= \mathscr{F}^{-1}((2\pi|\xi|)^{2}\mathscr{F}(u))(x) \\
&= \mathscr{F}^{-1}(\mathscr{F}(-\Delta u))(x) \\
&= -\Delta u(x) \text{ for all } x \in \mathbb{R}^n.
\end{aligned} \tag{1.41}$$

This concludes the proof. □

Remark 1.3.1. We may suppose instead that $u \in C_0^\infty(\mathbb{R}^n)$ and repeat the same proof so that Theorem 1.3.1 applies to the space C_0^∞ of test functions also. Using this fact, we may approximate using test functions to generalize to more abstract spaces have $C_0^\infty(\mathbb{R}^n)$ as a dense subset.

Theorem 1.3.1 shows that, for suitable u, the fractional Laplacian of u converges to the Laplacian of u (or to u itself) in an appropriate limit, establishing a direct connection between the local and nonlocal operators $-\Delta$ and $(-\Delta)^s$. In the subsequent chapter, we develop yet another connection between local and nonlocal problems for the Laplacian and fractional Laplacian using an extension problem developed by Caffarelli and Silvestre in 2007. We later use these results to prove a famous conjecture of De Giorgi for $(-\Delta)^s$.

CHAPTER 2: THE EXTENSION PROBLEM

In the past decade there has been an increased interest in the analysis of nonlocal operators such as $(-\Delta)^s$ in connection with the applied sciences, analysis, probability, and geometry. Such developments have been motivated largely by the extension paper published in 2007 by L. Caffarelli and L. Silvestre (see [13]). In this paper, the authors introduced a method allowing the conversion of nonlocal problems in $\mathbb{R}^n$ into problems involving a particular differential operator in $\mathbb{R}^{n+1}_+ := \{(x,y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n,\ y > 0\}$. In the present chapter, we summarize many of their results.

The motivation for realizing the fractional Laplacian as a local operator is as follows: When dealing with nonlocal operators such as $(-\Delta)^s$, a major difficulty one encounters stems from the fact that they do not act on functions like differential operators do. Instead, their action is often through nonlocal integral formulas such as (1.19). As a result, the tools of differential calculus are not readily available. It is therefore desirable to have a procedure allowing us to connect nonlocal problems to local problems, in which the rules of differential calculus are at our disposal.

Before diving head first into the general case (i.e., $s \in (0,1)$), we consider the particular case $s = \frac{1}{2}$. To motivate, we note that the fractional Laplacian satisfies the following *semigroup property*: For any $s, s' \in (0,1)$ satisfying $s + s' \leq 1$ we have

$$\begin{aligned}\mathscr{F}(-\Delta)^s(-\Delta)^{s'}u &= (2\pi|\xi|)^{2s}\mathscr{F}((-\Delta)^{s'}u)\\ &= (2\pi|\xi|)^{2s}(2\pi|\xi|)^{2s'}\hat{u}\\ &= (2\pi|\xi|)^{2(s+s')}\hat{u}\\ &= \mathscr{F}\Big((-\Delta)^{s+s'}u\Big).\end{aligned}$$

That is

$$(-\Delta)^s(-\Delta)^{s'}u = (-\Delta)^{s'}(-\Delta)^s u = (-\Delta)^{s+s'}u.$$

As a special case of the above, when $s = s' = \frac{1}{2}$, we obtain

$$((-\Delta)^{\frac{1}{2}})^2 = -\Delta. \tag{2.1}$$

From this observation, we see that if $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ is the harmonic extension of $f : \mathbb{R}^n \to \mathbb{R}$, that is, if u solves

$$\begin{cases} \Delta u = 0,\ (x,y) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = f(x),\ x \in \mathbb{R}^n \end{cases} \tag{2.2}$$

then

$$-\lim_{y\to 0^+} u_y(x,y) = (-\Delta)^{\frac{1}{2}} f(x). \tag{2.3}$$

Indeed, writing Δ_x to represent the Laplacian in the coordinates $x \in \mathbb{R}^n$, we may write the *total Laplacian* in the variables $(x,y) \in \mathbb{R}^n \times (0,+\infty)$ as

$$\Delta = \Delta_x + \frac{\partial^2}{\partial y^2}. \tag{2.4}$$

Given a smooth and bounded function $f : \mathbb{R}^n \to \mathbb{R}$, we take $u := U_f$ (smooth and bounded) solving (2.2) and consider the operator

$$L[f(x)] := -\partial_y U_f(x,0). \tag{2.5}$$

Set $v(x,y) := -u_y(x,y)$. Computing, we find that $\Delta v = -\partial_y \Delta u = 0$ in $\mathbb{R}^n \times (0,+\infty)$ and $v(x,0) = L[f(x)]$ for any $x \in \mathbb{R}^n$. Thus, we may recognize v as the harmonic extension of $L[f]$

and can write $v = U_{L[f]}$. Then,

$$\begin{aligned} L(L[f])(x) &= -\partial_y U_{L[f]}(x,0) \\ &= -v_y(x,0) \\ &= u_{yy}(x,0) \\ &= \Delta u(x,0) - \Delta_x u(x,0) \\ &= -\Delta_x u(x,0) \\ &= -\Delta f(x) \end{aligned}$$

which gives $L^2 = -\Delta$. This is consistent wih $L = (-\Delta)^{\frac{1}{2}}$, by (2.3).

We now fix $s \in (0,1)$ and $f \in \mathscr{S}(\mathbb{R}^n)$ and consider the function $u(x,y)$ that solves the following Dirichlet problem in divergence form in the half-space $\mathbb{R}^{n+1}_+$:

$$\begin{cases} \operatorname{div}(y^a \nabla u) = 0, \ (x,y) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = f(x), \ x \in \mathbb{R}^n. \end{cases} \tag{2.6}$$

Here, $a = 1 - 2s \in (-1,1)$. For simplicity, we sometimes may write $L_a u = \operatorname{div}(y^a \nabla u)$. Note that (2.6) may be written in nondivergence form as follows:

$$\begin{cases} -\Delta_x u = u_{yy} + \frac{a}{y} u_y, \ (x,y) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = f(x) \text{ and } u(x,y) \to 0 \text{ as } y \to \infty \text{ for each } x \in \mathbb{R}^n. \end{cases} \tag{2.7}$$

Direct computation yields the equivalency of problems (2.6) and (2.7).

The formulation of problem (2.7) is the product of a clever observation: For a nonnegative integer a, we may suppose $u : \mathbb{R}^n \times \mathbb{R}^{1+a} \to \mathbb{R}$ is radially symmetric in the y variable, with y residing in the "fractional dimension" $N = 1+a$. That is, if $|y| = |y'| = r$, then $u(x,y) = u(x,y')$. We can then think of u as a function of x and r, and in these variables write an expression for its

Laplacian:

$$\Delta u = \Delta_x u + \frac{a}{r} u_r + u_{rr}.$$

We thereby have obtained an identical expression for the equation in (2.7), with y replaced by r in the expression. However, as far as the expression is concerned, there is no need to consider only integer values of a. We can thus realize problem (2.7) as the harmonic extension problem of f in dimension $\mathbb{R}^{n+1+a}$ for suitably chosen a.

Going forward, the goal of this chapter is to present the fundamental solution for $(-\Delta)^s$ and to develop a Poisson kernel in order to obtain a representation formula for solutions to problem (2.7) and, likewise, problem (2.6). Furthermore, we will show that

$$(-\Delta)^s f = \lim_{y \to 0^+} -y^a u_y$$

up to a multiplicative constant, where $s = \frac{1-a}{2}$ (analogously, $a = 1 - 2s$). Note that setting $a = 0$ results in problem (2.3). Verification of the above limit will, however, be postponed until later. Instead, we first focus on representing solutions for $(-\Delta)^s$ and problems (2.6) and (2.7). The interested reader may refer to the original paper by Caffarelli and Silvestre [13], or the more recent note by Garofalo [30] for more on this subject.

2.1 The Fundamental Solution

It is appropriate to first make some remarks about what is meant by a fundamental solution for $(-\Delta)^s$. We follow the presentation laid out by Garofalo (see [30]).

With the definition of the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ as a guide, we may define a *fractional Schwartz space*. Indeed, given $0 < s < 1$, we may consider the linear space of functions $u \in C^\infty(\mathbb{R}^n)$ such that, for every multiindex α, we have

$$[u]_\alpha = \sup_{x \in \mathbb{R}^n} (1 + |x|^{n+2s}) |D^\alpha u(x)| < \infty.$$

We denote by $\mathscr{S}_s(\mathbb{R}^n)$ the space $C^\infty(\mathbb{R}^n)$ whose topology is generated by the countable family of seminorms $[\cdot]_\alpha$, and by $\mathscr{S}_s'(\mathbb{R}^n)$ its topological dual. We have the following inclusions

$$C_0^\infty(\mathbb{R}^n) \hookrightarrow \mathscr{S}(\mathbb{R}^n) \hookrightarrow \mathscr{S}_s(\mathbb{R}^n) \hookrightarrow C^\infty(\mathbb{R}^n),$$

as well as the dual inclusions

$$\mathscr{E}'(\mathbb{R}^n) \hookrightarrow \mathscr{S}_s'(\mathbb{R}^n) \hookrightarrow \mathscr{S}'(\mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n),$$

with $\mathscr{E}'(\mathbb{R}^n)$ denoting the space of distributions with compact support. To justify the introduction of the new space $\mathscr{S}_s(\mathbb{R}^n)$, we have the following proposition:

Proposition 2.1.1. *Let* $u \in \mathscr{S}(\mathbb{R}^n)$. *Then* $(-\Delta)^s u \in \mathscr{S}_s(\mathbb{R}^n)$.

Proposition 2.1.1 follows from Proposition 1.2.1, Proposition 1.3.3, and Corollary 1.3.1 by induction on the order of α, where α is a multiindex. With this fact in mind, we may extend the notion of a solution to the distributional sense.

Definition 2.1.1 (Distributional Solution)**.** Let $F \in \mathscr{S}'(\mathbb{R}^n)$. We say that a distribution $u \in \mathscr{S}_s'(\mathbb{R}^n)$ solves $(-\Delta)^s u = F$ if for every test function $\phi \in \mathscr{S}(\mathbb{R}^n)$ one has

$$\langle u, (-\Delta)^s \phi \rangle = \langle F, \phi \rangle.$$

In the special case for which F is the Dirac delta, Definition 2.1.1 yields the following:

Definition 2.1.2 (Fundamental Solution)**.** We say that a distribution $\Phi_s \in \mathscr{S}'_s(\mathbb{R}^n)$ is a fundamental solution of $(-\Delta)^s$ if $(-\Delta)^s\Phi_s = \delta$. This means that, for every $\phi \in \mathscr{S}(\mathbb{R}^n)$, one has

$$\langle \Phi_s, (-\Delta)^s\phi\rangle \equiv \phi(0).$$

From Definition 2.1.2, it is evident that if $\Phi_s \in \mathscr{S}'_s(\mathbb{R}^n)$ is a fundamental solution of $(-\Delta)^s$, then one has $(-\Delta)^s\Phi_s = 0$ in $\mathscr{D}'(\mathbb{R}^n \setminus \{0\})$. In fact, there exists an explicit fundamental solution $\Phi_s \in C^\infty(\mathbb{R}^n \setminus \{0\})$ for $(-\Delta)^s$.

Theorem 2.1.1. *Let $n \geq 2$ and $0 < s < 1$. Denote by*

$$\Psi_s(x) = \alpha(n,s)|x|^{-(n-2s)}, \tag{2.8}$$

where the normalizing constant $\alpha(n,s)$ above is given by

$$\alpha(n,s) = \frac{\Gamma(\frac{n}{2} - s)}{2^{2s}\pi^{\frac{n}{2}}\Gamma(s)}. \tag{2.9}$$

Then Ψ_s is a fundamental solution for $(-\Delta)^s$.

We henceforth refer to Ψ_s as defined in Theorem 2.1.1 as the fundamental solution for $(-\Delta)^s$. The proof of Theorem 2.1.1, which we present shortly, is a result of two technical lemmata.

Lemma 2.1.1. *Suppose that either $n \geq 2$, or $n = 1$ and $0 < s < \frac{1}{2}$. For every $y > 0$, consider the regularized fundamental solution*

$$\Psi_{s,y}(x) = \alpha(n,s)(y^2 + |x|^2)^{-\frac{n-2s}{2}}. \tag{2.10}$$

Then

$$\widehat{\Psi_{s,y}}(\xi) = \frac{y^s}{2^{2s-1}\pi^s\Gamma(s)}|\xi|^{-s}K_s(2\pi y|\xi|), \tag{2.11}$$

where K_s denotes the modified Bessel function of the third kind (see (C.21)*). Moreover, we obtain* *for every* $\xi \neq 0$

$$\widehat{\Psi_s}(\xi) = \lim_{y \to 0^+} \widehat{\Psi_{s,y}}(\xi) = (2\pi|\xi|)^{-2s}. \tag{2.12}$$

Proof. We will prove only (2.11), as (2.12) is a simple computation which follows from (2.11). To establish (2.11), it suffices to show that

$$\langle \widehat{\Psi_{s,y}}, f \rangle = \frac{y^s}{2^{2s-1}\pi^s \Gamma(s)} \int_{\mathbb{R}^n} |\xi|^{-s} K_s(2\pi y|\xi|) f(\xi)\, d\xi \tag{2.13}$$

for every $f \in \mathscr{S}(\mathbb{R}^n)$. Note that

$$\int_0^\infty e^{-tL} t^\alpha \frac{dt}{t} = \frac{\Gamma(\alpha)}{L^\alpha}$$

for any $L > 0$ and $\alpha > 0$. To see this, apply the change of variable $u = tL$ and compare the resultant integral to Euler's gamma function. Set $L = |\xi|^2 + y^2$ and fix $\alpha > 0$. By Fubini's Theorem, we have

$$\begin{aligned}\int_0^\infty t^\alpha \left(\int_{\mathbb{R}^n} e^{-t(|\xi|^2+y^2)} \hat{f}(\xi)\, d\xi \right) \frac{dt}{t} &= \int_{\mathbb{R}^n} \hat{f}(\xi) \left(\int_0^\infty t^\alpha e^{-t(|\xi|^2+y^2)} \frac{dt}{t} \right) d\xi \\ &= \Gamma(\alpha) \int_{\mathbb{R}^n} \hat{f}(\xi) \Big(|\xi|^2 + y^2 \Big)^{-\alpha} d\xi. \end{aligned} \tag{2.14}$$

By assumption, we have $n \geq 2$ or $n = 1$ and $0 < s < \frac{1}{2}$ so that $\frac{n}{2} - s > 0$. Thus, we may take $\alpha = \frac{n}{2} - s$ in (2.14) to find

$$\int_0^\infty t^{\frac{n}{2}-s} \left(\int_{\mathbb{R}^n} e^{-t(|\xi|^2+y^2)} \hat{f}(\xi)\, d\xi \right) \frac{dt}{t} = \Gamma\Big(\frac{n-2s}{2}\Big) \int_{\mathbb{R}^n} \hat{f}(\xi) \Big(|\xi|^2 + y^2 \Big)^{-(\frac{n-2s}{2})} d\xi. \tag{2.15}$$

Then, by the symmetry property of the Fourier transform, we find that for every $f \in \mathbb{R}^n$ and $y > 0$

$$\int_{\mathbb{R}^n} \mathscr{F}_{x \to \xi}\Big(e^{-t(|x|^2+y^2)}\Big) f(\xi)\, d\xi = \int_{\mathbb{R}^n} e^{-t(|\xi|^2+y^2)} \hat{f}(\xi)\, d\xi. \tag{2.16}$$

Multiplying each side of equation (2.16) by $t^{\frac{n}{2}-s}$ and integrating between 0 and ∞ with respect to the dilation invariant measure $\frac{dt}{t}$, we obtain

$$\int_0^\infty t^{\frac{n}{2}-s}\int_{\mathbb{R}^n}\mathscr{F}_{x\to\xi}\big(e^{-t(|x|^2+y^2)}\big)f(\xi)\,d\xi\frac{dt}{t}=\int_0^\infty t^{\frac{n}{2}-s}e^{-y^2t}\int_{\mathbb{R}^n}\widehat{\big(e^{-t|\cdot|^2}\big)}(\xi)f(\xi)\,d\xi\frac{dt}{t}. \quad (2.17)$$

Since

$$\widehat{\big(e^{-t|\cdot|^2}\big)}(\xi)=\frac{\pi^{\frac{n}{2}}}{t^{\frac{n}{2}}}e^{-\pi^2\frac{|\xi|^2}{t}},$$

substituting the above into (2.17) yields

$$\begin{aligned}\int_0^\infty t^{\frac{n}{2}-s}\int_{\mathbb{R}^n}\mathscr{F}_{x\to\xi}\big(e^{-t(|x|^2+y^2)}\big)f(\xi)\,d\xi\frac{dt}{t}&=\pi^{\frac{n}{2}}\int_0^\infty t^{-s}e^{-y^2t}\int_{\mathbb{R}^n}e^{-\pi^2\frac{|\xi|^2}{t}}f(\xi)\,d\xi\frac{dt}{t}\\&=\pi^{\frac{n}{2}}\int_{\mathbb{R}^n}f(\xi)\bigg(\int_0^\infty t^{-s}e^{-y^2t}e^{-\pi^2\frac{|\xi|^2}{t}}\frac{dt}{t}\bigg)\,d\xi.\end{aligned} \quad (2.18)$$

Applying (C.29) with $\nu=-s$, $\beta=\pi^2|\xi|^2$, and $\gamma=y^2$, and recalling that $K_\nu=K_{-\nu}$, we find

$$\int_0^\infty t^{-s}e^{-y^2t}e^{-\pi^2\frac{|\xi|^2}{t}}\frac{dt}{t}=2\Big(\frac{y}{\pi|\xi|}\Big)^sK_s(2\pi y|\xi|). \quad (2.19)$$

Substituting (2.19) into (2.18), we obtain

$$\int_0^\infty t^{\frac{n}{2}-s}\int_{\mathbb{R}^n}\widehat{\big(e^{-t(|\cdot|^2+y^2)}\big)}(\xi)f(\xi)\,d\xi\frac{dt}{t}=2\pi^{\frac{n}{2}-s}y^s\int_{\mathbb{R}^n}|\xi|^{-s}K_s(2\pi y|\xi|)f(\xi)\,d\xi. \quad (2.20)$$

Putting this all together, we see that

$$\alpha(n,s)\int_{\mathbb{R}^n}\hat{f}(\xi)(|\xi|^2+y^2)^{-(\frac{n-2s}{y})}\,d\xi=\alpha(n,s)\frac{2\pi^{\frac{n}{2}-s}y^s}{\Gamma(\frac{n-2s}{2})}\int_{\mathbb{R}^n}|\xi|^{-s}K_s(2\pi y|\xi|)f(\xi)\,d\xi. \quad (2.21)$$

Now, since

$$\alpha(n,s)=\frac{\Gamma(\frac{n}{2}-s)}{2^{2s}\pi^{\frac{n}{2}}\Gamma(s)},$$

may replace $\alpha(n,s)$ on the right-hand side of (2.21) to obtain the expression

$$\alpha(n,s)\int_{\mathbb{R}^n}\hat{f}(\xi)(|\xi|^2+y^2)^{-(\frac{n-2s}{y})}\,d\xi=\frac{y^s}{2^{2s-1}\pi^s\Gamma(s)}\int_{\mathbb{R}^n}|\xi|^{-s}K_s(2\pi y|\xi|)f(\xi)\,d\xi. \tag{2.22}$$

We have thus shown

$$\langle\Psi_{s,y},\hat{f}\rangle=\frac{y^s}{2^{2s-1}\pi^s\Gamma(s)}\int_{\mathbb{R}^n}|\xi|^{-s}K_s(2\pi y|\xi|)f(\xi)\,d\xi. \tag{2.23}$$

Since $\langle\widehat{\Psi_{s,y}},f\rangle=\langle\Psi_{s,y},\hat{f}\rangle$ by definition, we conclude that (2.13) holds, as was to be shown. $\square$

Lemma 2.1.2. *For every $y>0$ the function $\Psi_{s,y}$ satisfies the equation*

$$(-\Delta)^s\Psi_{s,y}(x)=y^{2s}\frac{\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(s)}(y^2+|x|^2)^{-(\frac{n}{2}+s)} \tag{2.24}$$

Proof. Define

$$F_{s,y}(x):=(-\Delta)^s\Psi_{s,y}(x). \tag{2.25}$$

By Proposition 1.2.1 and Lemma 2.1.1, we readily observe that

$$\begin{aligned}\widehat{F_{s,y}}(\xi)&=\widehat{(-\Delta)^s\Psi}_{s,y}(\xi)\\&=(2\pi|\xi|)^{2s}\widehat{\Psi_{s,y}}(\xi)\\&=(2\pi|\xi|)^{2s}\frac{y^s}{2^{2s-1}\pi^s\Gamma(s)}|\xi|^{-s}K_s(2\pi y|\xi|)\\&=\frac{2y^s\pi^s}{\Gamma(s)}|\xi|^sK_s(2\pi y|\xi|).\end{aligned} \tag{2.26}$$

Then, the Fourier-Bessel representation (C.25) implies

$$F_{s,y}(x)=\frac{4y^s\pi^{s+1}}{\Gamma(s)}\frac{1}{|x|^{\frac{n}{2}-1}}\int_0^\infty t^{\frac{n}{2}+s}K_s(2\pi yt)J_{\frac{n}{2}-1}(2\pi|x|t)\,dt. \tag{2.27}$$

Letting $\gamma = -\frac{n}{2} - s$, $\mu = s$, and $\nu = \frac{n}{2} - 1$, we may write the right-hand side of (2.27) in the form

$$\int_0^\infty t^{-\lambda} K_\mu(at) J_\nu(bt)\, dt$$

with $a = 2\pi y$ and $b = 2\pi|x|$. The assumption $\nu - \gamma + 1 > |\mu|$ is then analogous to $n + s > s$ which clearly holds, so we may use (C.28) to obtain

$$\int_0^\infty t^{\frac{n}{2}+s} K_s(2\pi y t) J_{\frac{n}{2}-1}(2\pi|x|t)\, dt = \frac{(2\pi|x|)^{\frac{n}{2}-1}\Gamma(\frac{n}{2}+s)}{2^{-\frac{n}{2}-s+1}(2\pi y)^{n+1}} \cdot F\Big(\frac{n}{2}+s, \frac{n}{2}; \frac{n}{2}; -\frac{|x|^2}{y^2}\Big). \tag{2.28}$$

Since

$$F\Big(\frac{n}{2}+s, \frac{n}{2}; \frac{n}{2}; -\frac{|x|^2}{y^2}\Big) = \Big(1 + \frac{|x|^2}{y^2}\Big)^{-(\frac{n}{2}+s)},$$

the right-hand side of (2.28) is equivalent to

$$\frac{(2\pi|x|)^{\frac{n}{2}-1}\Gamma(\frac{n}{2}+s)}{2^{-\frac{n}{2}-s+1}(2\pi y)^{n+s}}\Big(1 + \frac{|x|^2}{y^2}\Big)^{-(\frac{n}{2}+s)}. \tag{2.29}$$

Replacing in (2.28), we find

$$\begin{aligned} F_{s,y}(x) &= \frac{\Gamma(\frac{n}{2}+s)}{y^n \pi^{\frac{n}{2}}\Gamma(s)}\Big(1 + \frac{|x|^2}{y^2}\Big)^{-(\frac{n}{2}+s)} \\ &= \frac{y^{2s}\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(s)}\Big(y^2 + |x|^2\Big)^{-(\frac{n}{2}+s)}, \end{aligned} \tag{2.30}$$

which is (2.24). □

With Lemma 2.1.1 and Lemma 2.1.2 in our arsenal, Theorem 2.1.1 now follows without too much effort.

Proof of Theorem 2.1.1. We need to establish

$$\int_{\mathbb{R}^n} \Psi_s(x)(-\Delta)^s \phi(x)\, dx = \phi(0) \tag{2.31}$$

for all test functions $\phi \in \mathscr{S}(\mathbb{R}^n)$. The hypothesis $n \geq 2$ implies $0 < s < \frac{n}{2}$. For fixed $y > 0$, consider the regularization $\Psi_{s,y}$ of Ψ_s. By observation, we see that $\Psi_{s,y} \in C^\infty(\mathbb{R}^n)$ for all $y > 0$ and decays at infinity like $|x|^{-(n-2s)}$. Since $\phi \in \mathscr{S}(\mathbb{R}^n)$ implies $(-\Delta)^s\phi \in \mathscr{S}_s(\mathbb{R}^n)$, Lebesgue's dominated convergence theorem gives

$$\int_{\mathbb{R}^n} \Psi_{s,y}(x)(-\Delta)^s\phi(x)\,dx \to \int_{\mathbb{R}^n} \Psi_s(x)(-\Delta)^s\phi(x)\,dx \tag{2.32}$$

as $y \to 0^+$. By the symmetry property,

$$\int_{\mathbb{R}^n} \Psi_{s,y}(x)(-\Delta)^s\phi(x)\,dx = \int_{\mathbb{R}^n} (-\Delta)^s\Psi_{s,y}(x)\phi(x)\,dx. \tag{2.33}$$

Therefore, it suffices to show

$$\int_{\mathbb{R}^n} (-\Delta)^s\Psi_{s,y}(x)\phi(x)\,dx \to \phi(0) \tag{2.34}$$

as $y \to 0^+$. Applying Lemma 2.1.2, we have

$$\int_{\mathbb{R}^n} (-\Delta)^s\Psi_{s,y}(x)\phi(x)\,dx = \frac{\Gamma(\frac{n}{2}+s)}{y^n\pi^{\frac{n}{2}}\Gamma(s)} \int_{\mathbb{R}^n} \left(1+\frac{|x|^2}{y^2}\right)^{-(\frac{n}{2}+s)} \phi(x)\,dx$$

With the change of variable $x' = \frac{x}{y}$ on the right-hand side of the above expression, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} (-\Delta)^s\Psi_{s,y}(x)\phi(x)\,dx &= \frac{\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(s)} \int_{\mathbb{R}^n} \left(1+|x'|^2\right)^{-(\frac{n}{2}+s)} \phi(yx')\,dx' \\ &\to \phi(0)\frac{\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(s)} \int_{\mathbb{R}^n} \left(1+|x'|^2\right)^{-(\frac{n}{2}+s)} dx' \end{aligned} \tag{2.35}$$

as $y \to 0^+$ by the dominated convergence theorem. Then, applying (C.10) with $a = n + 2s$ and $b = 0$, we find that the integral in (2.35) is equal to

$$\frac{\pi^{\frac{n}{2}}\Gamma(s)}{\Gamma(\frac{n}{2}+s)}$$

so that

$$\int_{\mathbb{R}^n} (-\Delta)^s \Psi_{s,y}(x)\phi(x)\, dx \to \phi(0) \tag{2.36}$$

as $y \to 0^+$. Since this yields (2.31), the proof of the theorem is complete. □

Before we derive the Poisson kernel for $(-\Delta)^s$, some remarks are necessary.

Remark 2.1.1. In the proof of Theorem 2.1.1, we used the fact that $n - 2s > 0$. Thus, the proof provided works for the cases $n = 1$ and $0 < s < \frac{1}{2}$, as well as the situation $n \geq 2$. However, we must also consider the following two cases:

- $n = 1$ and $\frac{1}{2} < s < 1$;
- $n = 1$ and $s = \frac{1}{2}$.

In the first case, formulas (2.8) and (2.9) continue to hold while in the second case they must be replaced with the following:

$$\Psi_s(x) = -\frac{1}{\pi} \log |x|.$$

For more on these cases, see [30].

2.2 The Poisson Kernel

We have now reached the heart of this chapter, the derivation of the Poisson kernel for problem (2.7). We follow the proof provided by Garofalo in [30].

Theorem 2.2.1. *Let $f \in \mathscr{S}(\mathbb{R}^n)$. Then the solution u to the extension problem* (2.7) *(likewise,* (2.6)*) is given by*

$$u(x, y) = (P_{s,y} * f)(x) = \int_{\mathbb{R}^n} P_{s,y}(x - z) f(z)\, dz, \tag{2.37}$$

where

$$P_{s,y}(x) = \frac{\Gamma(\frac{n}{2} + s)}{\pi^{\frac{n}{2}}\Gamma(s)} \frac{y^{2s}}{(y^2 + |x|^2)^{\frac{n+2s}{2}}} \tag{2.38}$$

is the Poisson kernel for the extension problem in the half-space $\mathbb{R}^{n+1}_+$.

Proof. Applying the Fourier transform in the variable $x \in \mathbb{R}^n$ to problem (2.7), we obtain the transformed problem

$$\begin{cases} \frac{\partial^2}{\partial y^2}\hat{u}(\xi,y) + \frac{a}{y}\frac{\partial}{\partial y}\hat{u}(\xi,y) - 4\pi^2|\xi|^2\hat{u}(\xi,y) = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \hat{u}(\xi,0) = \hat{f}(\xi) \text{ for } \xi \in \mathbb{R}^n,\ \hat{u}(\xi,y) \to 0 \text{ as } y \to +\infty. \end{cases} \tag{2.39}$$

Fix $\xi \in \mathbb{R}^n \setminus \{0\}$ and write $Y(y) = Y_\xi(y) = \hat{u}(\xi,y)$ to rewrite problem (2.39) as an ODE in the variable $y \in \mathbb{R}^+$ as

$$\begin{cases} y^2Y''(y) + ayY'(y) - 4\pi^2|\xi|^2y^2Y(y) = 0 \text{ for } y \in \mathbb{R}^+, \\ Y(0) = \hat{f}(\xi),\ Y(y) \to 0 \text{ as } y \to +\infty. \end{cases} \tag{2.40}$$

Comparing the ODE in (2.40) with the generalized modified Bessel equation in (C.22) with $\alpha = s$, $\gamma = 1$, $\nu = s$, and $\beta = 2\pi|\xi|$, we see that two linearly independent solutions to (2.40) are given by

$$Y_1(y) = y^sI_s(2\pi|\xi|y) \tag{2.41}$$

and

$$Y_2(y) = y^sK_s(2\pi|\xi|y). \tag{2.42}$$

Thus, for all $\xi \neq 0$ we see that the general solution to problem (2.40) is given by

$$Y(y) = \hat{u}(\xi,y) = Ay^sI_s(2\pi|\xi|y) + By^sK_s(2\pi|\xi|y) \tag{2.43}$$

for constants A and B. In order to satisfy the condition $\hat{u} \to 0$ as $y \to +\infty$, we must have that $A = 0$ by the asymptotic behavior of K_s and I_s at $+\infty$. We thereby conclude that

$$\hat{u}(\xi,y) = By^sK_s(2\pi|\xi|y) \tag{2.44}$$

for some constant B to be determined. Using the asymptotics (C.23) and(C.24), we find that

$$
\begin{aligned}
By^s K_s(2\pi|\xi|y) &\to \frac{B\pi 2^{s-1}(2\pi|\xi|)^{-s}}{\Gamma(1-s)\sin(\pi s)} \\
&= B2^{s-1}\Gamma(s)(2\pi|\xi|)^{-s}
\end{aligned}
\tag{2.45}
$$

as $y \to 0^+$. Applying the initial condition, we see that

$$
B = \frac{(2\pi|\xi|)^s \hat{g}(\xi)}{2^{s-1}\Gamma(s)}.
$$

Substituting the above expression for B into (2.45), we find that the solution to the transformed problem (2.40) is given by

$$
\hat{u}(\xi, y) = \frac{(2\pi|\xi|)^s \hat{g}(\xi)}{2^{s-1}\Gamma(s)} y^s K_s(2\pi|\xi|y). \tag{2.46}
$$

To complete the proof, we must apply the inverse Fourier transform to obtain an expression for the solution in the original variables. From (2.46) and the convolution property of the Fourier transform, it is clear that the solution $u(x, y)$ to the problem (2.7) will be given by (2.37), with $P_s(x, y)$ to be determined. Therefore, to conclude the proof we must prove (2.38). This amounts to showing

$$
\mathscr{F}^{-1}_{\xi\to x}\left(\frac{(2\pi|\xi|)^s \hat{g}(\xi)}{2^{s-1}\Gamma(s)} y^s K_s(2\pi|\xi|y)\right) = \frac{\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(s)} \frac{y^{2s}}{(y^2+|x|^2)^{\frac{n+2s}{2}}}. \tag{2.47}
$$

Since the right-hand side of(2.46) is spherically symmetric in the variable ξ, showing (2.47) is equivalent to establishing

$$
\mathscr{F}_{\xi'\to x}\left(\frac{2\pi^s|\xi'|^s y^s}{\Gamma(s)} K_s(2\pi|\xi'|y)\right) = \frac{\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(s)} \frac{y^{2s}}{(y^2+|x|^2)^{\frac{n+2s}{2}}}. \tag{2.48}
$$

By the Fourier-Bessel representation (C.25), this is the same as showing

$$
\frac{2^s\pi^{s+1}y^s}{|x|^{\frac{n}{2}-1}\Gamma(s)} \int_0^\infty t^{\frac{n}{2}+s} K_s(2\pi yt) J_{\frac{n}{2}-1}(2\pi|x|t)\, dt = \frac{\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(s)} \frac{y^{2s}}{(y^2+|x|^2)^{\frac{n+2s}{2}}}, \tag{2.49}
$$

where J_ν denotes the Bessel function of order ν (see (C.12)). Fortunately, the identity (2.49) follows immediately from (2.28), (2.29), and (2.30). Thus, combining (2.48) and (2.49) we conclude

$$P_{s,y}(x) = \frac{\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(s)} \frac{y^{2s}}{(y^2+|x|^2)^{\frac{n+2s}{2}}}. \tag{2.50}$$

Since we have shown that (2.38) is indeed the correct formula for $P_{s,y}(x)$, the theorem follows. $\square$

Remark 2.2.1. We remark that formula (2.37) continues to hold under less stringent conditions on f. For instance, one may check that u as in (2.37) is well-defined and solves problem (2.7) if f is bounded and $f \in C^2_{loc}(\mathbb{R}^n)$. In fact, (2.37) is the *unique* solution (up to an additive constant) in $C(\overline{\mathbb{R}^{n+1}_+}) \cap L^\infty(\overline{\mathbb{R}^{n+1}_+})$ to problem (2.7) assuming only that $f \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ (see Corollary 3.5 and Remark 3.8 in [9]).

Let us check that the Poisson kernel for the operator L_a in the upper half-space $\mathbb{R}^{n+1}_+$ posseses the properties we would expect our Poisson kernel to have.

After setting $b = 0$ and $a = n + 2s$ in (C.10), from (2.38) we see that

$$\|P_{s,y}(x)\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} P_{s,y}(x)\, dx = 1 \tag{2.51}$$

for every $y > 0$. Furthermore, comparing the expression for the Poisson kernel (2.38) with (2.24) in Lemma 2.1.2, we conclude that we have, in fact, shown

$$P_{s,y}(x) = (-\Delta)^s \Psi_{s,y}(x), \tag{2.52}$$

where $y > 0$ is fixed, the function $\Psi_{s,y}$ being the y-regularization of the fundamental solution of $(-\Delta)^s$. We have therefore proved that

$$\lim_{y\to 0^+} P_{s,y}(x) = \delta(x) \tag{2.53}$$

in $\mathscr{S}'(\mathbb{R}^n)$. Finally, setting $s = \frac{1}{2}$ we have $a = 1 - 2s = 0$, and the extension operator L_a reduces to the standard Laplacian $L_a = \Delta_x + \frac{\partial^2}{\partial y^2}$ in $\mathbb{R}^{n+1}_+$. Thus, from the formula (2.38), we see that

$$P_{\frac{1}{2},y}(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{y}{(y^2 + |x|^2)^{\frac{n+1}{2}}}, \tag{2.54}$$

which is the standard Poisson kernel for Δ in the upper half-space $\mathbb{R}^{n+1}_+$. It follows that formula (2.38) is consistent with the classical theory, and is an appropriate generalization for the Poisson kernel for the upper-half space corresponding to fractional powers of the Laplacian.

2.3 Local realization of $(-\Delta)^s$

So far, we have derived a Poisson kernel for the extension problem (2.7) (likewise, problem (2.6)) from which we may obtain the classical Poisson kernel. What remains to be shown is the relationship between our solution (2.37) of (2.7) and the fractional Laplacian of our boundary function f.

Theorem 2.3.1 (Local Realization of $(-\Delta)^s$)**.** *For u as in* (2.37)*, we have*

$$(-\Delta)^s f(x) = -\frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)} \lim_{y\to 0^+} y^a u_y(x,y). \tag{2.55}$$

Remark 2.3.1. We henceforth write $d_s := \frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)}$.

Proof. By (2.51), we may write

$$u(x,y) = \frac{\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(s)} \int_{\mathbb{R}^n} \frac{y^{2s}}{(y^2 + |x-z|^2)^{\frac{n+2s}{2}}} (f(z) - f(x))\, dz + f(x).$$

Differentiating both sides with respect to y, we find

$$u_y(x,y) = 2s\frac{\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(s)} \int_{\mathbb{R}^n} \frac{y^{-a}}{(y^2 + |x-z|^2)^{\frac{n+2s}{2}}} (f(z) - f(x))\, dz + O(y^2) \text{ as } y \to 0^+.$$

Here we have differentiated under the integral sign using the product rule and the fact that $2s =$

$1-a$. It follows that

$$y^a u_y(x,y) = 2s\frac{\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(s)}\int_{\mathbb{R}^n}\frac{f(z)-f(x)}{(y^2+|x-z|^2)^{\frac{n+2s}{2}}}\,dz + O(y^2). \tag{2.56}$$

Then, appealing to Corollary 1.3.1 and letting $y \to 0^+$, Lebesgue's dominated convergence theorem gives

$$\begin{aligned}\lim_{y\to 0^+} y^a u_y(x,y) &= 2s\frac{\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(s)}P.V.\int_{\mathbb{R}^n}\frac{f(z)-f(x)}{|x-z|^{n+2s}}\,dz \\ &= -2s\frac{\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(s)}\cdot C(n,s)^{-1}(-\Delta)^s f(x). \end{aligned}\tag{2.57}$$

Noting that $C(n,s) = \frac{s2^{2s}\Gamma(\frac{n+2s}{2})}{\pi^{\frac{n}{2}}\Gamma(1-s)}$ for $0 < s < 1$, we obtain the result. $\square$

There are several proofs of Theorem 2.3.1, and Caffarelli and Silvestre presented two in their original paper[1] [13]. We have chosen the above for its simplicity.

Now that we have established a direct link between local and nonlocal problems via the extension problem (2.6), we are ready to focus our attention on a conjecture of Enrico De Giorgi in both the local and nonlocal case.

[1]One similar to above and one via the Fourier transform.

CHAPTER 3: DE GIORGI'S CONJECTURE FOR THE ALLEN-CAHN EQUATION

In the present chapter, we study an extended version of Enrico De Giorgi's conjecture for the Allen-Cahn equation[1] for a more general class of nonlinearities which includes the Allen-Cahn nonlinearity. Much of the background material in this chapter has been adapted from [14], [20], and [50]. After introducing and motivating the problem, we present the proof, as given by Ambrosio and Cabré in [2], for dimension $n = 3$. The techniques used by Ambrosio and Cabré when $n = 3$ generalizes to the case $n = 2$, however, the proof for this case was originally provided by Ghoussoub and Gui in [31]. We will not go into too much detail concerning dimensions $4 \leq n \leq 8$, proved by Savin in [49] under a mild limit assumption, as the proof incorporates techniques from minimal surface theory and is outside the scope of this thesis. Nonetheless, the author feels that this topic constitutes an intriguing subject for future study. We conclude the chapter by discussing current directions. In the chapter that follows, we will consider De Giorgi's conjecture for the nonlocal operator $(-\Delta)^s$ introduced in Chapter 2.

3.1 Background

Of central importance in theory of PDE is the classification of solutions. For example, in an introductory course much time is dedicated to existence and uniqueness of solutions for elliptic, parabolic, and hyperbolic PDE, as well as Liouville-type theorems[2], maximum principles, regularity, and related topics. In the same spirit, De Giorgi's conjecture is a Liouville-type theorem for the nonlinear *Allen-Cahn equation*

$$-\Delta u = u - u^3 \text{ in } \mathbb{R}^n, \tag{3.1}$$

[1] We henceforth refer to the conjecture as De Giorgi's conjecture.

[2] These are theorems which aim to classify solutions to a given PDE due to some boundedness properties that resemble the classic Liouville theorem one studies in a first course in complex analysis.

which is the Euler-Lagrange equation for the *Ginzburg-Landau energy*

$$J_\Omega(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2\right) dx. \tag{3.2}$$

More generally, a Ginzburg-Landau energy is a functional of the form

$$\mathscr{J}_\Omega(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + W(u)\, dx, \tag{3.3}$$

where W is a *double-well potential* having minima at $u = \pm 1$ satisfying the following:

1. $W \in C^2([-1,1])$,
2. $W(-1) = W(1) = 0$,
3. $W > 0$ on $(-1,1)$,
4. $W'(-1) = W'(1) = 0$,
5. $W''(-1) > 0$ and $W''(1) > 0$.

Comparing with (3.2), one easily verifies that $W(u) := \frac{1}{4}(1-u^2)^2$ has minima at $u = \pm 1$ and satisfies the above criteria so that W is a double-well potential and (3.2) is, in fact, a Ginzburg-Landau energy.

To motivate Conjecture 3.2.1 below, we turn to the study of equation (3.1) in the context of phase transitions (mathematical physics). Consider a pure body contained in a bounded region of space Ω, in which the state (thermodynamic, say) may change from one to another. To each phase, we may assign the value $u = -1$ or $u = 1$, while the transient state of the body is assigned a value $u \in (-1,1)$. We are interested in the description of the interface between these two states. In fact, such an interface should be close to a *minimal surface*, that is, a surface that locally minimizes its area[3]. As a concrete example, one may think of a surface formed after dipping a frame in water and soap solution. Though this physical experiment is easy to perform, the mathematics is very intricate. For example, there may be more than one locally minimizing surface, and they may have non-trivial topology. Let us formulate our working definition of a minimal surface more precisely.

[3]Aside from phase transitions, the study of minimal surface problems are also important in other areas of mathematical physics, such as general relativity. As this is not the focus of the current project, we refer the reader to [55].

Definition 3.1.1. A surface $M \subset \mathbb{R}^n$ is *minimal* if and only if every point $p \in M$ has a neighborhood with least area relative to its boundary.

For our purposes, it is necessary to consider the variational formulation of Definition 3.1.1. The calculus of variations uses small changes in functions and functionals (i.e. *variations*) to find maxima and minima of functionals. In fact, many problems are simplified in this manner and a perfect example is given by Dirichlet's principle[4] from classical PDE theory.

Definition 3.1.2 (Variational Formulation)**.** A surface $M \subset \mathbb{R}^n$ is said to be *minimal* if and only if it is the critical point of the area functional for all compactly supported variations.

Consider the Lagrangian $L : \mathbb{R}^n \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$

$$L(p, z, x) = \left(1 + |p|^2\right)^{\frac{1}{2}}$$

and define the area functional

$$A[u] = \int_\Omega \left(1 + |\nabla u|^2\right)^{\frac{1}{2}} dx \tag{3.4}$$

giving the area of the graph of the function $u : \Omega \to \mathbb{R}$. The associated Euler-Lagrange equation is given by

$$\operatorname{div}\left(\frac{\nabla u}{\left(1 + |\nabla u|^2\right)^{\frac{1}{2}}}\right) = 0 \text{ in } \Omega. \tag{3.5}$$

Noting that the lefthand side in (3.5) is a constant multiple of the mean curvature of the graph of u, we see that minimal surfaces also have zero mean curvature[5]. In fact, this condition is equivalent to Definition 3.1.1 and Definition 3.1.2. Simple examples of minimal surfaces include hyperplanes, catenoids $\{\cosh^2(x_3) = x_1^2 + x_2^2\}$, and helicoids $\{(r\cos t, r\sin t, t) : r > 0,\ t \in \mathbb{R}\}$.

Let us return to the phase transition model. In a typical phase transition model, the unknown

[4] Dirichlet's principle asserts that if $u \in \mathcal{A} := \{w \in C^2(\overline{\Omega}) : w = g \text{ on } \partial\Omega\}$ solves $-\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$, then u minimizes the energy

$$I(w) := \int_\Omega \frac{1}{2}|\nabla w|^2 - wf\, dx.$$

[5] The mean curvature of a surface, denoted by H, is an extrinsic measure of curvature that locally describes the curvature of an embedded surface in some ambient space.

function $u := u(x)$ represents the density of a two-phase fluid at a point x in a domain $\Omega \subset \mathbb{R}^n$. The double–well potential $W(u(x))$ represents the energy density of the fluid in the domain Ω and has minima at u_1, u_2 with $W(u_1) = W(u_2) = 0$ [6]. In fact, for $u \neq u_1, u_2$, we find $W(u) > 0$. Note that the densities u_1, u_2 represent stable fluid phases.

At this point, it is tempting to conclude that the total energy of the fluid in Ω is given by

$$\int_\Omega W(u(x))\, dx,$$

however, this is not satisfactory since any density function $u(x)$ that takes only the values u_1, u_2 minimizes the energy. Indeed, for such a u, we see that $W(u) \equiv 0$. Moreover, in this scenario it is possible for the stable phases u_1, u_2 to coexist along any complicated interface. The issue is that we ignored the kinetic energy term which takes into account interactions at small scales, such as friction. As such, we introduce the rescaled energy

$$\mathscr{J}_{\epsilon,\Omega}(u) = \frac{\epsilon^2}{2} \int_\Omega |\nabla u|^2\, dx + \int_\Omega W(u)\, dx, \tag{3.6}$$

where $\epsilon > 0$ is taken to be small.

The presence of the gradient squared term in (3.6) coupled with the additional assumption that u belong to an appropriate Sobolev space prevents instantaneous jumps from a region of density u_1 to a region of density u_2. In general, these stable states are only attained asymptotically by u. Moreover, the transition region between the two phases occurs in a thin region of width ϵ. We want to understand this transition at length-scale ϵ. To do so, we must dilate by a factor of $\frac{1}{\epsilon}$ and consider the rescaled density $u_\epsilon(x) := u(\epsilon x)$ which minimizes the energy (3.3) and thereby solves the Euler-Lagrange equation

$$\Delta u = W'(u)$$

in the rescaled domain $\frac{\Omega}{\epsilon}$. Indeed, by a simple change of variable argument, we see that u minimizes (3.6) if and only if u_ϵ minimizes (3.3) allowing us to work with the normalized equation

[6] In our case, we have set $u_1 = 1$ and $u_2 = -1$.

(3.3) in lieu of equation (3.6). In fact, letting $\epsilon \to 0$ in the original domain[7] Ω, the transition region converges to a minimal surface inside Ω which motivated De Giorgi to conjecture that global solutions to the Euler-Lagrange equation above should have similar properties to the seemingly unrelated concept, global minimal surfaces.

3.1.1 Minimal Surfaces and De Giorgi

Before we continue, it is appropriate to make clear De Giorgi's approach. The idea of De Giorgi is to view hypersurfaces in $\mathbb{R}^n$ as boundaries of "nice" subsets of $\mathbb{R}^n$. As one might imagine, the nice sets we are referring to are the measurable subsets of $\mathbb{R}^n$. Let $E \subset \mathbb{R}^n$ be measurable and define the *perimeter of* E (or area of ∂E) in a domain $\Omega \subset \mathbb{R}^n$ as the total variation[8] of the characteristic function of E, χ_E, in Ω. That is,

$$P_\Omega(E) = \int_\Omega |\nabla \chi_E| = \sup_{|g| \leq 1} \left| \int_E \operatorname{div} g \, dx \right|, \tag{3.7}$$

where the supremum appearing in the above expression is taken over all vector fields $g \in C_0^1(\Omega)$ satisfying $|g| \leq 1$. When ∂E is a C^1 hypersurface we see that (3.7) is equivalent to the usual notion of the area of ∂E via Green's theorem. Moreover, with (3.7) at our disposal, we may reformulate our definition of a minimal surface once more.

Definition 3.1.3 (Minimal Perimeter & Minimal Surface)**.** We say that E is a set with *minimal perimeter* in Ω, equivalently, ∂E is a *minimal surface* in Ω if, for every open set $U \subset \Omega$ relatively compact (i.e. having compact closure) in Ω,

$$P_U(E) \leq P_U(F) \tag{3.8}$$

[7] In this case, we write $u := u(\frac{x}{\epsilon})$.

[8] The total variation of an integrable function u defined on an open set $\Omega \subset \mathbb{R}^n$, where $n \geq 2$, is given by

$$V(u, \Omega) := \sup \Big\{ \int_\Omega u(x) \operatorname{div} \phi(x) \, dx : \phi \in C_0^1(\Omega) \text{ and } \|\phi\|_{L^\infty(\Omega)} \leq 1 \Big\}.$$

whenever E and F coincide outside of a compact set included in U.

Now that the background and motivations are established, let us take a short detour to discuss some results that, aside from being of independent interest, will be useful in establishing some key bounds and sharpness of estimates.

3.1.2 A Monotonicity Formula of L. Modica

In 1985, Modica proved in [42] that if $W \geq 0$ in $\mathbb{R}$, then every bounded solution $u \in C^2(\mathbb{R}^n)$ of

$$-\Delta u = f(u) \text{ in } \mathbb{R}^n \tag{3.9}$$

satisfies the gradient bound

$$\frac{1}{2}|\nabla u|^2 \leq W(u) \text{ pointwise in } \mathbb{R}^n. \tag{3.10}$$

Here, $W'(u) = -f(u)$, and we may take W to be a double-well potential, for example. Such a result is now commonly referred to as a *Modica-type estimate*, and they have been useful in establishing *monotonicity formulae* for both the local and nonlocal operators Δ and $(-\Delta)^s$, for instance. In particular, (3.10) says that the kinetic energy of a phase transition system, say, is bounded at every point by the potential energy everywhere in $\mathbb{R}^n$.

Our goal in this section is to present an estimate by Modica for problem (3.9) and use this to establish a monotonicity formula for the associated Ginzburg-Landau energy (3.3), also due to Modica. The Modica estimate and monotonicity formula will be useful in the next chapter to establish a key gradient bound and prove sharpness of an important estimate.

To accomplish this goal, we need to first establish the Pohozaev identity. This result is well known and can be found in the texts of Dupaigne [20] and Evans [21]. The Pohozaev identity will be crucial in the proof of the Modica estimate.

Lemma 3.1.1 (Pohozaev Identity)**.** *Let $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ denote a smooth and bounded domain and let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then,*

$$\int_\Omega \Delta u(x \cdot \nabla u)\, dx = \frac{n-2}{2} \int_\Omega |\nabla u|^2\, dx - \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 x \cdot \nu\, dS + \int_{\partial\Omega} \partial_\nu u \cdot (\nabla u \cdot x)\, dS, \tag{3.11}$$

where ν denotes the outward unit normal to $\partial\Omega$. If, in addition, u is constant on the boundary of Ω, then

$$\int_\Omega \Delta u(x \cdot \nabla u)\, dx = \frac{n-2}{2} \int_\Omega |\nabla u|^2\, dx + \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 x \cdot \nu\, dS. \tag{3.12}$$

Proof. The proof is simply successive applications of the integration by parts formula. We have

$$\begin{aligned}
\int_\Omega \Delta u(x \cdot \nabla u)\, dx &= \sum_{i,j=1}^n \int_\Omega u_{x_i x_i} x_j u_{x_j}\, dx \\
&= \sum_{i,j=1}^n \left(- \int_\Omega u_{x_i} (x_j u_{x_j})_i\, dx + \int_{\partial\Omega} u_{x_i} \nu_i x_j u_{x_j}\, dS \right) \\
&= \sum_{i,j=1}^n \left(- \int_\Omega u_{x_i} \delta_{ij} u_{x_j}\, dx - \int_\Omega u_{x_i} u_{x_i x_j} x_j\, dx + \int_{\partial\Omega} u_{x_i} \nu_i x_j u_{x_j}\, dS \right) \\
&= - \int_\Omega |\nabla u|^2\, dx - \frac{1}{2} \int_\Omega \nabla(|\nabla u|^2) \cdot x\, dx + \int_{\partial\Omega} (\nabla u \cdot \nu)(\nabla u \cdot x)\, dS \\
&= \frac{n-2}{2} \int_\Omega |\nabla u|^2\, dx - \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 x \cdot \nu\, dS + \int_{\partial\Omega} \partial_\nu u (\nabla u \cdot x)\, dS.
\end{aligned}$$

This is (3.11). If, in addition, u is constant on $\partial\Omega$, then $\nabla u = |\nabla u| \nu$ on $\partial\Omega$ so (3.12) follows. $\square$

The proof of the Modica estimate below proceeds in four steps, which we label for clarity. Due to the length of the proof, we leave out some of the details. However, all of the details can easily be reconstructed from the proof given below.

Proposition 3.1.1 (Modica Estimate; see [41])**.** *Let* $n \geq 1$. *Let* $u \in C^3(\mathbb{R}^n)$ *denote a bounded solution to the problem*

$$-\Delta u = f(u) \text{ in } \mathbb{R}^n, \tag{3.13}$$

where $W \in C^2(\mathbb{R}^n)$ *is nonnegative and* $W'(u) = -f(u)$. *Then,*

$$\frac{1}{2}|\nabla u|^2 \leq W(u) \text{ in } \mathbb{R}^n. \tag{3.14}$$

Proof. The idea is to show that the function $P(x) := |\nabla u|^2 - 2W(u)$ is nonpositive, as this will give us (3.14). Since $u \in C^3(\mathbb{R}^n)$ is bounded, we see that P is bounded and, by continuity, $\inf_{\mathbb{R}^n} |\nabla u| = 0$. In particular, given $\delta > 0$, we may assume up to a translation of space that

$$|\nabla u|^2(0) < \delta. \tag{3.15}$$

1. We claim that P satisfies the inequality

$$|\nabla u|^2 \Delta P \geq \frac{1}{2}|\nabla P|^2 - 2f(u)\nabla u \cdot \nabla P \tag{3.16}$$

in $\mathbb{R}^n$. Indeed, for each $i = 1, \ldots, n$, we have

$$P_{x_i} = 2\sum_{j=1}^{n} u_{x_j} u_{x_i x_j} + 2f(u)u_{x_i}. \tag{3.17}$$

From (3.17) and the Cauchy-Schwarz inequality, we see that

$$\begin{aligned} \sum_{i=1}(P_{x_i} - 2f(u)u_{x_i})^2 &= 4\sum_{i=1}^{n}\Big(\sum_{j=1}^{n} u_{x_j} u_{x_i x_j}\Big)^2 \\ &\leq 4\sum_{j=1}^{n} u_{x_j}^2 \sum_{i,j=1}^{n} u_{x_i x_j}^2 \\ &= 4|\nabla u|^2 \|Hu\|_{\text{FR}}^2, \end{aligned} \tag{3.18}$$

where we have denoted Hu the Hessian matrix of u and $\|\cdot\|_{\text{FR}}$ the Frobenius matrix norm. Differentiating (3.17) with respect to x_i once more and summing over i, we find

$$\begin{aligned}\Delta P &= 2\sum_{i,j=1}^{n} u_{x_i x_j}^2 + 2\sum_{j=1}^{n} u_{x_j}(\Delta u)_{x_j} + 2f'(u)|\nabla u|^2 + 2f(u)\Delta u \\ &= 2\|Hu\|_{\text{FR}}^2 - 2[f(u)]^2, \end{aligned}\tag{3.19}$$

with the last equality (3.19) obtained from the fact that u solves (3.13). Whence, from (3.18) and (3.19) we have

$$\begin{aligned}|\nabla u|^2 \Delta P &\geq \frac{1}{2}\sum_{i=1}^{n}(P_{x_i} - 2f(u)u_{x_i})^2 - 2[f(u)]^2|\nabla u|^2 \\ &= \frac{1}{2}|\nabla P|^2 - 2f(u)\nabla u \cdot \nabla P,\end{aligned}$$

which is (3.16).

2. Let $\epsilon > 0$ and $R > 0$. Then, there exists a radial cutoff function $\eta(r) := \eta_{\epsilon,R}(|x|) \in C^2(\mathbb{R}^n)$ having the following properties:

$$\eta(R) = 1,\ \eta > 0,\ \eta' < 0,\ \text{and}\ \lim_{r\to\infty}\eta(r) = 0, \tag{3.20}$$

$$\lim_{\epsilon\to 0^+}\eta(r) = 1 \text{ for all } r \geq R, \tag{3.21}$$

$$\frac{\eta^2}{(\eta')^2}\Big(\frac{2\eta'}{\eta} - \frac{M}{\epsilon}\eta' - \eta'' - \frac{(n-1)\eta'}{r}\Big) \leq \frac{\epsilon}{L} \text{ for all } r \geq R, \tag{3.22}$$

where

$$M = \sup_{\mathbb{R}^n} 2|f(u)||\nabla u| \text{ and } L = \sup_{\mathbb{R}^n} 2|\nabla u|^2. \tag{3.23}$$

Indeed, set

$$g_\epsilon(t) = \int_t^1 \frac{e^{-\frac{\epsilon}{Ls}}}{s^2}\,ds \text{ for } 0 \leq t \leq 1$$

and

$$h_{\epsilon,R}(t) = \int_R^t \frac{e^{-(\frac{M}{\epsilon})s}}{s^{n-1}}\, ds \text{ for } t \geq R.$$

Take

$$\eta(r) = g_\epsilon^{-1}(c \cdot h_{\epsilon,R}(r)) \text{ for } r \geq R,$$

where

$$c = \frac{g_\epsilon(0)}{h_{\epsilon,R}(+\infty)}.$$

Then (3.20) follows immediately. For (3.21), it suffices to note that $h_{\epsilon,R} \to 0$ as $\epsilon \to 0$ and $g_\epsilon^{-1}(0^+) = 1$. For (3.22), differentiate, take the log, and differentiate again with respect to r the equality

$$\int_\eta^1 \frac{e^{-\frac{\epsilon}{Ls}}}{s^2}\, ds = \int_R^r \frac{e^{-(\frac{M}{\epsilon})s}}{s^{n-1}}\, ds.$$

3. Set $v = \eta P$. We want to show that

$$v(x) \leq \max\{\epsilon, \max_{|x|=R} P\}, \tag{3.24}$$

for all $|x| \geq R$. It is clear that (3.24) holds if $\sup_{|x|\geq R} v \leq 0$, so we may assume that v is positive somewhere. Since P is bounded and $\lim_{r\to\infty} \eta = 0$, we deduce that $\lim_{|x|\to\infty} v = 0$. Thus, either v achieves its maximum on $|x| = R$, from which (3.24) follows, or v achieves its maximum at some point x_0 with $|x_0| > R$. Then, at x_0 we have $0 = \nabla v = \eta\nabla P + P\nabla\eta$, from which we find $\nabla P = -P\frac{\nabla\eta}{\eta}$. Using this and (3.16), we obtain the estimate

$$|\nabla u|^2 \Delta v \geq \left(|\nabla u|^2\Delta\eta - 2\frac{|\nabla u|^2|\nabla\eta|^2}{\eta} + 2f(u)\nabla u\cdot\nabla\eta\right)P + \frac{P^2|\nabla\eta|^2}{2\eta}. \tag{3.25}$$

Since x_0 is an interior point at which v attains its maximum, we have $\Delta v(x_0) \leq 0$. Furthermore, we see that $P(x_0) > 0$, since $\eta(x_0) > 0$ and $v(x_0) > 0$. Therefore, at x_0

$$\frac{P|\nabla u|^2}{2\eta} \leq 2\frac{|\nabla u|^2|\nabla\eta|^2}{\eta} - 2f(u)\nabla u\nabla\eta - |\nabla u|^2\Delta\eta. \tag{3.26}$$

Now, if $|\nabla u|^2(x_0) \leq \epsilon$, then, since $\eta \leq 1$ and $W \geq 0$, we have $v(x) \leq v(x_0) \leq P(x_0) \leq |\nabla u|^2(x_0) \leq \epsilon$ for $|x| \geq R$ and (3.24) holds. Otherwise, since $\eta' < 0$, (3.23) and (3.26) imply that, at x_0,

$$v = \eta P \leq L \frac{\eta^2}{|\nabla \eta|^2} \left(\frac{2|\nabla \eta|^2}{\eta} + \frac{M|\nabla \eta|}{\epsilon} - \Delta \eta \right). \tag{3.27}$$

From (3.22) and (3.27), we conclude that $v(x_0) \leq \epsilon$ proving (3.24).

4. Let $\epsilon \to 0^+$ in (3.24). Then

$$P(x) \leq \max\{0, \max_{|x|=R} P\}$$

for $|x| \geq R$. Now, letting $R \to 0^+$ and using (3.15), we see that

$$P(x) \leq \max\{0, P(0)\} < \delta$$

for all $x \in \mathbb{R}^n$. Since $\delta > 0$ is arbitrary, we find that $P \leq 0$, as desired.

This concludes the proof. □

The proof of the monotonicity formula now follows from a simple differentiation.

Theorem 3.1.1 (Monotonicity Formula; see [43])**.** *Let* $n \geq 1$. *Let* $u \in C^3(\mathbb{R}^n)$ *denote a bounded solution to the problem* (3.13)*. Then,*

$$I(R) = R^{1-n} \mathscr{J}_{B_R}(u), \tag{3.28}$$

is a nondecreasing function of R*, where* $\mathscr{J}_{B_R}(u)$ *is defined by the Ginzburg-Landau energy* (3.3)*.*

Proof. By Lemma 3.1.1 (Pohozaev Identity), we have

$$\int_{B_R} \Delta u (x \cdot \nabla u)\, dx = \frac{n-2}{2} \int_{B_R} |\nabla u|^2\, dx - \frac{R}{2} \int_{\partial B_R} |\nabla u|^2\, dS^{n-1} + r \int_{\partial B_R} u_r^2\, dS^{n-1}. \tag{3.29}$$

Since u solves equation (3.13), we also have

$$\begin{aligned}\int_{B_R} \Delta u(x\cdot\nabla u)\,dx &= -\int_{B_R} W'(u)(x\cdot\nabla u)\,dx\\ &= -n\int_{B_R} W(u)\,dx + R\int_{\partial B_R} W(u)\,dS^{n-1}\end{aligned} \tag{3.30}$$

after integrating by parts, where $W(u)$ is as above. Combining (3.29) and (3.30), we obtain

$$\int_{B_R}\Big((n-2)|\nabla u|^2 + 2nW(u)\Big)\,dx = R\int_{\partial B_R}\Big(|\nabla u|^2+2W(u)\Big)\,dS^{n-1} - 2R\int_{\partial B_R} u_r^2\,dS^{n-1}. \tag{3.31}$$

Upon differentiation of I given by (3.28), we find

$$2I'(R) = -(n-1)R^{-n}\int_{B_R}\Big(|\nabla u|^2+2W(u)\Big)\,dx + R^{1-n}\int_{\partial B_R}\Big(|\nabla u|^2+2W(u)\Big)\,dS^{n-1}.$$

Then, from (3.31) we deduce

$$2R^nI'(R) = \int_{B_R}\Big(2W(u)-|\nabla u|^2\Big)\,dx + 2R\int_{\partial B_R} u_r^2\,dS^{n-1}. \tag{3.32}$$

Applying Proposition 3.1.1 (Modica Estimate) to (3.32), we obtain the result. $\square$

After closer inspection, we see that the assumption that $u\in C^3(\mathbb{R}^n)$ is superflous. Indeed, the computations above show we only require that $u\in C^2(\mathbb{R}^n)$ and $f\in C^1(\mathbb{R}^n)$. Since $u\in C^2(\mathbb{R}^n)$ satisfies (3.9) and $f\in C^1(\mathbb{R})$, we obtain the result due to the fact that the partials $(\Delta u)_{x_i} = f'(u)u_{x_i}$ exist and are continuous for each $i=1\ldots,n$. Thus, the claim at the beginning of this section that the Modica estimate holds for all bounded solutions of (3.9) is valid allowing us to apply the estimate and resulting monotonicity formula freely in the sequel. For a brief overview of monotonicity formulae and their applications, the interested reader should see [22].

3.2 The Conjecture

De Giorgi's conjecture aims to classify bounded solutions to the Allen-Cahn equation (3.1) and has close connections with the theory of minimal surfaces. The conjecture, proposed in 1978 by Enrico De Giorgi, is as follows:

Conjecture 3.2.1 (De Giorgi's Conjecture)**.** *Let $n \geq 1$ and let $u \in C^2(\mathbb{R}^n)$ be a bounded solution of* (3.1)*. Further suppose*

$$u_{x_n} > 0 \textit{ in all of } \mathbb{R}^n. \tag{3.33}$$

Then the level sets of u are hyperplanes, at least if $n \leq 8$.

Remark 3.2.1. The level sets of u are given by $\{x \in \mathbb{R}^n : u(x) = s\}$, where $s \in \mathbb{R}$ is a fixed constant. Thus, De Giorgi's conjecture states that the x-values mapped to s by u form hyperplanes. Moreover, they must be parallel since no two level sets can cross. It follows that the level sets share a common normal, say τ. Supposing τ is of unit length, we see that for each $x \in \mathbb{R}^n$ u depends solely on the projection of x along τ, $x \cdot \tau$. It follows that Conjecture 3.2.1 is equivalent to the requirement that u is one-dimensional, that is, u is a function of one variable.

Proposition 3.2.1. *Let $n \geq 1$ and $u \in C^2(\mathbb{R}^n; [-1, 1])$. Assume that u is a one-dimensional solution to* (3.1)*. Then, there exists a unit vector $\tau \in \mathbb{R}^n$ with $\tau_n > 0$ and a constant $c \in \mathbb{R}$ such that*

$$u(x) = g_0(\tau \cdot x + c) \tag{3.34}$$

for all $x \in \mathbb{R}^n$. Here, g_0 is given by the formula

$$g_0(s) = \tanh\left(\frac{s}{\sqrt{2}}\right). \tag{3.35}$$

As justification for Proposition 3.2.1, we observe that the function $u(s) = \tanh\left(\frac{s}{\sqrt{2}}\right)$ is the unique solution of the ODE problem

$$\begin{cases} u'' + u - u^3 = 0, \\ u(0) = 0,\ u(\pm\infty) = \pm 1, \end{cases}$$

which increases from -1 to 1. We also note that it is common to see De Giorgi's conjecture presented with the additional assumption

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1. \tag{3.36}$$

This additional hypothesis is natural when viewed from the phase transition viewpoint discussed earlier, as we can view the solutions as passing through "layers" of intermediary states in the x_n direction. Accordingly, it is now customary to call solutions u satisfying (3.36) *layer solutions*. Layer solutions will be of primary interest in this project.

The first positive result was given by Ghoussoub and Gui in [31] for dimension $n = 2$. Ambrosio and Cabré subsequently proved the conjecture for dimension $n = 3$ (see [2]). In 2009, Savin showed in [49] that, for dimensions $4 \leq n \leq 8$, the conjecture is true under the additional hypothesis (3.36). Finally del Pino et al. constructed counterexamples for dimensions $n \geq 9$ in [47].

Note that the threshold dimension is $n = 8$. Interestingly, this has a deep connection to the theory of minimal surfaces and strikingly resembles the *Bernstein Conjecture* for minimal graphs. By minimal graphs, we mean graphs

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = F(x_1, \ldots, x_{n-1})\}$$

with vanishing mean curvature H in $\mathbb{R}^{n-1}$.

By direct computation, one may verify that any hyperplane defined by

$$F(x_1, \ldots, x_{n-1}) = \sum_{j=1}^{n-1} c_j x_j + c$$

has zero mean curvature. Bernstein conjectured much more.

Conjecture 3.2.2 (Bernstein Conjecture)**.** *All entire minimal graphs are hyperplanes. Namely, any entire solution of* (3.5) *must be an affine function.*

The Bernstein conjecture turned out to be true in dimensions $n \leq 8$, so that the parallel between the Bernstein conjecture and De Giorgi's conjecture is immediate. In particular, an intimate relationship between De Giorgi's conjecture and the theory of minimal surfaces was elucidated.

3.2.1 Dimensions $n = 2$ and $n = 3$

Here, we present the results of Ambrosio and Cabré in [2] for dimension $n = 3$ for layer solutions of the problem (3.37) below, whose work was based on that of Ghoussoub and Gui (see [31]) for dimension $n = 2$. Since the same techniques used by Ambrosio and Cabré can be applied to the case when $n = 2$, we will present only the work of Ambrosio and Cabré.

In [2], the authors studied bounded solutions of semilinear elliptic equations

$$-\Delta u = f(u) \text{ in } \mathbb{R}^n, \tag{3.37}$$

under the assumption that u is monotone in one direction, say $u_{x_n} > 0$ in $\mathbb{R}^n$. Here, we have assumed that $F \in C^2(\mathbb{R})$ satisfies $F'(u) = -f(u)$. We further note that nonlinearities of this form include the Allen-Cahn nonlinearity introduced prior. As in the case of the Allen-Cahn equation, the goal was to establish that u depends only on one variable, or, equivalently, that the level sets of u are hyperplanes.

The first positive result regarding De Giorgi came in 1998 by Ghoussoub and Gui in [31]. They too considered the more general nonlinearity f under the hypotheses that u is a bounded solution of (3.37) with $F \in C^2(\mathbb{R})$. Specifically, they showed that when the dimension $n = 2$, then u is a function of one variable only.

In [2], Ambrosio and Cabré obtained similar results in dimension $n = 3$ by generalizing the methods of Ghoussoub and Gui in [31]. They began by first examining the simpler case when the solution satisfies the limit assumption (3.36). Throughout this section, we will also assume the limit (3.36). Generalizations of the below results will then be discussed before switching to the nonlocal case.

Theorem 3.2.1 (Theorem 1.1 in [2])**.** *Let u be a bounded solution of*

$$-\Delta u = f \textit{ in } \mathbb{R}^3 \tag{3.38}$$

satisfying

$$u_{x_3} > 0 \textit{ in } \mathbb{R}^3 \textit{ and } \lim_{x_3 \to \pm\infty} u(x', x_3) = \pm 1 \textit{ for all } x' \in \mathbb{R}^2. \tag{3.39}$$

Assume that $F \in \mathbb{C}^2(\mathbb{R})$ satisfies $F'(u) = -f(u)$ and that

$$F \geq \min\{F(-1), F(1)\} \textit{ in } (-1, 1). \tag{3.40}$$

Then the level sets of u are planes. In particular, there exists a unit vector $\tau \in \mathbb{R}^3$ and $g \in C^2(\mathbb{R})$ such that

$$u(x) = g(\tau \cdot x) \textit{ for all } x \in \mathbb{R}^3. \tag{3.41}$$

Moreover, $\nabla u \cdot \tau > 0$ in $\mathbb{R}^3$.

Remark 3.2.2. Note that the direction variable τ on which u depends is not known a priori. However, if we instead assume that the limits in (3.39) are uniform in $x' \in \mathbb{R}^{n-1}$, then we are imposing an a priori choice of the direction τ given by $\tau \cdot x = x_n$. Under this assumption, it has been shown that for every dimension n, that u depends only on the variable x_n (see [4,6,31]) and we may write $u = u(x_n)$. This result applies to equation (3.38) for various classes of nonlinearities f which always include the Ginzburg-Landau model considered earlier.

To establish Theorem 3.2.1, a significant amount of work needs to be done. We will first establish some preliminary results and then use them to tackle the theorem.

Recall from the previous section the Modica estimate. That is, if $F \geq 0$ in $\mathbb{R}$, then every bounded solution u of $-\Delta u = f(u)$ in $\mathbb{R}^n$ satisfies the gradient bound

$$\frac{1}{2}|\nabla u|^2 \leq F(u) \text{ in } \mathbb{R}^n. \tag{3.42}$$

As an example of their uitility, we mention the work of Caffarelli, Garofalo, and Segala in [11]. Here, they not only established this bound to more general equations, but also showed that if equality occurs in (3.42) for some $x \in \mathbb{R}^n$, then the conclusion of De Giorgi's Conjecture is true.

The proof of De Giorgi's conjecture in dimension $n = 3$ proceeds as the proof given by Ghoussoub and Gui for dimension $n = 2$. Specifically, Ambrosio and Cabré show that for every coordinate x_i, the function $\sigma_i := \frac{u_{x_i}}{u_{x_n}}$ is constant. This is done using the following Liouville-type theorem for the equation $\operatorname{div}(\varphi^2 \nabla \sigma_i) = 0$, where $\varphi = u_{x_n}$:

Proposition 3.2.2 (Liouville Theorem; Proposition 2.1 in [2])**.** *Let $\varphi \in L^\infty_{loc}(\mathbb{R}^n)$ be a positive function. Suppose that $\sigma \in H^1_{loc}(\mathbb{R}^n)$ satisfies*

$$\sigma \operatorname{div}(\varphi^2 \nabla \sigma) \geq 0 \textit{ in } \mathbb{R}^n \tag{3.43}$$

in the distributional sense. For every $R > 1$, assume that

$$\int_{B_R} (\varphi\sigma)^2 \, dx \leq CR^2, \tag{3.44}$$

for some constant C independent of R. Then σ is constant.

The following energy estimate will allow us to apply Proposition 3.2.2 when $n = 3$ to conclude that σ_i is constant for each $i \in \{1, 2\}$.

Proposition 3.2.3 (Energy Estimate; Theorem 1.3 in [2]). *Let u be a bounded solution of*

$$-\Delta u = f(u) \text{ in } \mathbb{R}^n,$$

where $F \in C^2(\mathbb{R})$ satisfies $F'(u) = -f(u)$. Assume that

$$u_{x_n} > 0 \text{ in } \mathbb{R}^n \text{ and } \lim_{x_n \to +\infty} u(x', x_n) = 1 \text{ for all } x' \in \mathbb{R}^{n-1}. \tag{3.45}$$

Then, for every $R > 1$,

$$\int_{B_R} \left(\frac{1}{2} |\nabla u|^2 + F(u) - F(1) \right) dx \leq C R^{n-1} \tag{3.46}$$

for some constant C independent of R.

Before proving the propositions and theorem, we make some remarks and establish some simple bounds and regularity results for bounded solutions $u \in C^2(\mathbb{R}^n)$ of (3.37). We then prove the propositions, and the theorem will follow.

It is important to note that the energy functional in B_R,

$$J_R(u) = \int_{B_R} \left(\frac{1}{2} |\nabla u|^2 + F(u) - F(1) \right) dx, \tag{3.47}$$

has $-\Delta u = f(u)$ as the Euler-Lagrange equation, which can be checked by a simple differentiation and integration by parts. Recall from the previous section that Modica proved in [43] a monotonicity formula for the Ginzburg-Landau energy (3.46). Since $F \geq F(1)$ in $\mathbb{R}$ and u is a bounded solution of $\Delta u + F'(u) = 0$, then the quantity

$$I(R) = \frac{J_R(u)}{R^{n-1}} \tag{3.48}$$

is a nondecreasing function of R. In particular, the monotonicity formula shows that the upper bound in Proposition 3.2.3 is optimal. To prove this, we will show that $I(R)$ is bounded below by a positive constant. Note that if $I(R) \to 0$ as $R \to \infty$, then $J_R(u) \equiv 0$ by the monotonicity of $I(R)$. We claim that u is constant. Indeed, since $J_R(u) = 0$ for all $R > 0$, we have

$$\int_{B_R} \Big(\frac{1}{2}|\nabla u|^2 + F(u) - F(1)\Big)\, dx = 0.$$

Since the integrand is nonnegative, we find

$$\frac{1}{2}|\nabla u|^2 + F(u) - F(1) = 0$$

which implies

$$\frac{1}{2}|\nabla u|^2 = F(1) - F(0) \leq 0.$$

We conclude that $|\nabla u| = 0$ and u is constant, a contradiction.

The following estimates will be key. Assuming only that $u \in C^2(\mathbb{R}^n)$ is a bounded solution of $-\Delta u = f(u)$ and $F \in C^2(\mathbb{R})$ satisfies $F'(u) = -f(u)$, we may apply Proposition 3.1.1 (Modica Estimate) to see that

$$|\nabla u| \in L^\infty(\mathbb{R}^n).$$

Next, we show that $u \in W^{3,p}_{loc}(\mathbb{R}^n)$ for all $1 \leq p < \infty$. Since F' is of class C^1 and u and ∇u are bounded, we find that $F'(u) \in W^{1,p}_{loc}(\mathbb{R}^n)$ and $\nabla F'(u) = F''(u)\nabla u$. It follows that

$$\Delta u_{x_j} - F''(u) u_{x_j} = 0 \text{ in the weak sense in } \mathbb{R}^n,$$

and this holds for any index j. Since $F''(u)u_{x_j} \in L^\infty(\mathbb{R}^n) \subset L^p_{loc}(\mathbb{R}^n)$, we conclude that $u_{x_j} \in W^{2,p}_{loc}(\mathbb{R}^n)$ for each index j implying $u \in W^{3,p}_{loc}(\mathbb{R}^n)$. In particular, Sobolev embedding gives $u \in C^{2,\alpha}(\mathbb{R}^n)$ for $\alpha \in (0,1)$. Note that we may obtain the same conclusion assuming u is only a bounded distributional solution of (3.37) by applying interior $W^{2,p}$ estimates to $\Delta u = F'(u) \in L^\infty(B_2(y))$ for every $y \in \mathbb{R}^n$.

Let us summarize the key points from above. For a bounded (distributional) solution u of

(3.37), the following hold:

1. $u \in C^{2,\alpha}(\mathbb{R}^n)$ for $\alpha \in (0,1)$,

2. $|\nabla u| \in L^\infty(\mathbb{R}^n)$, and

3. For each $i \in \{1,2,\dots,n\}$, the partials u_{x_i} satisfy the linearized equation

$$\Delta w - F''(u)w = 0 \text{ in the weak sense in } \mathbb{R}^n.$$

We are ready to prove the propositions. We will begin with Proposition 3.2.2. The proof of Proposition 3.2.3 will follow and, finally, the theorem. Throughout the proofs, we will use C to denote different positive constants. The constant C will always be independent of R.

Proof of Proposition 3.2.2. Let $\zeta := \zeta(t)$ be a C^∞ function on $\mathbb{R}^+$ with bounded derivative such that $0 \le \zeta \le 1$. Further suppose that

$$\zeta(t) = \begin{cases} 1, \ 0 \le t \le 1 \\ 0, \ t \ge 2. \end{cases}$$

For $R > 1$, set

$$\zeta_R(x) := \zeta\Big(\frac{|x|}{R}\Big) \text{ for } x \in \mathbb{R}^n.$$

Multiply (3.43) by $\zeta_R^2(x)$ on both sides and integrate by parts in $\mathbb{R}^n$ to find

$$\begin{aligned}\int_{\mathbb{R}^n} \zeta_R^2 \varphi^2 |\nabla\sigma|^2 \, dx &\le -2\int_{\mathbb{R}^n} \zeta_R \varphi^2 \sigma \nabla\zeta_R \cdot \nabla\sigma \, dx \\ &\le 2\bigg(\int_{R<|x|<2R} \zeta_R^2 \varphi^2 |\nabla\sigma|^2 \, dx\bigg)^{\frac{1}{2}} \bigg(\int_{\mathbb{R}^n} \varphi^2\sigma^2 |\nabla\zeta_R|^2 \, dx\bigg)^{\frac{1}{2}}\end{aligned}$$

by the Cauchy-Schwarz inequality. Note that

$$\begin{aligned}\nabla\zeta_R(x) &= \nabla\zeta\Big(\frac{|x|}{R}\Big)\\ &= \zeta'\Big(\frac{|x|}{R}\Big)\cdot\nabla\Big(\frac{|x|}{R}\Big)\\ &= \frac{1}{R}\zeta'\Big(\frac{|x|}{R}\Big)\cdot\frac{x}{|x|},\end{aligned}$$

so that

$$|\nabla\zeta_R(x)|^2 \le \frac{|\zeta'\big(\frac{|x|}{R}\big)|^2}{R^2} \le \frac{C}{R^2}$$

where C is constant. Hence,

$$\begin{aligned}&\bigg(\int_{R<|x|<2R}\zeta_R^2\varphi^2|\nabla\sigma|^2\,dx\bigg)^{\frac{1}{2}}\bigg(\int_{\mathbb{R}^n}\varphi^2\sigma^2|\nabla\zeta_R|^2\,dx\bigg)^{\frac{1}{2}}\\ &\qquad\qquad \le C\bigg(\int_{R<|x|<2R}\zeta_R^2\varphi^2|\nabla\sigma|^2\,dx\bigg)^{\frac{1}{2}}\bigg(\frac{1}{R^2}\int_{B_{2R}}(\varphi\sigma)^2\,dx\bigg)^{\frac{1}{2}}\end{aligned}$$

for some constant C. We note that the second integral in the product on the right-hand side can be truncated to the integral over the ball B_{2R} since, for any $R>0$, ζ_R vanishes for $|x|>2R$ along with its gradient. Applying the hypothesis (3.45), we find

$$\begin{aligned}\int_{\mathbb{R}^n}\zeta_R^2\varphi^2|\nabla\sigma|^2\,dx &\le C\bigg(\int_{R<|x|<2R}\zeta_R^2\varphi^2|\nabla\sigma|^2\,dx\bigg)^{\frac{1}{2}}\\ &\le C\bigg(\int_{\mathbb{R}^n}\zeta_R^2\varphi^2|\nabla\sigma|^2\,dx\bigg)^{\frac{1}{2}}\end{aligned} \tag{3.49}$$

for some constant C. It follows that

$$\bigg(\int_{\mathbb{R}^n}\zeta_R^2\varphi^2|\nabla\sigma|^2\,dx\bigg)^{\frac{1}{2}} \le C$$

so that the integral on the right-hand side of (3.49) tends to zero as R increases to infinity. This gives that

$$\int_{\mathbb{R}^n} \varphi^2 |\nabla\sigma|^2 \, dx = 0.$$

Since φ is strictly positive, we conclude that $\nabla\sigma = 0$ so that σ is constant. $\square$

The proof of Proposition 3.2.3 is fundamental and will be discussed in more detail below. Next, we prove the energy estimate.

Proof of Proposition 3.2.3. Consider the functions

$$u^t(x) := u(x', x_n + t)$$

defined for $x = (x', x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then for each $t \in \mathbb{R}$ we see that u^t satisfies

$$\Delta u^t - F'(u^t) = 0. \tag{3.50}$$

Note also that

$$\lim_{t\to+\infty} u^t(x) = 1 \text{ for all } x \in \mathbb{R}^n. \tag{3.51}$$

Denoting the derivative of $u^t(x)$ with respect to t by $\partial_t u^t(x)$, we have

$$\partial_t u^t(x) = u_{x_n}(x', x_n + t) > 0 \text{ for all } x \in \mathbb{R}^n. \tag{3.52}$$

Consider the energy of u^t in the ball $B_R = B_R(0)$ defined by

$$J_R(u^t) = \int_{B_R} \Big(\frac{1}{2}|\nabla u^t|^2 + F(u^t) - F(1)\Big)\, dx.$$

We claim that

$$\lim_{t\to+\infty} J_R(u^t) = 0. \tag{3.53}$$

By (3.51), a simple application of the dominated convergence theorem shows that the term

$$\int_{B_R} \Big(F(u^t) - F(1)\Big)\, dx$$

tends to zero at $t \to +\infty$. To show that the gradient term tends to zero also, we multiply (3.50) by $u^t - 1$ and integrate by parts in B_R. Doing so, we find that

$$\int_{B_R} |\nabla u^t|^2\, dx = \int_{\partial B_R} \partial_\nu u^t(u^t - 1)\, dS^{n-1} - \int_{B_R} F'(u^t)(u^t - 1)\, dx.$$

Due to the L^∞ bounds established prior, we may apply the dominated convergence theorem to find that the two integrals on the right-hand side converge to zero as $t \to +\infty$. We conclude that (3.53) holds.

Next, we compute and bound the derivative of $J_R(u^t)$ with respect to t. Before doing so, we point out that the L^∞ bounds derived for solutions u of (3.37) apply to the functions u^t by (3.50). Observe that

$$\partial_t J_R(u^t) = \int_{B_R} \nabla u^t \cdot \nabla(\partial_t u^t)\, dx + \int_{B_R} F'(u^t)\partial_t u^t\, dx.$$

Since $F'(u^t) = \Delta u^t$, we may integrate by parts in the second term in the sum to find

$$\partial_t J_R(u^t) = \int_{\partial B_R} \partial_\nu u^t \partial_t u^t\, dS^{n-1}$$

after cancellations. Using the L^∞ bound for ∇u^t and (3.52), we find

$$\partial_t J_R(u^t) \geq -C \int_{\partial B_R} \partial_t u^t\, dS^{n-1} \tag{3.54}$$

for some constant C independent of R. Thus, for all $T > 0$ we have

$$\begin{aligned} J_R(u) &= J_R(u^T) - \int_0^T \partial_t J_R(u^t)\,dt \\ \text{(By (3.54))} \quad &\leq J_R(u^T) + C\int_0^T \left(\int_{\partial B_R} \partial_t u^t(x)\,dS^{n-1} \right) dt \\ &= J_R(u^t) + C\int_{\partial B_R} \left(\int_0^T \partial_t [u^t(x)]\,dt \right) dS^{n-1} \\ &= J_R(u^T) + C\int_{\partial B_R} (u^T - u)(x)\,dS^{n-1} \\ \text{(By the } L^\infty \text{ bounds)} \quad &\leq J_R(u^T) + C|\partial B_R| \\ &= J_R(u^T) + CR^{n-1}. \end{aligned}$$

Letting $T \to +\infty$ and appealing to (3.53), we conclude that

$$J_R(u) \leq CR^{n-1},$$

as desired. □

Remark 3.2.3. We can actually relax the condition in Proposition 3.2.2 and Proposition 3.2.3 that $F \in C^2(\mathbb{R}^n)$ to only require F' is Lipschitz. For more details, we refer the reader to the original paper [2].

Having established Proposition 3.2.2 and Proposition 3.2.3, we may now prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Set $\varphi = u_{x_3}$ and, for each $i \in \{1, 2\}$, consider the functions $\sigma_i := \frac{u_{x_i}}{u_{x_3}}$ which are well defined by (3.39). Moreover, since $u \in C^{2,\alpha}(\mathbb{R}^n)$, we have enough regularity to compute $\nabla \sigma_i$. Doing so, we find

$$\nabla \sigma_i = \frac{\nabla u_{x_i} \cdot u_{x_3} - \nabla u_{x_3} \cdot u_{x_i}}{u_{x_3}^2}$$

so that

$$\varphi^2 \nabla \sigma_i = \nabla u_{x_i} \cdot u_{x_3} - \nabla u_{x_3} \cdot u_{x_i}. \tag{3.55}$$

Note that the right-hand side of (3.55) belongs to $W^{1,p}_{loc}(\mathbb{R}^3)$ since $u \in W^{3,p}_{loc}(\mathbb{R}^3)$ by the bounds derived earlier. Furthermore, it was shown that both u_{x_i} and u_{x_3} satisfy the linearized equation

$$\Delta w - F''w = 0 \tag{3.56}$$

in $\mathbb{R}^n$. We claim that

$$\operatorname{div}(\varphi^2\sigma_i) = 0 \text{ in } \mathbb{R}^3 \text{ in the weak sense for each } i \in \{1,2\}. \tag{3.57}$$

Indeed, for $i \in \{1,2\}$ fixed we have

$$\begin{aligned}
\operatorname{div}(\varphi^2\sigma_i) &= \operatorname{div}(\nabla u_{x_i} \cdot u_{x_3} - \nabla u_{x_3} \cdot u_{x_i}) \\
&= \operatorname{div}(\nabla u_{x_i} \cdot u_{x_3}) - \operatorname{div}(\nabla u_{x_3} \cdot u_{x_i}) \\
&= \Delta u_{x_i} u_{x_3} - \Delta u_{x_3} u_{x_i}.
\end{aligned}$$

Since u_{x_i} and u_{x_3} solve (3.56), we may write

$$\Delta u_{x_i} u_{x_3} - \Delta u_{x_3} u_{x_i} = F''(u) u_{x_i} u_{x_3} - F''(u) u_{x_i} u_{x_3} = 0.$$

Since $i \in \{1,2\}$ is arbitrary, we have proven (3.57). We need to show

$$\int_{B_R} |\nabla u|^2 \, dx \leq CR^2$$

where C is a constant independent of R. Since $|u_{x_i}|^2 \leq |\nabla u|^2$, this will show that

$$\int_{B_R} (\varphi\sigma_i)^2 \, dx \leq CR^2 \tag{3.58}$$

allowing us to apply Proposition 3.2.2 to the equation given in (3.57).

By assumption, we have $F \geq \min\{F(-1), F(1)\}$ in $(-1,1)$. First, suppose $\min\{F(-1), F(1)\} =$

$F(1)$. Then $F(u) - F(1) \geq 0$ in $\mathbb{R}^3$, and by Proposition 3.2.3 we obtain

$$\frac{1}{2}\int_{B_R} |\nabla u|^2\,dx \leq \int_{B_R} \Big(\frac{1}{2}|\nabla u|^2 + F(u) - F(1)\Big)\,dx \leq CR^2.$$

Thus, (3.58) holds. If we assume instead that $\min\{F(-1), F(1)\} = F(-1)$, we may obtain the same result by replacing $u(x', x_3)$ with $u(x', -x_3)$ and $F(v)$ with $F(-v)$. Then, by Proposition 3.2.2 we see that σ_i is constant for each $i \in \{1, 2\}$ and we may write $u_{x_i} = c_i u_{x_3}$ for some constant c_i and each $i \in \{1, 2\}$. We conclude that u is constant along the directions $(1, 0, -c_1)$ and $(0, 1, -c_2)$ since

$$\nabla u \cdot (1, 0, -c_1) = \nabla u \cdot (0, 1, -c_2) = 0.$$

Set $\tau_1 = (c_1, c_2, 1)$ and note that τ_1 is perpendicular to both of the vectors $(1, 0, -c_1)$ and $(0, 1, -c_2)$ with $\nabla u = u_{x_3}\tau$. We conclude that u depends solely on one variable. Setting $\tau := \frac{\tau_1}{|\tau_1|}$, we see that $u = g(x \cdot \tau)$ where g solves the one-dimensional equation and

$$\nabla u \cdot \tau = u_{x_3}|\tau_1| > 0,$$

completing the proof. $\square$

The proof above, provided by Cabré and Ambrosio, is essentially the same proof given by Ghoussoub and Gui in [31] for the extended version of De Giorgi's conjecture in dimension $n = 2$. Note that, in this case, there is no need to apply Proposition 3.2.3 since the condition

$$\int_{B_R} |\nabla u|^2\,dx \leq CR^2$$

is trivially satisfied by the bound on ∇u. Thus, De Giorgi is resolved, modulo the mild limit assumption, at least for dimension $n = 2$ and $n = 3$.

The authors next sought to establish an extended De Giorgi result for dimensions $n = 2, 3$ without imposing the limit assumption (3.40).

Theorem 3.2.2 (Theorem 1.2 in [2]). *Let u be a bounded solution of*

$$-\Delta u = f(u) \textit{ in } \mathbb{R}^3 \tag{3.59}$$

satisfying

$$u_{x_3} > 0 \textit{ in } \mathbb{R}^3. \tag{3.60}$$

Assume that $F \in C^2(\mathbb{R})$ is such that $F'(u) = -f(u)$ and that

$$F \geq \min\{F(m), F(M)\} \textit{ in } (m, M) \tag{3.61}$$

for each pair of real numbers $m < M$ satisfying $F'(m) = F'(M) = 0$, as well as $F''(m) \geq 0$ and $F''(M) \geq 0$. Then the level sets of u are planes. In particular, there exists $\tau \in \mathbb{R}^3$ and $g_0 \in C^2(\mathbb{R})$ such that

$$u(x) = g_0(\tau \cdot x) \textit{ for all } x \in \mathbb{R}^3. \tag{3.62}$$

Once again, the goal was to establish the energy estimate[9] $J_R(u) \leq CR^2$ for some constant C. After analyzing the proof of Proposition 3.2.3, it becomes evident that the difficulties arise when trying to show $\lim_{t\to+\infty} J_R(u^t) = 0$ since we longer are imposing a limit condition as $x_n \to +\infty$. To overcome this issue, the authors considered the function

$$\overline{u}(x') = \lim_{x_3\to+\infty} u(x', x_3) \text{ where } x' \in \mathbb{R}^2,$$

which is a solution of the same semilinear equation but in $\mathbb{R}^2$. Even more, using a technique developed by Berestycki, Caffarelli, and Nirenberg in [5], they showed that $\overline{u}$ is a solution depending on one variable only. As a consequence, they showed that the energy of $\overline{u}$ in a two-dimensional ball of radius R is bounded by CR, hence, that

$$\lim_{t\to+\infty} \sup J_R(u^t) \leq CR^2.$$

[9]Note that, in the definition of $J_R(u)$, we replace the term $F(1)$ with $F(\sup u)$.

Proceeding exactly as in the proof of Proposition 3.2.3, they then established the estimate $J_R(u) \leq CR^2$, thus proving the conjecture under the more general assumptions made on F.

3.2.2 Stable Solutions and Global Minimizers

One natural question to consider is whether one may replace the limit assumption or the monotonicity condition $u_{x_n} > 0$ by other more physical conditions and still obtain De Giorgi type results. With this in mind, we turn to the notion of *global minimizers* of the energy (3.2). Global minimizers of (3.2) are defined precisely below.

Definition 3.2.1 (Global Minimizer)**.** We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is a *global minimizer* of the energy (3.2) if

$$J(u) \leq J(u + \phi) \text{ for all } \phi \in C_0^1(\mathbb{R}^n). \tag{3.63}$$

In other words, u is a global minimizer if u minimizes the energy under any compactly supported perturbation. Note that we may similarly define global minimizers for the energy (3.3). In addition, it is well known that imposing both monotonicity ($u_{x_n} > 0$) together with the limit condition (3.36) on solutions u of the problem (3.1) imply that u is a global minimizer of the energy (3.2) [10].

In [49], Savin proved a rigidity result for global minimizers of (3.2).

Theorem 3.2.3. *Let $n \leq 7$. If u is a global minimizer of the energy* (3.2)*, then u is a function of one-variable.*

However, for dimensions $n \geq 8$, there exist global minimizers of (3.2) that are not one-dimensional (see [40]). Hence, global minimizers of (3.37) are completely classified.

Another subject of interest is whether De Giorgi type results exist for a class of solutions called *stable solutions*.

[10] See Chapter 7 in L. Dupaigne's text [20], for example.

Definition 3.2.2 (Stable Solution)**.** Let $f \in C^1(\mathbb{R})$ and let Ω be an open subset of $\mathbb{R}^n$, $n \geq 1$. A bounded solution u of the problem

$$-\Delta u = f(u) \text{ in } \Omega$$

is *stable* if

$$J_u(\phi) := \int_\Omega (|\nabla \phi|^2 - f'(u)\phi^2)\, dx \geq 0 \text{ for all } \phi \in C_0^\infty(\Omega). \tag{3.64}$$

Remark 3.2.4. Loosely speaking, stable solutions are solutions that recover after compact perturbations. Applying the theory of the maximum principle, one may show that bounded monotone solutions are stable.

In fact, in order to prove Theorem 3.2.2, Cabré and Ambrosio relied on the following stability property of $\overline{u}$.

Proposition 3.2.4 (Lemma 3.1 in [2])**.** *Let $F \in C^2(\mathbb{R})$ be such that $F'(u) = -f(u)$, and let u be a bounded solution of*

$$-\Delta u = f(u) \textit{ in } \mathbb{R}^n$$

satisfying

$$u_{x_n} > 0 \textit{ in } \mathbb{R}^n.$$

Then the function

$$\overline{u}(x') = \lim_{x_n \to +\infty} u(x', x_n) \textit{ where } x' \in \mathbb{R}^{n-1}$$

is a bounded solution of

$$-\Delta \overline{u} = f(\overline{u}) \textit{ in } \mathbb{R}^{n-1}.$$

In addition, there exists a positive function $\varphi \in W^{2,p}_{loc}(\mathbb{R}^{n-1})$ for every $p < \infty$ such that

$$\Delta \varphi - F''(\overline{u})\varphi \leq 0 \textit{ in } \mathbb{R}^{n-1}. \tag{3.65}$$

As a consequence, if $n = 3$, then $\overline{u}$ is a function of one variable only.

Remark 3.2.5. Applying once more the theory of the maximum principle, one may show that $u_{x_n} > 0$ and (3.65) lead to (3.64). Thus, Proposition 3.2.4 proves the *stability conjecture* for bounded monotone solutions in dimension $n = 2$. That is, bounded solutions u satisfying $u_{x_n} > 0$ and (3.64) are necessarily one-dimensional.

For the proof of Proposition 3.2.4, we refer the reader to the original paper. Next, we consider an improvement of the Liouville-type theorem, Proposition 3.2.2.

3.2.3 An Improvement of the Liouville-type Theorem

For future considerations, it is worthwhile to note that a sharper estimate can be formulated than that given in Proposition 3.2.2. This was done by L. Moschini in [44].

Set

$$\mathcal{F} := \Big\{ F : \mathbb{R}^+ \to \mathbb{R}^+ : F \text{ is nondecreasing and } \sum_{j=0}^{\infty} \frac{1}{F(2^{j+1})} = +\infty \Big\}, \tag{3.66}$$

and note that $\mathcal{F}$ is nonempty since it includes the function $\log R$ defined for $R \in (1, \infty)$. We have the following theorem.

Theorem 3.2.4 (Theorem 5.1 in [44])**.** *Let $\varphi \in L^\infty_{loc}(\mathbb{R}^n)$ be a positive function. Suppose that $\sigma \in H^1_{loc}(\mathbb{R}^n)$ satisfies*

$$\sigma\, \mathrm{div}(\varphi^2 \nabla \sigma) \geq 0 \textit{ in } \mathbb{R}^n \tag{3.67}$$

in the distributional sense, $n \geq 1$. Further suppose

$$\limsup_{R \to +\infty} \frac{1}{R^2 F(R)} \int_{B_R} (\varphi\sigma)^2 \, dx = C \tag{3.68}$$

where C is constant and $F \in \mathcal{F}$. Then σ is constant.

Remark 3.2.6. Observe that setting $F(R) \equiv 1$ in (3.68) yields Proposition 3.2.2 above.

Proof. Suppose σ satisfies (3.67). Then,

$$\begin{aligned} \operatorname{div}(\sigma\varphi^2\nabla\sigma) &= \varphi^2|\nabla\sigma|^2 + \sigma\operatorname{div}(\varphi^2\nabla\sigma) \\ &\geq \varphi^2|\nabla\sigma|^2. \end{aligned} \tag{3.69}$$

Also, after two applications of the Cauchy-Schwarz inequality, we find

$$\begin{aligned} \int_{\partial B_R} \sigma\varphi^2\langle\nabla\sigma,\nu\rangle\,dS^{n-1} &\leq \int_{\partial B_R} |\sigma|\varphi^2|\nabla\sigma|\,dS^{n-1} \\ &\leq \left(\int_{\partial B_R} \varphi^2|\nabla\sigma|^2\,dS^{n-1}\right)^{\frac{1}{2}}\left(\int_{\partial B_R} (\varphi\sigma)^2\,dS^{n-1}\right)^{\frac{1}{2}}, \end{aligned} \tag{3.70}$$

where ν is the radial unit normal on ∂B_R. Set

$$D(R) := \int_{B_R} \varphi^2|\nabla\sigma|^2\,dx.$$

Thus, integrating (3.69) in B_R, we have

$$\begin{aligned} \int_{B_R} \varphi^2|\nabla\sigma|^2\,dx + \int_{B_R} \sigma\operatorname{div}(\varphi^2\nabla\sigma)\,dx &= \int_{\partial B_R} \sigma\varphi^2\langle\nabla\sigma,\nu\rangle\,dS^{n-1} \\ &\geq \int_{B_R} \varphi^2|\nabla\sigma|^2\,dx, \end{aligned}$$

where we have integrated by parts in the first integral. Observe that

$$\int_{\partial B_R} \sigma\varphi^2\langle\nabla\sigma,\nu\rangle\,dS^{n-1} \leq D'(R)^{\frac{1}{2}}\left(\int_{\partial B_R} (\varphi\sigma)^2\,dS^{n-1}\right)^{\frac{1}{2}} \tag{3.71}$$

by (3.70). Furthermore, since $\varphi^2 > 0$, if σ is not identically constant, then there exists $R_0 > 0$ such that $D(R) > 0$ for all $R \geq R_0$. In addition,

$$\frac{\mathrm{d}}{\mathrm{d}R}\big(D^{-1}(R)\big) = -D^{-2}(R)\cdot D'(R).$$

By (3.71),

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}R}\big(D^{-1}(R)\big) &= \frac{D'(R)}{D^2(R)} \\
&\geq \left(\int_{\partial B_R} (\varphi\sigma)^2 \, dS^{n-1} \right)^{-1}.
\end{aligned} \tag{3.72}
$$

Let $r_2 > r_1 > R_0$ and integrate from r_1 to r_2 above to get

$$
\begin{aligned}
\frac{1}{D(r_1)} - \frac{1}{D(r_2)} &= -\int_{r_1}^{r_1} \frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{1}{D(R)}\right) dR \\
&\geq \int_{r_1}^{r_2} \left(\int_{\partial B_R} (\varphi\sigma)^2 \, dS^{n-1} \right)^{-1} dR
\end{aligned} \tag{3.73}
$$

for all $r_1 > r_2 > R_0$. We claim

$$
(r_2 - r_1)^2 \left(\int_{r_1}^{r_2} \left(\int_{\partial B_R} (\varphi\sigma)^2 \, dS^{n-1} \right) dR \right)^{-1} \leq \left(\int_{r_1}^{r_2} \left(\int_{\partial B_R} (\varphi\sigma)^2 \, dS^{n-1} \right)^{-1} dR \right) \tag{3.74}
$$

holds. Indeed, the Cauchy-Schwarz inequality yields, for positive and integrable $f : \mathbb{R}^n \to \mathbb{R}$,

$$
\begin{aligned}
\left(\int_{r_1}^{r_2} dR \right)^2 &= \left(\int_{r_1}^{r_2} f(R)^{\frac{1}{2}} f(R)^{-\frac{1}{2}} \, dR \right)^2 \\
&\leq \left(\int_{r_1}^{r_2} f^{-1}(R) \, dR \right) \cdot \left(f(R) \, dR \right).
\end{aligned} \tag{3.75}
$$

Setting

$$
f(R) = \int_{\partial B_R} (\varphi\sigma)^2 \, dS^{n-1},
$$

we obtain (3.74) by (3.75). Now, applying (3.74) and (3.75), we find

$$\begin{aligned}\frac{1}{D(r_1)} - \frac{1}{D(r_2)} &\geq \int_{r_1}^{r_2} \left(\int_{\partial B_R} (\varphi\sigma)^2 \, dS^{n-1} \right)^{-1} dR \\ &\geq (r_2 - r_1)^2 \left(\int_{r_1}^{r_2} \left(\int_{\partial B_R} (\varphi\sigma)^2 \, dS^{n-1} \right) dR \right)^{-1} \\ &= (r_2 - r_1)^2 \left(\int_{B_{r_2} \setminus B_{r_1}} (\varphi\sigma)^2 \, dx \right)^{-1}. \end{aligned} \tag{3.76}$$

For each $j = 0, 1, \ldots, n-1$, set $r_{2,j} := 2^{j+1} r^*$ and $r_{1,j} := 2^j r^*$ where r^* is chosen so that $r^* > R_0$. Then (3.76) gives

$$\begin{aligned}\frac{r_{2,j}^2}{4} \left(\int_{B_{r_{2,j}}} (\varphi\sigma \, dx \right)^{-1} &\leq r_{1,j}^2 \left(\int_{B_{r_{2,j}} \setminus B_{r_{1,j}}} (\varphi\sigma)^2 \, dx \right)^{-1} \\ &\leq \frac{1}{D(r_{2,j})} - \frac{1}{D(r_{1,j})}. \end{aligned}$$

Note that

$$\frac{1}{D(r^*)} \geq \frac{1}{D(r^*)} - \frac{1}{D(r_{2,n-1})},$$

since $\frac{1}{D(2^n r^*)} > 0$. Summing over j above and applying (3.68), we find

$$\frac{1}{D(r^*)} - \frac{1}{D(r_{2,n-1})} \geq \frac{1}{4C} \sum_{j=0}^{n-1} \frac{1}{F(r_{2,j})}. \tag{3.77}$$

Since $F \in \mathcal{F}$, we see that $F(r_{2,j}) \leq F(2^{j+j_0+1})$ if j_0 is such that $r^* \leq 2^{j_0}$. Thus, the sum on the right-hand side of (3.77) diverges as $n \to \infty$. We conclude that $D(r^*) = 0$, for otherwise the sum on the right-hand side of (3.77) is bounded. It follows that $D(r^*) = 0$ for all $r^* > R_0$. Since $\varphi^2 > 0$, we see that $|\nabla\sigma| = 0$ so that σ is constant. □

From Theorem 3.2.4, we may obtain the following corollary which is proven by taking $F(R) = \log R$ for $R > 1$ in (3.68).

Corollary 3.2.1 (Corollary 5.3 in [44]). *Let $\varphi \in L^\infty_{loc}(\mathbb{R}^n)$ be a positive function. Suppose that $\sigma \in H^1_{loc}(\mathbb{R}^n)$ satisfies*

$$\sigma \operatorname{div}(\varphi^2 \nabla \sigma) \geq 0 \textit{ in } \mathbb{R}^n \tag{3.78}$$

in the distributional sense, $n \geq 1$. Further suppose that for every $R > 1$

$$\int_{B_R} (\varphi\sigma)^2 \, dx \leq CR^2 \log R \tag{3.79}$$

for some constant C. Then σ is constant.

Proof. Take $F(R) := \log R$ defined for $R > 1$. Then F is nondecreasing in R. Furthermore,

$$\sum_{j=0}^{\infty} \frac{1}{F(2^{j+1})} = \sum_{j=0}^{\infty} \frac{1}{\log(2^{j+1})}$$

which diverges by the ratio test, for example. Then $F \in \mathcal{F}$ and Theorem 3.2.3 gives the result. $\square$

Remark 3.2.7. An interesting open problem is to either prove or to give a counterexample to Theorem 3.2.4 when $F(R) = R^{n-3}$, thus $R^2F(R) = R^{n-1}$, and $4 \leq n \leq 8$. It is known that counterexamples exist for this choice of F when $n \geq 9$ (see [2]). Further counterexamples exist when $F(R) = R^{n-2}$ for any $n \geq 3$ (see [3]). When $n \geq 7$, a different counterexample to this case was given explicitly by Ghoussoub and Gui in [31] (see Proposition 2.8).

Unfortunately, the sharpness of the estimate (3.44) in higher dimensions is improved only by a factor of $\log R$. As a result, we still face complications when trying to extend the proof of De Giorgi to dimensions $4 \leq n \leq 8$ via the techniques employed by Ghoussoub-Gui and Ambrosio-Cabré, since, to extend to higher dimensions using the same techniques, one must obtain additional multiples of R in the estimate (3.79). Though Theorem 3.2.4 and, hence, Corollary 3.2.1 are not immediately useful, a similar Liouville-type theorem will be important in Chapter 4. We thereby include the proofs above as instruction for what follows.

3.3 Current Directions

From the above considerations, we see that De Giorgi's conjecture is entirely resolved apart from the limit assumption imposed by Savin in dimensions $4 \leq n \leq 8$. We also note that De Giorgi type results have been obtained in the systems case (see, for instance, Fazly and Ghoussoub in [23]). As shown prior, the stability conjecture has been proven true for dimension $n = 2$ by Ambrosio and Cabré (see [2]). In dimensions $n \geq 8$, however, the conjecture has been shown to be false (see [45] and [40]). Thus, the stability conjecture remains wide open for dimensions $3 \leq n \leq 7$. Recently, authors have also considered the problem of how one classifies unstable solutions (i.e. solutions that are not stable), for example, those having *finite Morse index* (see [14]).

Though the topics above constitute fascinating avenues for future study, these considerations are outside of the scope of this project. For a recent overview of related open problems, we refer the reader to [14]. For the remainder of this project, we focus our attention on the extension of De Giorgi's conjecture to the nonlocal operator introduced prior, the fractional Laplacian $(-\Delta)^s$.

CHAPTER 4: THE FRACTIONAL DE GIORGI CONJECTURE

Our goal in this final chapter is to lay out what is known for the fractional De Giorgi conjecture, that is, the counterpart to Conjecture 3.2.1 for the nonlocal operator $(-\Delta)^s$. We will again focus on the lower dimensional cases. The cases dimension $n = 2$ with $s \in (0,1)$ and $n = 3$ with $s \in [\frac{1}{2}, 1)$ will be proven in full detail for a particular class of solutions called *layer solutions* using the methods of Cabré et al. developed in a series of papers. The case dimension $n = 2$ and general s without the limit assumption was proved by Sire and Valdinoci in [57]. We will only provide an overview of the case $n = 3$ with $s \in (0, \frac{1}{2})$, as it is based on the work of Dipierro et al. in [19] which requires knowledge of minimal surface theory. We will also summarize what is known for fractional stable solutions and fractional global minimizers. At the end of the chapter, we prove nonlocal versions of the Pohozaev identity, Modica estimate, and monotonicity formula introduced at the beginning of Chapter 3. To close, we discuss current directions for the fractional De Giorgi conjecture and related topics, as well as problems of interest to the author for future study.

Let $0 < s < 1$. The *fractional Allen-Cahn equation* is given by

$$(-\Delta)^s u = u - u^3 \text{ in } \mathbb{R}^n, \tag{4.1}$$

and is the Euler-Lagrange equation for the energy functional

$$J_{s,\mathbb{R}^n}(u) := \frac{C(n,s)}{2} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy + \frac{1}{4}\int_{\mathbb{R}^n} (1-u^2)^2\, dx \tag{4.2}$$

As in the local case, we may consider the more general equation

$$(-\Delta)^s u = f(u) \text{ in } \mathbb{R}^n, \tag{4.3}$$

where $F(u) = \int_u^1 f$ (i.e. $F'(u) = -f(u)$) and $F \geq 0$ in $\mathbb{R}$, with $F(\pm 1) = 0$ and $F > 0$ in $(-1, 1)$

(e.g. F is a double-well potential). In this setting, the associated energy is

$$\mathscr{J}_{s,\mathbb{R}^n}(u) := \frac{C(n,s)}{2} \int\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_{\mathbb{R}^n} F(u(x))\,dx \tag{4.4}$$

Mentioned earlier, nonlocal equations arise in many pure and applied fields including differential geometry (fractional minimal surfaces), probability (Levy processes), and physics (nonlocal phase transitions, image processing, and general relativity). In this setting, De Giorgi's conjecture can be stated precisely as follows:

Conjecture 4.0.1 (Fractional De Giorgi Conjecture)**.** *Let u be a bounded solution of* (4.1) *which is monotone in one direction, say $u_{x_n} > 0$. Then u is a function of one variable.*

Before we continue, let us recall the extension problem presented in Chapter 2 and discuss some preliminaries. By the results in Chapter 2, we observe that problem (4.3) is equivalent to the following boundary value problem in $\mathbb{R}^{n+1}_+$

$$\begin{cases} \operatorname{div}(y^a\nabla v) = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ -d_s \lim_{y\to 0^+} y^a v_y = f(v) \text{ on } \partial\mathbb{R}^{n+1}_+, \end{cases} \tag{4.5}$$

to which we associate the energy functional

$$J^+_{s,C_R}(w) := d_s \int_{C_R} \frac{1}{2} y^a |\nabla w|^2\,dx\,dy + \int_{B_R\times\{0\}} F(w(x,0))\,dx. \tag{4.6}$$

The domain C_R is the cylinder in the upper half-space given by $C_R := B_R \times (0,R)$. We call the solution v of (4.5) the *s-extension of u in $\mathbb{R}^{n+1}_+$*.

As before, $\mathbb{R}^{n+1}_+ := \{(x,y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n \text{ and } y > 0\}$. Then, we see that $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n\times\{0\}$ which we view as $\mathbb{R}^n$ embedded in $\mathbb{R}^{n+1}$. We take $a = 1-2s$ where $s \in (0,1)$ and $v(x,0) := u(x)$ on $\partial\mathbb{R}^{n+1}_+$, $n \geq 1$. Weak solutions for the problem (4.5) are defined below.

Definition 4.0.1 (Weak Solution)**.** Consider the problem

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 \text{ in } C_R, \\ -\lim_{y\to 0^+} y^a v_y = h \text{ on } B_R \times \{0\}. \end{cases} \tag{4.7}$$

Given $R > 0$ and a function $h \in L^1(B_R \times \{0\})$, we say that v is a *weak solution of* (4.7) provided

$$y^a |\nabla v|^2 \in L^1(C_R) \tag{4.8}$$

and

$$\int_{C_R} y^a \nabla v \nabla \phi \, dx\, dy - \int_{B_R \times \{0\}} h\phi \, dx = 0 \tag{4.9}$$

for all $\phi \in C^1(\overline{C_R})$ such that $\phi \equiv 0$ on ∂C_R.

We may define weak solutions on other domains in a similar manner.

Remark 4.0.1. Let us comment on the regularity of solutions to the problem (4.5). By Lemma D.1.1 in Appendix D (Lemma 4.4 in [9]), weak solutions to (4.5) are $C^\beta(\overline{\mathbb{R}^{n+1}_+})$ for some $\beta \in (0, 1)$. In general, solutions to (4.5) have no further regularity in the y variable. For example, the function $(x, y) \mapsto y^{2s}$ is a weak solution to (4.5) with f identically constant even though it is unbounded. Note that the trace $v(x, 0)$ on $\partial\mathbb{R}^{n+1}_+$ of any bounded weak solution of (4.5) belongs to $C^{2,\beta}(\mathbb{R}^n)$ (see Lemma D.1.1 also).

By Theorem 2.3.1, we find that

$$(-\Delta)^s u(x) = (-\Delta)^s v(x, 0) = -d_s \lim_{y\to 0^+} y^a v_y \tag{4.10}$$

where d_s is a positive constant depending only on s. In fact, comparing with (2.55), we see that

$$d_s = \frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)}. \tag{4.11}$$

As mentioned in the opening of the chapter, we will be concerned with certain bounded solutions of (4.3) called *layer solutions*. These are bounded, monotone increasing solutions u connecting -1 to 1 at $\mp\infty$ in one of the coordinates x_i, $i = 1, \ldots, n$. In [9], Cabré and Sire proved a Modica-type estimate[1] which allowed them to derive a necessary condition on the nonlinearity f for the existence of a layer solution in $\mathbb{R}$.

Theorem 4.0.1 (Theorem 2.2(i) in [9])**.** *Let* $s \in (0,1)$ *and* $f \in C^{1,\alpha}(\mathbb{R})$ *for some* $\alpha > \max\{0, 1-2s\}$. *Assume that there exists a layer solution* u *of*

$$(-\partial^2_{xx})^s u = f(u) \text{ in } \mathbb{R}, \tag{4.12}$$

that is, u *is a solution of* (4.12) *satisfying*

$$u' > 0 \text{ in } \mathbb{R} \text{ and } \lim_{x\to\pm\infty} u(x) = \pm 1.$$

Then,

$$F'(-1) = F'(1) = 0 \tag{4.13}$$

and

$$F > F(-1) = F(1) \text{ in } (-1,1). \tag{4.14}$$

After establishing the necessary conditions (4.13) and (4.14) for the existence layer solutions to (4.12), Cabré and Sire proved in [10] that these conditions are sufficient.

Theorem 4.0.2 (Existence of Layer Solutions; Theorem 2.4 in [10])**.** *Let* $f \in C^{1,\alpha}(\mathbb{R})$ *with* $\alpha > \max\{0, 1-2s\}$, *where* $s \in (0,1)$. *Then there exists a solution* u *of* (4.12) *such that* $u' > 0$ *in* $\mathbb{R}$ *and* $\lim_{x\to\pm\infty} u(x) = \pm 1$ *if and only if* (4.13) *and* (4.14) *are satisfied. In addition, if* $f'(-1) < 0$ *and* $f'(1) < 0$, *then this solution is unique up to translations.*

Combining Theorem 4.0.1 and Theorem 4.0.2, we see that one-dimensional layer solutions of (4.12) are completely classified.

[1] See Theorem 4.3.1.

Remark 4.0.2. We point out that the uniqueness of a layer solution also holds for any nonlinearity f of class $C^1([-1,1])$ satisfying $f'(-1) < 0$ and $f'(1) < 0$. In fact, by following the proof of Theorem 2.4 in [10], one finds that we only need f to be Lipschitz in $[-1,1]$ and nonincreasing in a neighborhood of -1 and of 1. Moreover, we see that if f is odd and $f'(\pm 1) < 0$, then the solution is odd with respect to some point. That is, $u(x+c) = -u(-x+c)$ for some $c \in \mathbb{R}$.

Let us introduce some important definitions to be used in the sequel and make precise some of the notions mentioned above.

Definition 4.0.2. Let $H^1(\Omega, y^a)$ denote the space of functions $w : \Omega \to \mathbb{R}$ such that $y^a w \in H^1(\Omega)$

1. We say that a bounded $C^\alpha_{loc}(\overline{\mathbb{R}^{n+1}_+}) \cap H^1_{loc}(\overline{\mathbb{R}^{n+1}_+}, y^a)$ function u is a *global minimizer of* (4.3) if for all $R > 0$,

$$J^+_{s,C_R}(u) \leq J^+_{s,C_R}(w) \tag{4.15}$$

   for every $w \in H^1(C_R, y^a)$ such that $u \equiv w$ in $\partial C_R \setminus \{y = 0\}$.

2. We say that a bounded function v is a *global minimizer of* (4.5) if its trace $v(x,0) = u(x)$ on $\partial\mathbb{R}^{n+1}_+$ is a global minimizer of (4.3).

3. Assume that $v \in C^\alpha(\overline{\mathbb{R}^{n+1}_+})$ for some $\alpha \in (0,1)$ satisfying $-1 < v < 1$ in $\overline{\mathbb{R}^{n+1}_+}$ and such that, for all $R > 0$, $v \in H^1(C_R, y^a)$. We say that v is a *local minimizer of problem* (4.5) if

$$J^+_{s,C_R}(v) \leq J^+_{s,C_R}(v + \xi) \tag{4.16}$$

   for every $R > 0$ and every $\xi \in C^1(\overline{\mathbb{R}^{n+1}_+})$ having compact support in $C_R \cup (B_R \times \{0\})$ and such that $-1 \leq v + \xi \leq 1$ in C_R.

4. We say that a bounded function u is a *layer solution of* (4.3) if u is a solution, it is monotone increasing in one of the coordinates, say $u_{x_n} > 0$ in $\mathbb{R}^n$, and

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \text{ for each } x' \in \mathbb{R}^{n-1}. \tag{4.17}$$

5. A bounded function v is a *layer solution of* (4.5) if its trace $v(x,0) = u(x)$ on $\partial\mathbb{R}^{n+1}_+$ is a layer solution of (4.3).

Note that layer solutions are global minimizers, while local minimizers are not necessarily global minimizers. In addition, we would like to point out that some authors consider the problem

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ -(1+a)\lim_{y\to 0^+} y^a v_y = f(v) \text{ on } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{4.18}$$

in lieu of problem (4.5). Replacing f with $(1+a)d_s^{-1}f$ in (4.18) gives the equivalency of problems (4.18) and (4.5). To be consistent with the literature, we will sometimes consider (4.18) instead of (4.5). In each subsequent section, we will state clearly which formulation of the problem is being used. When considering problem (4.18), we will write

$$J^+_{s,C_R}(w) := \int_{C_R} \frac{1}{2} y^a |\nabla w|^2 \, dx\, dy + \frac{1}{1+a}\int_{B_R\times\{0\}} F(w(x,0))\, dx. \tag{4.19}$$

in place of (4.6). This will cause no difficulties since we may easily switch between the two via the mapping $f \mapsto (1+a)d_s^{-1}f$ and vice versa.

Naturally, when trying to tackle the nonlocal De Giorgi conjecture, one will wonder if Proposition 3.2.2 and Proposition 3.2.3 can be extended to the nonlocal operator $(-\Delta)^s$. The following results, obtained by Cabré and Sire in [9] and Cabré and Cinti in [7] via the extension problem, show that they can.

4.1 A Nonlocal Liouville-type Theorem and Nonlocal Energy Estimate via the Extension Problem

Our goal in this section is to prove nonlocal versions of Proposition 3.2.2 and Proposition 3.2.3 via the extension problem in order to obtain De Giorgi type results for the fractional Laplacian. We first consider the nonlocal analogue to Proposition 3.2.2, proven by X. Cabré and Y. Sire in [9].

Proposition 4.1.1 (Liouville Theorem; Theorem 4.10 in [9]). *Let $\varphi \in L^{\infty}_{loc}(\overline{\mathbb{R}^{n+1}_+})$ be a positive function. Suppose that $\sigma \in H^1_{loc}(\overline{\mathbb{R}^{n+1}_+}, y^a)$ is such that*

$$\begin{cases} -\sigma \operatorname{div}(y^a \varphi^2 \nabla \sigma) \leq 0 \text{ in } \mathbb{R}^{n+1}_+ \\ -\lim_{y \to 0^+} \sigma y^a \sigma_y \leq 0 \text{ on } \partial\mathbb{R}^{n+1}_+ \end{cases} \tag{4.20}$$

in the weak sense. Assume that for every $R > 1$,

$$\int_{C_R} y^a (\sigma\varphi)^2 \, dx \, dy \leq C R^2 \tag{4.21}$$

for some constant C independent of R. Then σ is constant.

As one might imagine, a difficulty that often arises when considering problems for $(-\Delta)^s$ via the extension problem is dealing with the boundary terms that inevitably show up after integration by parts. The proof of the Liouville-type theorem provides us with a perfect example.

Proof of Proposition 4.1.1. Let $\zeta := \zeta(t)$ be the nonnegative C^∞ function defined as in the proof of Proposition 3.2.2. For $R > 1$ and $(x, y) \in \mathbb{R}^{n+1}_+$, set $\zeta_R := \zeta\left(\frac{r}{R}\right)$ where $r = |(x, y)|$.

Multiplying the first expression in (4.20) by ζ_R^2 and integrating by parts in $\mathbb{R}^{n+1}_+$, we find

$$\begin{aligned} \int_{\mathbb{R}^{n+1}_+} y^a \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy - \lim_{y \to 0^+} \int_{\partial\mathbb{R}^{n+1}_+} \zeta_R^2 \varphi^2 \sigma y^a \partial_\nu \sigma \, dx \\ \leq -2 \int_{\mathbb{R}^{n+1}_+} y^a \zeta_R \varphi^2 \sigma \nabla \zeta_R \nabla \sigma \, dx \, dy, \end{aligned} \tag{4.22}$$

where ν is the outer unit normal to $\mathbb{R}^{n+1}_+$. Since $\nu = (0, 0, \ldots, -1)$, we find

$$\lim_{y \to 0^+} \int_{\partial\mathbb{R}^{n+1}_+} \zeta_R^2 \varphi^2 \sigma y^a \partial_\nu \sigma \, dx = -\lim_{y \to 0^+} \int_{\partial\mathbb{R}^{n+1}_+} \zeta_R^2 \varphi^2 \sigma y^a \sigma_y \, dx \leq 0$$

This allows us to drop the boundary term in in (4.22) to conclude

$$\int_{\mathbb{R}^{n+1}_+} y^a \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy \le -2 \int_{\mathbb{R}^{n+1}_+} y^a \zeta_R \varphi^2 \sigma \nabla \zeta_R \nabla \sigma \, dx \, dy.$$

Then the Cauchy-Schwarz inequality gives

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} y^a \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy &\le 2 \left(\int_{R^{n+1}_+ \cap \{R<r<2R\}} y^a \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy \right)^{\frac{1}{2}} \cdot \\ &\quad \left(\int_{\mathbb{R}^{n+1}_+} y^a \varphi^2 \sigma^2 |\nabla \zeta_R|^2 \, dx \, dy \right)^{\frac{1}{2}} \\ &\le C \left(\int_{R^{n+1}_+ \cap \{R<r<2R\}} y^a \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy \right)^{\frac{1}{2}} \cdot \\ &\quad \left(\frac{1}{R^2} \int_{C_R} y^a (\varphi \sigma)^2 \, dx \, dy \right)^{\frac{1}{2}},\end{aligned}$$

for some constant C. Thus, by (4.21) we have

$$\int_{\mathbb{R}^{n+1}_+} y^a \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy \le C \left(\int_{R^{n+1}_+ \cap \{R<r<2R\}} y^a \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy \right)^{\frac{1}{2}}, \tag{4.23}$$

where C is constant. It follows that $\int_{\mathbb{R}^{n+1}_+} y^a \zeta_R^2 \varphi^2 |\nabla \sigma|^2 \, dx \, dy \le C$ and, letting $R \to \infty$, we conclude that $\int_{\mathbb{R}^{n+1}_+} y^a \varphi^2 |\nabla \sigma|^2 \, dx \, dy \le C$ so that the right-hand side of (4.23) goes to zero in the limit as $R \to +\infty$. Therefore, we have $\int_{\mathbb{R}^{n+1}_+} y^a \varphi^2 |\nabla \sigma|^2 \, dx \, dy = 0$ so that σ is constant. $\square$

To prove a nonlocal version of Proposition 3.2.3 via the extension problem, the main difficulty to overcome is, once again, the boundary terms that appear after integration by parts. Moreover, care must be taken when dealing with the fractional exponent $s \in (0,1)$. Using the identities $s\Gamma(s) = \Gamma(s+1)$ and $(1-s)\Gamma(1-s) = \Gamma(2-s)$, one may easily show that $\frac{d_s}{2s^{-1}} \to 1$ as $s \downarrow 0$ and $\frac{d_s}{2(1-s)} \to 1$ as $s \uparrow 1$. Hence, the constant d_s blows up as $s \downarrow 0$ and tends linearly to zero as $s \uparrow 1$.

Let us state the main estimates.

Proposition 4.1.2 (Nonlocal Energy Estimate; Theorem 2.1 in [8])**.** *Let $f \in C^{1,\alpha}(\mathbb{R}^n)$ with $\alpha > \max\{0, 1-2s\}$. Suppose F satisfies* (4.13) *and* (4.14) *above. Suppose also that u is a layer solution of* (4.3)*. Let v be the s-extension of u in $\mathbb{R}^{n+1}_+$, that is, v solves* (4.5)*. Given any $s \in (0,1)$, choose $s_0 \in (0, \frac{1}{2})$ so that $s_0 < s < 1$. Then v satisfies the following estimates depending on s:*

$$J^+_{s,C_R}(v) \leq \frac{C}{1-2s} R^{n-2s} \text{ if } s_0 < s < \frac{1}{2} \tag{4.24}$$

$$J^+_{s,C_R}(v) \leq C R^{n-1} \log R \text{ if } s = \frac{1}{2} \tag{4.25}$$

$$J^+_{s,C_R}(v) \leq \frac{C}{2s-1} R^{n-1} \text{ if } \frac{1}{2} < s < 1 \tag{4.26}$$

for any $R > 2$. The constant C depends only on n, s_0, and $\|f\|_{C^{1,\alpha}([-1,1])}$.

The following lemmata are key to the proof of Proposition 4.1.2.

Lemma 4.1.1. *Every bounded solution of* (4.5) *satisfies*

$$|\nabla_x v(x,y)| \leq c_s \text{ for all } x \in \mathbb{R}^n,\ y \geq 0 \tag{4.27}$$

$$|\nabla v(x,y)| \leq \frac{c_s}{y} \text{ for all } x \in \mathbb{R}^n,\ y > 0 \tag{4.28}$$

$$|y^a v_y(x,y)| \leq c_s \text{ for all } x \in \mathbb{R}^n,\ y > 0, \tag{4.29}$$

where, in each case, the constant c_s depends only on s and is uniformly bounded in s away from zero. In addition, if we define the translates

$$v^t(x,y) := v(x', x_n + t, y),$$

then

$$\lim_{t\to\infty} |\nabla v^t(x,y)| = 0. \tag{4.30}$$

Proof. The estimates (4.27), (4.28), and (4.29) are immediate consequences of Proposition D.1.2.

On the other hand, (4.30) is a consequence of (D.8) and (D.9) in Proposition D.1.3. □

Lemma 4.1.2. *Let u be a layer solution of* (4.5) *and v its s-extension in $\mathbb{R}^{n+1}_+$. Then $v \in C^\infty(\mathbb{R}^{n+1}_+)$ and $v_{x_n} > 0$ in $\overline{\mathbb{R}^{n+1}_+}$.*

Proof. Since v solves $\operatorname{div}(y^a \nabla v) = 0$ in $\mathbb{R}^{n+1}_+$, we see that $v \in C^\infty(\mathbb{R}^{n+1}_+)$ by standard elliptic regularity. For the second claim, we observe that $v_{x_n}(x, 0) = u_{x_n} > 0$ for every $x \in \mathbb{R}^n$. Since v is the s-extension of u in $\mathbb{R}^{n+1}_+$, v_{x_n} is bounded and continuous in $\overline{\mathbb{R}^{n+1}_+}$ by Proposition D.1.1. In addition, $v_{x_n}(x, y) = (u_{x_n} * P_{s,y})(x)$ by Theorem 2.2.1. Letting $|x| \to \infty$, we see that $v_{x_n}(x, y) \to 0$ pointwise for $y > 0$. Hence, we may apply the maximum principle to show that $v_{x_n} > 0$ in all of $\overline{\mathbb{R}^{n+1}_+}$. □

We may now prove Proposition 4.1.2. Note that in the computations that follow, and the remainder of this chapter, the set B_R will always denote the ball of radius R centered at the origin in $\mathbb{R}^n$ (i.e. $B_R \times \{0\}$ in $\mathbb{R}^{n+1}_+$).

Proof of Proposition 4.1.2. Consider the translates

$$v^t(x, y) := v(x', x_n + t, y)$$

defined for $(x, y) = (x', x_n, y) \in \mathbb{R}^{n+1}_+$ and $t \geq 0$. Then, for all $t \geq 0$, v^t satisfies (4.5) by translation invariance of $(-\Delta)^s$. Thus, the a priori bounds for v above hold for v^t for each $t \geq 0$. Furthermore, $\lim_{t\to\infty} |\nabla v^t(x, y)| = 0$ by Lemma 4.1.1. Then,

$$\lim_{t\to\infty} [|v^t(x, y) - 1| + |\nabla v^t(x, y)|] = 0 \tag{4.31}$$

for all $x \in \mathbb{R}^n$ and $y \geq 0$ by the limit condition on u (hence, on the v^t). Moreover, $v_{x_n} > 0$ in $\overline{\mathbb{R}^{n+1}_+}$ by Lemma 4.1.2. Differentiating v^t with respect to t, we find

$$\partial_t v^t(x, y) = v_{x_n}(x', x_n + t, y) > 0 \tag{4.32}$$

for all $x = (x', x_n) \in \mathbb{R}^n$, $x' := (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, and $y \geq 0$. Then a direct application of (4.31) and the dominated convergence theorem shows that

$$\lim_{t\to\infty} J^+_{s,C_R}(v^t) = 0. \tag{4.33}$$

Our goal is to compute and bound $\partial_t J^+_{s,C_R}(v^t)$. Differentiating $J^+_{s,C_R}(v^t)$ with respect to t, we find

$$\partial_t J^+_{s,C_R}(v^t) = d_s \int_{C_R} y^a \nabla v^t \nabla(\partial_t v^t)\, dx\, dy + \int_{B_R\times\{0\}} F'(v^t(x,0))\partial_t v^t(x,0)\, dx := I_1 + I_2. \tag{4.34}$$

Consider the unit outer normal ν to the lateral boundary of C_R. Computing, we see that $\nu(x,y) = \frac{(x_1,\ldots,x_n,0)}{R}$. Thus, on the lateral boundary of C_R, we have $\partial_\nu v^t = \nabla v^t(x,y)\cdot\nu(x,y) = \nabla_x v^t(x,y)$. On the top and bottom of the cylinder C_R, the unit outer normal is the upward pointing and downward pointing unit vector in the y direction, respectively. Then, integrating by parts in I_1 and noting that v^t solves (4.5), we find

$$\begin{aligned} I_1 = &- d_s \lim_{y\to 0^+} \int_{B_R\times\{0\}} y^a v^t_y \partial_t v^t\, dx + \\ &d_s \int_0^R y^a\, dy \int_{\partial B_R} \nabla_x v^t \partial_t v^t\, dS^{n-1} + d_s \int_{B_R\times\{R\}} y^a v_y \partial_t v^t\, dx. \end{aligned}$$

Since

$$-d_s \lim_{y\to 0^+} \int_{B_R\times\{0\}} y^a v^t_y \partial_t v^t\, dx = -\int_{B_R\times\{0\}} F(v^t(x,0))\partial_t v^t\, dx,$$

we conclude that

$$\begin{aligned} \partial_t J^+_{s,C_R}(v^t) &= d_s \int_0^R y^a\, dy \int_{\partial B_R} \nabla_x v^t \partial_t v^t\, dS^{n-1} + d_s \int_{B_R\times\{R\}} y^a v_y \partial_t v^t\, dx \\ &\geq -C d_s \int_0^1 y^a\, dy \int_{\partial B_R} \partial_t v^t\, dS^{n-1} - \\ &\quad C d_s \int_1^R y^{a-1}\, dy \int_{\partial B_R} \partial_t v^t\, dS^{n-1} - C d_s R^{-2s} \int_{B_R\times\{R\}} \partial_t v^t\, dx, \end{aligned} \tag{4.35}$$

for some constant C, where we have split the integral over the cylinder into two integrals, one from height 0 to 1 and one from height 1 to R, and applied the gradient bounds (4.27), (4.28), and (4.29). In addition, the constant C is bounded away from zero. Then, for every $T > 0$,

$$\begin{aligned}
J^+_{s,C_R}(v) &= J^+_{s,C_R}(v^T) - \int_0^T \partial_t J^+_{s,C_R}(v^t)\,dt \\
\text{(By (4.35))} &\leq J^+_{s,C_R}(v^T) + Cd_s \int_0^T dt \int_0^1 y^a\,dy \int_{\partial B_R} \partial_t v^t\,dS^{n-1} + \\
&\quad Cd_s \int_0^T dt \int_1^R y^{a-1}\,dy \int_{\partial B_R} \partial_t v^t\,dS^{n-1} + \\
&\quad Cd_s R^{-2s} \int_0^T dt \int_{B_R\times\{R\}} \partial_t v^t\,dx \\
&= J^+_{s,C_R}(v^T) + Cd_s \int_{\partial B_R} dS^{n-1} \int_0^1 y^a\,dy \int_0^T \partial_t[v^t]\,dt + \\
&\quad Cd_s \int_{\partial B_R} dS^{n-1} \int_1^R y^{a-1}\,dy \int_0^T \partial_t[v^t]\,dt + \\
&\quad Cd_s R^{-2s} \int_{B_R\times\{R\}} dx \int_0^T \partial_t[v^t]\,dt \\
&= J^+_{s,C_R}(v^T) + Cd_s \int_{\partial B_R} dS^{n-1} \int_0^1 y^a(v^T - v^0)\,dy + \\
&\quad Cd_s \int_{\partial B_R} dS^{n-1} \int_1^R y^{a-1}(v^T - v^0)\,dy + \\
&\quad Cd_s R^{-2s} \int_{B_R\times\{R\}} (v^T - v^0)\,dx \\
\text{(By the } L^\infty \text{ bounds)} &\leq J^+_{s,C_R}(v^T) + Cd_s R^{n-1}\left(\int_0^1 y^{1-2s}\,dy + \int_1^R y^{-2s}\,dy\right) + Cd_s R^{n-2s} \\
&= J^+_{s,C_R}(v^T) + \frac{Cd_s}{2(s-1)} R^{n-1} + Cd_s R^{n-2s} \int_{\frac{1}{R}}^1 \rho^{-2s}\,d\rho + Cd_s R^{n-2s},
\end{aligned}$$

where we have applied the change of variables $y \mapsto \frac{\rho}{R}$ in the second integral in the sum above. Using the fact that $d_s \approx 2(s-1)$ as $s \uparrow 1$ and the fact that $s_0 < s < 1$, we may write

$$J^+_{s,C_R}(v) \leq J^+_{s,C_R}(v^T) + CR^{n-1} + C\cdot 2(1-s)R^{n-2s}\int_{\frac{1}{R}}^1 \rho^{-2s}\,d\rho + C\cdot(1-s)R^{n-2s}. \quad (4.36)$$

We conclude the proof by considering the three cases individually.

- $s_0 < s < \frac{1}{2}$: In this case,

$$\int_{\frac{1}{R}}^{1} \rho^{-2s}\, d\rho = \frac{1}{1-2s}(R^{2s-1}-1).$$

Combining this with the fact that $\frac{1}{1-2s} > 1$, $n-2s > n-1$, and $R > 2$, we obtain from (4.36) the estimate

$$J^{+}_{s,C_R}(v) \leq J^{+}_{s,C_R}(v^T) + C\frac{R^{n-2s}}{1-2s}.$$

- $s = \frac{1}{2}$: We have

$$\int_{\frac{1}{R}}^{1} \rho^{-2s}\, d\rho = \log R.$$

Since $2\log R > 1$ for all $R > 2$, (4.36) gives

$$J^{+}_{s,C_R}(v) \leq J^{+}_{s,C_R}(v^T) + CR^{n-1}\log R.$$

- $\frac{1}{2} < s < 1$: We have

$$\int_{\frac{1}{R}}^{1} \rho^{-2s}\, d\rho = \frac{1}{2s-1}(1-R^{2s-1}).$$

Since, in this case, we also have $n-2s < n-1$ and $\frac{1}{2s-1} > 1$, we find

$$J^{+}_{s,C_R}(v) \leq J^{+}_{s,C_R}(v^T) + C\frac{R^{n-1}}{2s-1}.$$

Taking the limit as $T \to \infty$ in each case and applying (4.33), we obtain the estimates (4.24), (4.25), and (4.26). This concludes the proof. $\square$

The estimates in Proposition 4.1.2 are sharp in the sense that they are bounded below by the same quantity multiplied by a smaller positive constant. To see this, note that, for every bounded solution u of (4.3), the energy is also bounded below by $C_1 R^{n-2s}$ for some constant $C_1 > 0$ and $s \in (0, \frac{1}{2})$ as a consequence of the nonlocal monotonicity formula in Theorem 4.3.2 below. When $s = \frac{1}{2}$, sharpness follows from the fact that the estimate is sharp in one dimension, hence, for layer solutions (see Section 2.1 in [7]). For $\frac{1}{2} < s < 1$, we may apply Fubini's theorem to obtain boundedness from below. Indeed, let v be a one-dimensional layer solution of (4.5). Then $v := v(x \cdot \tau, y)$ where $\tau \in \mathbb{R}^n$ is a unit vector, so for $R > 1$ we see that

$$
\begin{aligned}
J^+_{s,C_R}(v) &\geq C_2 J^+_{s,(-R,R)^n \times (0,R)}(v) \\
&= C_2 d_s \int_{(-R,R)^n \times (0,R)} \frac{1}{2} y^a |\nabla v(x_n, y)|^2 \, dx \, dy + \int_{(-R,R)^n} F(v(x_n, 0)) \, dx_n \\
&= C_2 R^{n-1} \left(d_s \int_0^R y^a \, dy \int_{-R}^R \frac{1}{2} |\nabla v(x_n, y)|^2 \, dx_n + \int_{-R}^R F(v(v_n, 0)) \, dx_n \right) \\
&\geq C_2 R^{n-1} \left(d_s \int_0^1 y^a \, dy \int_{-1}^1 \frac{1}{2} |\nabla v(x_n, y)|^2 \, dx_n \right) \\
&= C_2 R^{n-1}.
\end{aligned}
$$

since ∇v is bounded and integrable in $\mathbb{R}^{n+1}_+$ and $F \geq 0$, the positive constant C_2 depending on n and s. Hence, $J^+_{s,C_R} \geq C_2 R^{n-1}$ for $\frac{1}{2} < s < 1$.

Now that we have nonlocal versions of Proposition 3.2.2 and Proposition 3.2.3, the fractional De Giorgi conjecture is within reach, at least for dimensions $n = 2$ and $n = 3$. However, before we prove the conjecture in these cases, we discuss a generalization of the estimates in Proposition 4.1.2.

It is natural to wonder whether the estimates in Proposition 4.1.2 can be extended to a more general class of solutions. We can, in fact, by considering global minimizers of (4.3). For the proposition that follows, we introduce the constant c_u, dependent on u, given by

$$
c_u := \min\{F(s) : \inf_{\mathbb{R}^n} u \leq r \leq \sup_{\mathbb{R}^n} u\}.
$$

The following proposition holds.

Proposition 4.1.3 (Theorem 1.2 in [7]). *Let $f \in C^{1,\alpha}(\mathbb{R}^n)$ with $\alpha > \max\{0, 1-2s\}$, and let u be a bounded global minimizer of* (4.3)*. Let v be the s-extension of u in $\mathbb{R}^{n+1}_+$. Then, for all $R > 2$,*

$$J^{c_u}_{s,C_R} := d_s \int_{C_R} \frac{1}{2} y^a |\nabla v|^2 \, dx \, dy + \int_{B_R \times \{0\}} (F(u(x)) - c_u) \, dx \leq CR^{n-2s} \int_{\frac{1}{R}}^{1} \rho^{-2s} \, d\rho, \tag{4.37}$$

where C is a positive constant depending only on n, s, $\|f\|_{C^{1,\alpha}[(\inf_{\mathbb{R}^n} u, \sup_{\mathbb{R}^n} u)]}$, and $\|u\|_{L^\infty(\mathbb{R}^n)}$. As a consequence, for some constant C depending on the same quantities as above, we have

$$J^{c_u}_{s,C_R} \leq CR^{n-2s} \text{ if } 0 < s < \frac{1}{2}, \tag{4.38}$$

$$J^{c_u}_{s,C_R} \leq CR^{n-1} \log R \text{ if } s = \frac{1}{2}, \tag{4.39}$$

$$J^{c_u}_{s,C_R} \leq CR^{n-1} \text{ if } \frac{1}{2} < s < 1. \tag{4.40}$$

For $s = \frac{1}{2}$, the estimate was proved by Cabré and Cinti in [7]. For the case $s \in (0,1)$, these results were announced by Cinti in their Ph.D. Thesis (see [17]). These estimates are sharp (see Remark 1.3 in [8]). We also note that, in dimension $n = 3$, the energy estimate (4.37) holds also for bounded monotone solutions without the limit assumption imposed by layer solutions. These solutions can only be guaranteed to be minimizers among a certain class of functions (see Proposition 6.2 in [8]), but could fail to be global minimizers in the sense of Definition 4.02. We additionally note that the estimates given in Proposition 4.1.1 for layer solutions are uniform as $s \uparrow 1$, whereas the estimates in Proposition 4.1.2 are not. Neither are uniform for s near zero. The estimates above have also been proven by Savin and Valdinoci without reliance on the extension problem (see [53]).

With the above propositions at hand, we can now prove the fractional De Giorgi conjecture for layer solutions via the extension problem outlined in Chapter 2.

4.2 The Fractional De Giorgi Conjecture via the Extension Problem

Here, we prove the fractional De Giorgi conjecture for layer solutions u in dimensions $n = 2$ (general s) and $n = 3$ with $s \in [\frac{1}{2}, 1)$, using the extension problem introduced in Chapter 2. The proof for the these cases proceeds in a similar manner as the proof for the local case presented in Chapter 3. We first prove the conjecture for dimension $n = 2$ with general s. We then proceed to the case $n = 3$ with $s \in [\frac{1}{2}, 1)$. In the following section, we will discuss the case $n = 3$ with $s \in (0, \frac{1}{2})$. We will not prove the conjecture in this case since the proof relies on techniques from the theory of minimal surfaces and is thereby outside of the scope of this project. We will, however, provide the reader with an overview of how one obtains the result in this case.

4.2.1 Dimension $n = 2$ with $s \in (0, 1)$ and Dimension $n = 3$ with $s \in [\frac{1}{2}, 1)$

We begin our study of the fractional De Giorgi conjecture via the extension problem by following the work of Cabré and Sire in [10] for dimension $n = 2$. In this section, we will consider the problem (4.18) along with the associated energy (4.19). In particular, we will focus on layer solutions of (4.18), in line with Definition 4.0.2. We will also assume that $f \in C^{1,\alpha}(\mathbb{R})$ for some $\alpha > \max\{0, 1 - 2s\}$, where $s \in (0, 1)$, and that the potential F satisfies the conditions (4.13) and (4.14).

Dimension $n = 2$

Using Proposition 4.1.1 and Proposition 4.1.2, we may immediately prove the fractional De Giorgi conjecture in dimension $n = 2$ for each $s \in (0, 1)$.

Theorem 4.2.1 (Theorem 2.12 in [10]). *Let u be a layer solution of*

$$(-\Delta)^s u = f(u) \text{ in } \mathbb{R}^2$$

which is monotone in the variable x_2. Then, u is a function of one variable. In particular,

$$u(x_1, x_2) = g_0(\cos(\theta)x_1 + \sin(\theta)x_2) \text{ in } \mathbb{R}^2 \tag{4.41}$$

for some angle $\theta \in [0, 2\pi)$ and some solution g_0 of the one-dimensional problem with the same nonlinearity f. Moreover, $\nabla u \cdot \tau > 0$ everywhere, where we have set $\tau := (\cos\theta, \sin\theta)$.

Observe that the s-extension v of u satisfies

$$\operatorname{div}(y^a \nabla w) = 0 \text{ in } \mathbb{R}^{n+1}_+ \text{ for each } n \geq 1,$$

so that $v \in C^\infty(\mathbb{R}^{n+1}_+)$ by standard elliptic regularity. Thus, for each $i = 1, \ldots, n$, v_{x_i} solves the following problem with linearized boundary equation:

$$\begin{cases} \operatorname{div}(y^a w) = 0 \text{ in } \mathbb{R}^{n+1}_+ \\ -\lim_{y\to 0^+} y^a w_y - (1+a)^{-1} f'(v(x,0))w = 0 \text{ on } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{4.42}$$

For each $i = 1, \ldots, n-1$, consider the function $\sigma_i := \frac{v_{x_i}}{v_{x_n}}$ and note that σ_i is well-defined since $v_{x_n} > 0$ in $\overline{\mathbb{R}^{n+1}_+}$ by Lemma 4.1.2. In addition, we see that $\sigma_i \in H^1_{loc}(\overline{\mathbb{R}^{n+1}_+}, y^a)$ due to the regularity inherited from problems (4.18) and (4.42). Thus, we may apply Proposition 4.1.1 with $\varphi := v_{x_n}$ and $\sigma := \sigma_i$. These observations allow us to formulate a straightforward proof of Theorem 4.2.1 resembling the proof in the local case.

Proof of Theorem 4.2.1. Set $\varphi := v_{x_2}$ and $\sigma := \frac{v_{x_1}}{\varphi}$ where v is the s-extension of u in $\mathbb{R}^3_+$. We show that σ is constant.

Computing, we see that

$$\varphi^2 \nabla \sigma = \varphi \nabla v_{x_1} - v_{x_1} \nabla \varphi.$$

We claim that

$$\operatorname{div}(y^a \varphi^2 \nabla \sigma) = 0 \text{ in } \mathbb{R}^3_+.$$

Indeed, by direct computation we find

$$\begin{aligned}
\operatorname{div}(y^a \varphi^2 \nabla \sigma) &= \operatorname{div}(y^a \varphi \nabla v_{x_1}) - \operatorname{div}(y^a v_{x_1} \nabla \varphi) \\
&= \operatorname{div}(y^a \nabla v_{x_1})\varphi - \operatorname{div}(y^a \nabla \varphi) v_{x_1} \\
&= 0 \text{ in } \mathbb{R}^3_+
\end{aligned}$$

since both v_{x_1} and φ solve $\operatorname{div}(y^a \nabla w) = 0$ in $\mathbb{R}^3_+$. In addition,

$$-\lim_{y \to 0^+} y^a \sigma_y = 0 \text{ on } \partial \mathbb{R}^3_+$$

since

$$-\lim_{y \to 0^+} y^a \sigma_y = -\lim_{y \to 0^+} \frac{y^a v_{x_1 y} \varphi - y^a \varphi_y v_{x_1 y}}{\varphi^2}$$

and φ and v_{x_1} satisfy the same linearized boundary condition in (4.42). Note that we are using the fact that φ is strictly positive.

Our goal is to apply Proposition 4.1.1 (nonlocal Liouville theorem) to deduce that σ is constant, provided

$$\int_{C_R} y^a (\varphi \sigma)^2 \leq C R^2$$

for all $R > 1$ and some constant C independent of R. Since $\varphi \sigma = v_{x_1}$, it suffices to show that

$$\int_{C_R} y^a |\nabla v|^2 \leq C R^2. \tag{4.43}$$

In fact, after examining the proof of Proposition 4.1.1, it is apparent that it suffices for (4.43) to hold for all $R > 2$. Thus, we may apply the energy estimates (4.24), (4.25), and (4.26) in Proposition 4.1.2 for s in each case and use the fact that $n = 2$ and $R > 2$ to conclude (4.43) holds. Thus, σ is constant.

Since σ is constant, we see that $v_{x_1} = cv_{x_2}$ for some constant c. In particular,

$$\nabla v \cdot (1, -c, 0) = v_{x_1} - cv_{x_2} = 0$$

so that v is constant along the direction $(1, -c, 0)$. Note that $(c, 1, 0) \cdot (1, -c, 0) = 0$ so that $(c, 1, 0)$ is perpendicular to $(1, -c, 0)$. Then, $u(x) = v(x, 0)$ depends only on the variable parallel to $(c, 1)$. More precisely,

$$\begin{aligned} u(x_1, x_2) &= g\Big((x_1, x_2) \cdot \frac{1}{\sqrt{c^2+1}}(c, 1)\Big) \\ &= g\left(\frac{cx_1}{\sqrt{c^2+1}}, \frac{x_2}{\sqrt{c^2+1}}\right) \\ &= g(\cos(\theta)x_1, \sin(\theta)x_2) \end{aligned}$$

for some angle $\theta \in [0, 2\pi)$. Set $\tau := (\cos\theta, \sin\theta)$. Then,

$$\nabla u \cdot \tau = \frac{v_{x_2}}{\sqrt{c^2+1}} > 0$$

on $\mathbb{R}^2$, so the proof is complete. $\square$

We have thereby proven the fractional De Giorgi conjecture for layer solutions in dimension $n = 2$. Using the same technique, we may extrapolate this result to dimension $n = 3$ for $s \in [\frac{1}{2}, 1)$, though the case $s = \frac{1}{2}$ requires quite a bit more work.

Dimension $n = 3$ with $s \in [\frac{1}{2}, 1)$

Following the proof above, we see that the computation at the beginning remains valid in dimension $n = 3$, aside from slight adjustments. Furthermore, for $s \in (\frac{1}{2}, 1)$, the estimate (4.43) continues to hold by (4.26) so we may obtain the result in this case.

For $s = \frac{1}{2}$ the estimate (4.25) is no longer satisfactory in that we cannot apply Proposition 4.1.1 due to the factor of $\log R$. In order to rectify this, we prove a more general version of Proposition 4.1.1, given by Cabré and Cinti in [7], which is the nonlocal version of Theorem 3.2.4.

To do so, the authors define

$$\mathcal{F} := \left\{ F : \mathbb{R}^+ \to \mathbb{R}^+ : F \text{ is nondecreasing and } \int_2^\infty \frac{1}{RF(R)}\, dR = \infty \right\}, \tag{4.44}$$

which is nonempty since it includes the function $F(R) = \log R$. Note that this is equivalent to the condition on F in (3.66). To see this, observe that, since the function $j \mapsto F(2^{j+1})$ is nondecreasing in j, we have that

$$\sum_{j=3}^\infty \frac{1}{F(2^{j+1})} \leq \int_2^\infty \frac{ds}{F(2^{s+1})} = \frac{1}{\log 2} \int_8^\infty \frac{dR}{RF(R)} \leq \sum_{j=2}^\infty \frac{1}{F(2^{j+1})},$$

which proves the claim.

Theorem 4.2.2 (Proposition 6.1 in [7]). *Let $\varphi \in L^\infty_{loc}(\mathbb{R}^{n+1}_+)$ be a positive function. Suppose that $\sigma \in H^1_{loc}(\overline{\mathbb{R}^{n+1}_+}, y^a)$ satisfies*

$$\begin{cases} -\sigma \operatorname{div}(\varphi^2 \nabla \sigma) \leq 0 \textit{ in } \mathbb{R}^{n+1}_+, \\ -\lim_{y \to 0^+} \sigma y^a \sigma_y \leq 0 \textit{ on } \partial \mathbb{R}^{n+1}_+ \end{cases} \tag{4.45}$$

in the weak sense. Let the following condition hold:

$$\limsup_{R \to \infty} \frac{1}{R^2 F(R)} \int_{C_R} y^a (\varphi \sigma)^2 \, dx\, dy = C \tag{4.46}$$

for some $F \in \mathcal{F}$ and C a positive constant independent of R. Then σ is constant.

The proof is identical to that of Theorem 3.2.4 in Chapter 3 after the initial computation, aside from minor alterations. As such, we only demonstrate the initial computation. We denote by ν

the unit outer normal to C_R and ν_+ the unit outer normal to $C_R^+ := \partial C_R \setminus (B_R \times \{y = 0\})$. The measure on ∂C_R^+ we denote by $dS^+ := dS^{n-1}\, dy$, while the measure on ∂C_R we denote (formally) by dS_R.

Proof of Theorem 4.2.2. Since σ satisfies (4.45), we see that

$$\begin{aligned}\operatorname{div}(\sigma y^a \varphi^2 \nabla \sigma) &= y^a \varphi^2 |\nabla \sigma|^2 + \sigma \operatorname{div}(y^a \varphi^2 \nabla \sigma) \\ &\geq y^a \varphi^2 |\nabla \sigma|^2 \text{ in } \mathbb{R}^{n+1}_+. \end{aligned} \tag{4.47}$$

Also,

$$\int_{\partial C_R^+} y^a \sigma \varphi^2 \partial_\nu \sigma \, dS^+ \leq \left(\int_{\partial C_R^+} y^a \varphi^2 |\nabla \sigma|^2 \, dS^+ \right)^{\frac{1}{2}} \cdot \left(\int_{\partial C_R^+} y^a (\varphi \sigma)^2 \, dS^+ \right)^{\frac{1}{2}} \tag{4.48}$$

by the Cauchy-Schwarz inequality. Integrating over (4.47) in C_R, we see that

$$\int_{C_R} \operatorname{div}(\sigma y^a \varphi^2 \nabla \sigma) \, dx\, dy \geq \int_{C_R} y^a \varphi^2 |\nabla \sigma|^2 \, dx\, dy.$$

Integrating by parts on the left-hand side, we have

$$\begin{aligned}\int_{C_R} \operatorname{div}(\sigma y^a \varphi^2 \nabla \sigma) \, dx\, dy &= \int_{C_R} \sigma \operatorname{div}(y^a \varphi^2 \nabla \sigma) \, dx\, dy + \int_{C_R} y^a \varphi^2 |\nabla \sigma|^2 \, dx\, dy \\ &= \int_{\partial C_R} \sigma y^a \varphi^2 \langle \nabla \sigma, \nu \rangle \, dS_R. \end{aligned}$$

Hence,

$$\int_{\partial C_R} \sigma y^a \varphi^2 \langle \nabla \sigma, \nu \rangle \, dS_R \geq \int_{C_R} y^a \varphi^2 |\nabla \sigma|^2 \, dx\, dy. \tag{4.49}$$

Focusing on the left-hand side in (4.49), we see that

$$\begin{aligned}\int_{\partial C_R} \sigma y^a \varphi^2 \langle \nabla\sigma, \nu\rangle \, dS_R &= \int_{\partial C_R^+} \sigma y^a \varphi^2 \langle \nabla\sigma, \nu_+\rangle \, dS^+ - \lim_{y\to 0^+} \int_{\partial C_R^0} \sigma y^a \varphi^2 \sigma_y \, dS \\ &\le \int_{\partial C_R^+} \sigma y^a \varphi^2 \langle \nabla\sigma, \nu_+\rangle \, dS^+ \end{aligned} \tag{4.50}$$

by the boundary condition in (4.45). Set

$$D(R) := \int_{C_R} y^a \varphi^2 |\nabla\sigma|^2 \, dx \, dy.$$

Then,

$$\begin{aligned} D'(R) &= \int_0^R \int_{\partial B_R} y^a \varphi^2 |\nabla\sigma|^2 \, dS^{n-1} \, dy + \int_{B_R\times\{y=R\}} y^a \varphi^2 |\nabla\sigma|^2 \, dx \\ &= \int_{\partial C_R^+} y^a \varphi^2 |\nabla\sigma|^2 \, dS^+ + \int_{B_R\times\{y=R\}} y^a \varphi^2 |\nabla\sigma|^2 \, dx \\ &\ge \int_{\partial C_R^+} y^a \varphi^2 |\nabla\sigma|^2 \, dS^+ \end{aligned}$$

by the Leibniz formula and the fact that the second integral in the sum above is positive. By (4.49) and (4.50), we have

$$D(R) \le D'(R)^{\frac{1}{2}} \left(\int_{\partial C_R^+} y^a (\varphi\sigma)^2 \, dS^+ \right)^{\frac{1}{2}}. \tag{4.51}$$

Now, arguing as in the proof of Theorem 3.2.4 (with minor adjustments) we obtain the result. □

Corollary 4.2.1. *Let $\varphi \in L^\infty_{loc}(\overline{\mathbb{R}^{n+1}_+})$ be a positive function. Suppose that $\sigma \in H^1_{loc}(\overline{\mathbb{R}^{n+1}_+})$ is such that*

$$\begin{cases} -\sigma \operatorname{div}(y^a \varphi^2 \nabla\sigma) \le 0 \text{ in } \mathbb{R}^{n+1}_+ \\ -\lim_{y\to 0^+} \sigma y^a \sigma_y \le 0 \text{ on } \partial\mathbb{R}^{n+1}_+ \end{cases} \tag{4.52}$$

in the weak sense. Assume that for every $R > 1$,

$$\int_{C_R} y^a (\sigma\varphi)^2 \, dx \, dy \le C R^2 \log R \tag{4.53}$$

for some constant C independent of R. Then σ is constant.

The proof of Corollary 4.2.1 is omitted, as it is identical to that of Corollary 3.2.1.

Theorem 4.2.2 allows for a simple extension of Theorem 4.2.1 to the dimension $n = 3$ with $s \in [\frac{1}{2}, 1)$.

Theorem 4.2.3 (Theorem 1.4 in [7])**.** *Let u be a layer solution of*

$$(-\Delta)^s u = f(u) \textit{ in } \mathbb{R}^3$$

with $s \in [\frac{1}{2}, 1)$ which is monotone in the x_3 direction. Then, u depends only on one variable. In particular, there exists a unit vector $\tau \in \mathbb{R}^3$ and some solution g_0 of the one-dimensional problem with the same nonlinearity such that

$$u(x) = g_0(\tau \cdot x) \textit{ for all } x \in \mathbb{R}^3.$$

Furthermore,

$$\nabla u \cdot \tau > 0 \textit{ on } \mathbb{R}^3,$$

with $\tau := (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ for some angles ϕ and θ satisfying $0 \leq \phi \leq \pi$ and $0 \leq \theta < 2\pi$.

Proof. Fix $i \in \{1, 2\}$ and let $\varphi := v_{x_3}$ and $\sigma := \frac{v_{x_i}}{\varphi}$ where v is the s-extension of u in $\mathbb{R}^4_+$. We show that σ is constant. Repeating the same computation in $\mathbb{R}^4_+$ as in the beginning of the proof of Theorem 4.2.1 with v_{x_1} replaced by v_{x_i} gives

$$\operatorname{div}(y^a \varphi^2 \nabla\sigma) = 0 \text{ in } \mathbb{R}^4_+. \tag{4.54}$$

Moreover,

$$\lim_{y \to 0^+} y^a \sigma_y = 0 \text{ on } \partial\mathbb{R}^4_+ \tag{4.55}$$

since

$$-\lim_{y\to 0^+} y^a\sigma_y = -\lim_{y\to 0^+} \frac{y^a v_{x_i y}\varphi - y^a\varphi_y v_{x_i y}}{\varphi^2}$$

and φ and v_{x_i} satisfy the same linearized boundary condition in (4.42). We are again using the fact that φ is strictly positive. Note that the estimates (4.25) and (4.26) in Proposition 4.1.2 hold, where $s = \frac{1}{2}$ and $s \in (\frac{1}{2}, 1)$, respectively. Then, by (4.54) and (4.55), we see that the hypotheses for Proposition 4.1.1 and Corollary 4.2.1 are met so σ is constant for $s = \frac{1}{2}$ and $s \in (\frac{1}{2}, 1)$. Since $i \in \{1, 2\}$ is fixed, we deduce that $\sigma_i := \frac{v_{x_i}}{\varphi}$ is constant for each $i \in \{1, 2\}$ and each $s \in [\frac{1}{2}, 1)$. It follows that $v_{x_i} = c_i v_{x_3}$ for each $i \in \{1, 2\}$. We then find that the trace $u(x) = v(x, 0)$ of v on $\partial\mathbb{R}^4_+ = \mathbb{R}^3$ is constant along the directions in $\mathbb{R}^3$ perpendicular to $(c_1, c_2, 1)$ so that u is one-dimensional. Setting

$$\tau := \frac{1}{\sqrt{c_1^2 + c_2^2 + 1}}(c_1, c_2, 1) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

for some angles $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi)$, we see that

$$u(x) = g_0(x \cdot \tau) \text{ for all } x \in \mathbb{R}^3,$$

where g_0 is some solution of the one-dimensional problem with the same nonlinearity and τ as desired. Moreover,

$$\nabla u \cdot \tau = \frac{v_{x_3}}{\sqrt{c_1^2 + c_2^2 + 1}} > 0 \text{ on } \mathbb{R}^3,$$

concluding the proof. □

Together, Theorem 4.2.1 and Theorem 4.2.3 resolve the fractional De Giorgi conjecture for layer solutions in dimension $n = 2$, as well as dimension $n = 3$ with $s \in [\frac{1}{2}, 1)$. We note that we are not able to obtain the desired conclusion for the case dimension $n = 3$ and $s \in (0, \frac{1}{2})$ since the estimate (4.24) is no longer suitable in this case. To see this, we simply observe that $R^{n-2s} > R^2$ for all $R > 1$ after setting $n = 3$ and $s \in (0, \frac{1}{2})$ so that Proposition 4.1.1 is not applicable. Furthermore, setting $F(R) = R^{n-2(1-s)}$ for $R > 1$, we find that $RF(R) = R^{n-1-2s} = R^{2(1-s)}$

when $n = 3$. Then, for $s \in (0, \frac{1}{2})$, we see that $2(1-s) > 1$ so that $F \notin \mathcal{F}$. It follows that Theorem 4.2.2 may not be applied either. We also see that the above techniques fail for $4 \leq n \leq 8$ for similar reasons as in the local case.

4.2.2 Dimension $n = 3$ with $s \in (0, \frac{1}{2})$

Thus far, we have proven the fractional De Giorgi conjecture for layer solutions in dimension $n = 2$ for each $s \in (0, 1)$ and dimension $n = 3$ with $s \in [\frac{1}{2}, 1)$. We henceforth refer to the case $s \in [\frac{1}{2}, 1)$ as the *weakly nonlocal regime* while the case $s \in (0, \frac{1}{2})$ we call the *genuinely nonlocal regime*. In this section, we will work specifically with the fractional Allen-Cahn equation, that is, equation (4.1). The fractional De Giorgi conjecture in dimension $n = 3$ for layer solutions of (4.1) in the genuinely nonlocal regime has been proven by Dipierro et al. in [19], whose work was inspired by Savin's in the local case for dimensions $4 \leq n \leq 8$ (see [49]). More recently, Dipierro et al. have proven the result without the limit assumption in [18]. We will follow their work, in the hopes of providing an overview of the ideas used in higher dimensions, as well as the techniques used for extended fractional De Giorgi conjectures. The theorem to be considered is as follows.

Theorem 4.2.4 (Theorem 1.1 in [18])**.** *Let $n \leq 3$ with $s \in (0, \frac{1}{2})$ and $u \in C^2(\mathbb{R}^n, [-1, 1])$ be a solution of*

$$(-\Delta)^s u = u - u^3 \textit{ in } \mathbb{R}^n, \tag{4.56}$$

with $u_{x_n} > 0$ in $\mathbb{R}^n$. Then u is a function of one-variable.

Recalling the inadequacy of the estimate (4.24) for the case $n = 3$ and $s \in (0, \frac{1}{2})$, as well as the fact that this estimate is sharp, it becomes clear that the extension technique alone is not suitable if we wish to obtain De Giorgi type results in dimension $n = 3$ in the genuinely nonlocal regime.

Before we continue, we quickly recast the phase transition model described in the introduction of Chapter 3, as these ideas will briefly resurface. This time, however, we will focus on the nonlocal formulation. In this context, equation (4.56) represents a phase transition subject to long-range interactions. Precisely, the states $u = -1$ and $u = 1$ represent "pure phases" and equation (4.56) models the coexistence between intermediate phases and the separation between them.

As proved by Savin and Valdinoci in [52] and [53], if u is a local energy minimizer for (4.56) and $u_\epsilon(x) := u(\frac{x}{\epsilon})$, we have that u_ϵ approaches a pure phase step function as $\epsilon \downarrow 0$ having values in $\{-1, 1\}$. That is, we may write

$$\lim_{\epsilon \downarrow 0} u_\epsilon = \chi_E - \chi_{\mathbb{R}^n \setminus E},$$

up to subsequences, with the set E possessing a minimal interface criterion depending on s and a bifurcation[2] occuring at the threshold $s = \frac{1}{2}$. In particular, in the weakly nonlocal regime (i.e. $s \in [\frac{1}{2}, 1)$), the set E turns out to be a local minimizer for the classic perimeter functional (3.7). In other words, on a large scale, the weakly nonlocal regime is nearly indistinguishable from the classical case and, in spite of the nonlocality of equation (4.56), its limit interface behaves in a local way when $s \in [\frac{1}{2}, 1)$. In the genuinely nonlocal regime (i.e. $s \in (0, \frac{1}{2})$), the set E turns out to be a local minimizer for the nonlocal perimeter functional

$$P_{s,\Omega}(w) := \frac{C(n,s)}{2} \int\int_{Q_\Omega} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}} \, dx \, dy, \tag{4.57}$$

where $\Omega \subset \mathbb{R}^n$ and $w : \Omega \to \mathbb{R}$, in the sense below.

Definition 4.2.1. We say that a set $E \subset \mathbb{R}^n$ is a *s-perimeter minimizer in* $\mathbb{R}^n$ if its characteristic function is a minimizer for the functional (4.57) among characteristic functions. That is, if

$$P_{s,B}(\chi_E) < \infty$$

and

$$P_{s,B}(\chi_E) \leq P_{s,B}(\chi_F)$$

for any ball $B \subset \mathbb{R}^n$ and any $F \subset \mathbb{R}^n$ such that $F \setminus B = E \setminus B$.

[2] A bifurcation occurs when a small smooth change made to the parameter values of a system causes a sudden qualitative or topological change in its behavior.

In this section, we adopt the framework of Dipierro et. al in [18]. Set

$$Q_\Omega := (\Omega \times \Omega) \cup (\Omega \times \Omega^c) \cup (\Omega^c \times \Omega),$$

where $\Omega \subset \mathbb{R}^n$ and Ω^c denotes the complement of Ω in $\mathbb{R}^n$. Consider the energy

$$J_{s,\Omega}(w) := \frac{C(n,s)}{2} \int\int_{Q_\Omega} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega F(w(x,0))\,dx \tag{4.58}$$

where F is as before and $w : \mathbb{R}^n \to \mathbb{R}$, and notice that (4.58) is simply (4.4) on a general domain Ω. We have the following:

Definition 4.2.2 (Local Minimizer)**.** We say that u is a *local minimizer of* (4.58) if, for any $R > 0$ and any $\phi \in C_0^\infty(B_R)$, it holds that

$$J_{s,B_R}(u) \leq J_{s,B_R}(u+\phi).$$

Here, B_R denotes the ball of radius R centered at the origin in $\mathbb{R}^n$.

Now, given any $w : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^{n+1}$, we may define the *extended energy* on a general domain by

$$J^+_{s,\Omega}(w) := \frac{d_s}{2} \int_{\Omega^+} y^a |\nabla w(x,y)|^2\,dx\,dy + \int_{\Omega_0} F(w(x,0))\,dx, \tag{4.59}$$

where $\Omega^+ := \Omega \cap \mathbb{R}^{n+1}_+$, $\Omega_0 := \Omega \cap \{y = 0\}$. Here $w := w(x,y)$ where $x \in \mathbb{R}^n$ and $y > 0$.

Definition 4.2.3 (Extended Local Minimizer)**.** We say that v is an *extended local minimizer* if, for any $R > 0$ and any $\phi \in C_0^\infty(B_R)$, it holds that

$$J^+_{s,\mathcal{B}_R}(v) \leq J^+_{s,\mathcal{B}_R}(v+\phi).$$

Here, $\mathcal{B}_R$ denotes the ball of radius R in $\mathbb{R}^{n+1}$ centered at the origin [3].

[3] Once again, the reader should compare the extended energy (4.59) to (4.6).

Suppose

$$(-\Delta)^s u = f(u) \text{ in } \mathbb{R}^n. \tag{4.60}$$

Then, by Proposition 2.5 in [18], we see that the s-extension v of u in $\mathbb{R}^{n+1}_+$ is a local minimizer in the sense of Definition 4.2.3 if and only if u is a local minimizer in the sense of Definition 4.2.2. The authors also used the notion of a stable solution.

Definition 4.2.4 (Stable Solution)**.** Let u be a solution of

$$(-\Delta)^s u = f(u) \text{ in } \mathbb{R}^n.$$

We say that u is *stable*[4] if

$$\int_{\mathbb{R}^{n+1}_+} y^a |\nabla \phi(x,y)|^2 \, dx \, dy - \int_{\mathbb{R}^n} f'(u(x)) \phi^2(x,0) \, dx \geq 0 \tag{4.61}$$

for all $\phi \in C_0^\infty(\mathbb{R}^{n+1})$.

In order to prove Theorem 4.2.4, Dipierro et al. first considered the two profiles of a given solution at infinity. Precisely, if $s \in (0,1)$ and $u \in C^2(\mathbb{R}^n, [-1,1])$ is a solution of (4.60) with $u_{x_n} > 0$ in $\mathbb{R}^n$, we set

$$\underline{u}(x') := \lim_{x_n \to -\infty} u(x', x_n) \text{ and } \overline{u}(x') := \lim_{x_n \to +\infty} u(x', x_n),$$

where $x' \in \mathbb{R}^{n-1}$. The following holds.

Lemma 4.2.1 (Lemma 4.1 in [18])**.** *Assume that dimension $n = 3$. Then, both $\underline{u}$ and $\overline{u}$ are one-dimensional and local minimizers.*

To prove Lemma 4.2.1, the authors pass the equation to the limit to find both $\underline{u}$ and $\overline{u}$ are stable solutions in $\mathbb{R}^2$. By the classification of solutions in $\mathbb{R}^2$ (see below), we find that $\underline{u}$ and $\overline{u}$ are

[4] See also Definition 4.2.5 below.

one-dimensional and monotone. Then, using the fact that stable solutions of (4.60) in $\mathbb{R}$ satisfying $u \in C^2(\mathbb{R}, [-1,1])$ and $u' \geq 0$ are local minimizers for each $s \in (0,1)$ (see Lemma 3.1 in [18]), local minimality follows.

Dipierro et al. then proved the lemmata below.

Lemma 4.2.2 (Lemma 6.1 in [18]). *Let $s \in (0,1)$ and $u \in C^2(\mathbb{R}^n, [-1,1])$ be a solution of* (4.60) *such that $u_{x_n} > 0$ in $\mathbb{R}^n$. Suppose $\underline{u}$ and $\overline{u}$ are local minimizers in $\mathbb{R}^{n-1}$. Then, the s-extension v of u is a local minimizer in $\mathbb{R}^{n+1}_+$ and u is a local minimizer in $\mathbb{R}^n$.*

Now, setting $f(u) = u - u^3$, Dipierro et al. [18] relied on two earlier results of Dipierro et al. (see [19]) concerning solutions u of (4.56).

Lemma 4.2.3 (Lemma 8.1 in [19]). *Let $u \in C^2(\mathbb{R}^n, [-1,1])$ be a minimizing solution of* (4.56) *in $\mathbb{R}^n$ with $s \in (0, \frac{1}{2})$. For each $\epsilon > 0$, set $u_\epsilon := u(\frac{x}{\epsilon})$. Then, there exists a nonempty set $E \subset \mathbb{R}^n$ which is a minimizer of the s-perimeter* (4.57) *in $\mathbb{R}^n$ and, up to a subsequence,*

$$u_\epsilon \to \chi_E - \chi_{E^c}$$

a.e. in $\mathbb{R}^n$ as $\epsilon \downarrow 0$. In addition, the sets $\{u_\epsilon \leq 1 - \kappa\}$ and $\{u_\epsilon \leq -1 + \kappa\}$ converge locally uniformly to E^c in the sense of the Hausdorff distance[5].

Lemma 4.2.4 (Lemma 8.3 in [19]). *Assume that E is a minimizer of the s-perimeter that is contained in some half-space. Then, either E is empty or E is a parallel half-space.*

Using the above results and the fact that Theorem 4.2.4 holds for layer solutions in dimension $n = 3$ with $s \in (0, \frac{1}{2})$ (see [19]), Dipierro et al. obtain the fractional De Giorgi conjecture in dimension $n = 3$ in the genuinely nonlocal regime. For the details, we refer the reader to the original paper.

[5]Let X and Y be two nonempty subsets of a metric space (M, d). We define their *Hausdorff distance* $d_H(X,Y)$ by

$$d_H(X,Y) = \max\left\{ \sup_{x \in X} \inf_{y \in Y} d(x,y), \sup_{y \in Y} \inf_{x \in X} d(x,y) \right\}.$$

Loosely speaking, two sets are close in the Hausdorff distance if every point of either set is close to some point of the other set.

4.2.3 Fractional Stability and Fractional Global Minimizers

Similar to the local case, and alluded to above, we may drop the assumption that v is a layer solution of (4.5) and replace this condition with looser hypotheses and still obtain De Giorgi type results for $(-\Delta)^s$. We quickly remark on two classes of solutions already discussed: global minimizers and stable solutions for the fractional Allen-Cahn equation. This time, we consider these solutions without the extension problem. Moreover, we focus on these results for the Allen-Cahn nonlinearity.

In this context, we define stability as follows.

Definition 4.2.5. A solution u of the fractional Allen-Cahn equation (4.56) is said to be *stable in* Ω if for all $\phi \in C_0^1(\Omega)$, we have

$$C(n,s)\int\int_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\Omega^c\times\Omega^c)} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega (3u^2-1)\phi^2\,dx \geq 0. \tag{4.62}$$

Remark 4.2.1. We subtract $\Omega^c \times \Omega^c$ from the domain of integration so that the expression is well-defined.

The one-dimensional symmetry of stable solutions was proved for general $s \in (0,1)$ by Sire and Valdinoci in [57] and Dipierro et al. in [19] when $n = 2$. In addition, a breakthrough was made by Figalli and Serra in [28] when $n = 3$ and $s = \frac{1}{2}$.

We may likewise define global minimizers of (4.56).

Definition 4.2.6. A solution u is said to be a *global minimizer* on a compact set $\Omega \subset \mathbb{R}^n$ if

$$J_{s,\Omega}(u) \leq J_{s,\Omega}(u+\phi), \text{ for all } \phi \in C_0^1(\Omega), \tag{4.63}$$

where this time $J_{s,\Omega}$ is given by

$$J_{s,\Omega}(w) := \frac{C(n,s)}{2}\int\int_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\Omega^c\times\Omega^c)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \frac{1}{4}\int_\Omega (1-u^2)^2\,dx. \tag{4.64}$$

Classifications of global minimizers of (4.2) have been given by Dipierro et al. in [19] and Savin in [51]. Recently, Chan et al. constructed counterexamples in [15, 16] to the fractional De Giorgi conjecture for global minimizers and solutions that are monotone in x_n in dimensions $n = 8, 9$, respectively, with $s \in (\frac{1}{2}, 1)$. Taken together, this implies that Savin's classifications of global minimizers and monotone solutions in [51] are optimal. For more on these results, we refer the reader to the original papers.

4.3 A Nonlocal Monotonicity Formula and Modica-type Estimate

We turn once more to the subject of monotonicity formulae, this time for the nonlocal operator $(-\Delta)^s$. Specifically, we present a Modica-type estimate due to Cabré and Sire in dimension $n = 1$ (see [9]). After proving this result, we develop a nonlocal Pohozaev identity which we use to obtain a nonlocal monotonicity formula in the spirit Modica due to Cabré and Cinti (see [8]).

4.3.1 A Nonlocal Modica-type Estimate in Dimension $n = 1$

In this section, we work with problem (4.18), that is,

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ -(1+a) \lim_{y \to 0^+} y^a v_y = f(v(x,0)) \text{ on } \partial \mathbb{R}^{n+1}_+, \end{cases}$$

as well as the energy functional

$$J^+_{B^+_R} := \int_{B^+_R} \frac{1}{2} y^a |\nabla w|^2 \, dx \, dy + \int_{B_R \times \{0\}} \frac{1}{1+a} F(w(x,0)) \, dx. \tag{4.65}$$

Note that (4.65) is simply (4.19) with the domain of integration in the first term replaced by B^+_R. From the expression above, we see that the Lagrangian is given by

$$L(q,p) = \frac{1}{2} \|p\|^2_{2,a} + W(q),$$

where

$$W(q) = \frac{1}{2}\|\partial_y q\|_{2,a}^2 + \frac{1}{1+a}F(q(0)), \text{ and}$$
$$\|w\|_{2,a}^2 = \int_0^\infty y^a |w(y)|^2 \, dy.$$

Here, the time variable is $\tau = x$, the position q is the function $v(\tau, \cdot)$ in the half-line $\{y \geq 0\}$, and the momentum is $p = q' = v_x(\tau, \cdot)$.

The Legendre transform[6] of L with respect to p then gives the Hamiltonian

$$\begin{aligned} H(q,p) &= \frac{1}{2}\|p\|_{2,a}^2 - W(q) \\ &= \int_0^\infty \frac{t^a}{2}[v_x^2(x,t) - v_y^2(x,t)] \, dt - \frac{1}{1+a}F(v(x,0)), \end{aligned} \tag{4.66}$$

which will play an important role in the sequel.

Using the Hamiltonian (4.66), we obtain an analogue in dimension $n = 1$ to the Modica estimate introduced in Chapter 3 for the Laplacian. The proofs that follow will rely on some regularity results and maximum principles for the problem (4.18) which we have listed in Appendix D for convenience. In addition, we use the notation $L_a := \operatorname{div}(y^a \nabla(\cdot))$ to ease computations.

We begin by establishing integrability of the quantity $y^a |\nabla v|^2$ in the variable y, where v solves (4.18), and show that differentiability in x under the integral sign in $\int_0^\infty t^a |\nabla v|^2 \, dt$ is permitted.

Lemma 4.3.1 (Lemma 5.1 in [9]). *Let $v \in L^\infty(\mathbb{R}_+^{n+1})$ be a bounded solution of* (4.18). *Then, for all $x \in \mathbb{R}^n$ we have*

$$\int_0^\infty t^a |\nabla v(x,t)|^2 \, dt < \infty. \tag{4.67}$$

In addition, the integral can be differentiated with respect to $x \in \mathbb{R}^n$ under the integral sign.

[6]Recall that, by definition, the Hamiltonian H is given by the Legendre transform of the Lagrangian L. That is,

$$H = \sum_i p_i \frac{\partial L}{\partial p_i} - L.$$

Furthermore,

$$\lim_{M\to\infty}\int_M^\infty t^a|\nabla v(x,t)|^2\,dt = 0 \tag{4.68}$$

uniformly for $x\in\mathbb{R}^n$*. If, in addition,* v *is a layer solution, then*

$$\lim_{x_n\to\infty}\int_0^\infty t^a|\nabla v(x,t)|^2\,dt = 0. \tag{4.69}$$

Proof. Differentiability under the integral sign in x and (4.67) follow directly from (D.2), since

$$y^a|\nabla v(x,y)|^2 \le Cy^{a-2}$$

for some constant C independent of $(x,y)\in\mathbb{R}^{n+1}_+$, where $a-2<-1$. For the same reason, the limit (4.68) holds uniformly for $x\in\mathbb{R}^n$. The limit (4.69) is obtained by writing

$$\int_0^\infty t^a|\nabla v(x,t)|^2\,dt = \int_0^M t^a|\nabla v(x,t)|^2\,dt + \int_M^\infty t^a|\nabla v(x,t)|^2\,dt$$

for M large and applying (4.68), as well as (D.8) and (D.9). □

Using Lemma 4.3.1, we may prove conservation of the Hamiltonian (4.66) in dimension $n=1$.

Lemma 4.3.2 (Conservation of Hamiltonian; Lemma 5.2 in [9])**.** *Let* $n=1$ *and assume that* v *is a layer solution of* (4.18)*. Then, for all* $x\in\mathbb{R}$ *we have*

$$\int_0^\infty t^a|\nabla v(x,t)|^2\,dt < \infty \tag{4.70}$$

and the following Hamiltonian identity holds:

$$(1+a)\int_0^\infty \frac{t^a}{2}[v_x^2(x,t)-v_y^2(x,t)]\,dt = F(v(x,0))-F(1). \tag{4.71}$$

As a consequence $F(1)=F(-1)$*.*

Proof. The integrability of $y^a|\nabla v|^2$ is a direct consequence of Lemma 4.3.1 above. Thus, we need only to establish identity (4.71). Consider the function

$$h(x) = \int_0^\infty \frac{t^a}{2}[v_x^2(x,t) - v_y^2(x,t)]\,dt. \tag{4.72}$$

Applying Lemma 4.3.1 once more, we may differentiate under the integral sign above to find

$$h'(x) = \int_0^\infty t^a(v_{xx}v_x - v_{xy}v_y)(x,t)\,dt. \tag{4.73}$$

We note that $L_a v = \partial_y(y^a v_y) + y^a v_{xx} = 0$. Integrating by parts, we have

$$\begin{aligned}
\int_0^\infty t^a(v_{xx}v_x - v_{xy}v_y)\,dt &= \int_0^\infty t^a v_{xx}v_x\,dt - \int_0^\infty t^a v_{xy}v_y\,dt \\
&= \int_0^\infty t^a v_{xx}v_x\,dt + \int_0^\infty \frac{\partial}{\partial y}(t^a v_y)v_x\,dt \\
&= \int_0^\infty [t^a v_{xx} + \frac{\partial}{\partial y}(t^a v_y)]v_x\,dt - y^a v_y v_x|_0^\infty \\
&= \lim_{y\to 0^+} y^a v_y v_x.
\end{aligned}$$

Hence, we see that

$$\begin{aligned}
h'(x) &= \lim_{y\to 0^+} y^a v_y v_x \\
&= \frac{1}{1+a}\frac{\mathrm{d}}{\mathrm{d}x}[F(v(x,0))]
\end{aligned}$$

by continuity and the fact that $F'(v(x,0)) = -\lim_{y\to 0^+} y^a v_y(x,0)$. It follows that the function $(1+a)h(x) - [F(v(x,0)) - F(1)]$ is constant in the variable x. Letting $x \to \infty$, Lemma 4.3.1 shows that this constant is zero. Indeed, since $\frac{y^a}{2}(v_x^2 - v_y^2)(x,y) \le y^a|\nabla v|^2(x,y)$, the limit (4.69) gives the result. We now let $x \to -\infty$ so that $F(v(x,0)) - F(1) \to F(-1) - F(1)$ since v is monotone increasing, ranging from -1 to 1. Using the same estimate as above, we conclude that $F(1) = F(-1)$, as desired. $\square$

Remark 4.3.1. We note that, if for each $s \in (0,1)$ u_s is a layer solution of

$$(-\partial_{xx})^s u_s = f(u_s) \text{ in } \mathbb{R}$$

such that $u_s(0) = 0$, then there exists a function $\overline{u}$ such that $\lim_{s\uparrow 1} u_s = \overline{u}$ in the uniform C^2 norm on every compact subset of $\mathbb{R}$. Furthermore, $\overline{u}$ is a layer solution of

$$-\overline{u}'' = f(\overline{u}) \text{ in } \mathbb{R}$$

with $\overline{u}(0) = 0$ (see Theorem 2.2(ii) in [9]). In particular, $\overline{u}$ satisfies the Hamiltonian equality

$$\frac{1}{2}(\overline{u}')^2 = F(\overline{u}) - F(1) \text{ in } \mathbb{R}.$$

In other words, in the limit as $s \uparrow 1$, we obtain the usual conservation of the Hamiltonian for the Laplacian in dimension $n = 1$ (see also Theorem 6.1 in [9]).

Combining Lemmas 4.3.1 and 4.3.2 allows us to prove a Modica-type estimate for the problem (4.18) when $n = 1$. We emphasize that the following estimate is pointwise in x and nonlocal.

Theorem 4.3.1 (Modica-type Estimate; Theorem 2.3 in [9])**.** *Let $n = 1$ and assume that v is a layer solution of* (4.18). *Then, for every $y \geq 0$ and all $x \in \mathbb{R}$, we have*

$$(1+a)\int_0^y \frac{t^a}{2}[v_x^2(x,t) - v_y^2(x,t)]\,dt < F(v(x,0)) - F(1). \tag{4.74}$$

Proof. Consider the function

$$w(x,y) = \int_0^y \frac{t^a}{2}[v_x^2 - v_y^2](x,t)\,dt, \tag{4.75}$$

and note that w is bounded in all of $\mathbb{R}^2_+$ by Lemma 4.3.1. Consider also the function

$$\overline{w}(x,y) = \frac{1}{1+a}[F(v(x,0)) - F(1)] - w(x,y) \tag{4.76}$$

By the same reasoning, we see also that $\overline{w}$ is bounded in $\mathbb{R}^2_+$. We need to show that $\overline{w} > 0$ in $\overline{\mathbb{R}^2_+}$. To do so, we first derive some equations for $\overline{w}$ which will be helpful throughout the proof.

First, we observe that, for all $y > 0$,

$$\overline{w}_y(x,y) = -\frac{y^a}{2}(v_x^2 - v_y^2)(x,y). \tag{4.77}$$

Furthermore, using the fact that $L_a v = 0$, we may integrate by parts as in the proof of Lemma 4.3.2 to find that, for all $y > 0$,

$$\overline{w}_x(x,y) = y^a v_x(x,y) v_y(x,y). \tag{4.78}$$

Indeed, we have

$$\begin{aligned} \overline{w}_x(x,y) &= \frac{1}{1+a}\frac{\mathrm{d}}{\mathrm{d}x}[F(v(x,0)) - w(x,y)] \\ &= -\frac{\mathrm{d}}{\mathrm{d}x} w(x,y), \end{aligned}$$

where

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} w(x,y) &= \int_0^y t^a (v_{xx}v_x - v_{xy}v_y)\, dt \\ \text{(Integrate by Parts)} \quad &= -y^a v_y v_x + \lim_{y\to 0^+} y^a v v_y \\ &= -y^a v_y v_x + \frac{1}{1+a}\frac{\mathrm{d}}{\mathrm{d}x} F(v(x,0)). \end{aligned}$$

It follows that (4.78) holds. Then, by (4.77) and (4.78) combined with the fact that $L_a v = 0$, we see that

$$L_a \overline{w} = -a y^{2a-1} v_x^2 \tag{4.79}$$

and

$$L_{-a} \overline{w} = -a y^{-1} v_y^2 \tag{4.80}$$

for all $y > 0$. To see this, observe that

$$\begin{aligned}
L_a\overline{w} &= \mathrm{div}(y^a\nabla\overline{w}) \\
&= \frac{\partial}{\partial x}(y^a\overline{w}_x) + \frac{\partial}{\partial y}(y^a\overline{w}_y) \\
&= y^a\overline{w}_{xx} + ay^{a-1}\overline{w}_y + y^a\overline{w}_{yy} \\
&= y^a[y^a v_{xx}v_y + y^a v_x v_{xy}] + ay^{a-1}[-\frac{y^a}{2}(v_x^2 - v_y^2)] + y^a[-y^a v_x v_{xy} + y^a v_y v_{yy} - \frac{ay^{a-1}}{2}(v_x^2 - v_y^2)] \\
&= -ay^{2a-1}v_x^2 + y^{2a}v_{xx}v_y + ay^{2a-1}v_y^2 + y^{2a}v_y v_{yy} \\
&= -ay^{2a-1}v_x^2 + y^a v_y[y^a v_{xx} + ay^{a-1}v_y + y^a v_{yy}] \\
&= -ay^{2a-1}v_x^2 + y^a v_y L_a v \\
&= -ay^{2a-1}v_x^2,
\end{aligned}$$

and a similar computation works for $L_{-a}\overline{w}$.

We claim that $\overline{w}$ does not achieve its infimum at a point in $\overline{\mathbb{R}^2_+}$. To show this, we assume the contrary and reach a contradiction. Let (x_0, y_0) be a point where the infimum is achieved. There are two cases to consider, depending on if (x_0, y_0) is on the boundary (i.e. $y_0 = 0$) or not. We will also use the fact that $\overline{w}$ is not identically constant, for if it were, then

$$\text{constant} = \overline{w}(\cdot, 0) = \frac{1}{1+a}[F(v(\cdot, 0)) - F(1)]$$

since $w(\cdot, 0) \equiv 0$. Thus, F is constant in $(-1, 1)$, $f \equiv 0$ in $(-1, 1)$, and v is a bounded function satisfying (4.18) with $f \equiv 0$. Hence, after even reflection across $\{y = 0\}$, Proposition D.2.2 (Harnack inequality) shows that v is constant, contradicting the fact that $v_x > 0$.

- (Case 1: $y_0 = 0$). After a translation in x, we may assume $x_0 = 0$. Since $x_0 = 0$ is a global minimum of $(1 + a)\overline{w}(\cdot, 0) = F(v(\cdot, 0)) - F(1)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}F(v(x, 0))|_{x=0} = -f(v(0, 0))v_x(0, 0),$$

so that

$$\begin{aligned} 0 &= -f(v(0,0)) \\ &= (1+a) \lim_{y \to 0^+} y^a v_y(0,y), \end{aligned} \tag{4.81}$$

by the fact that v is a layer solution and, hence, $v_x(x,0) > 0$. Moreover, we have

$$v_x(x,y) > 0 \text{ in } \overline{\mathbb{R}^2_+} \tag{4.82}$$

by Lemma 4.1.2. We now consider two subcases. The first of these is the case $a \geq 0$. By (4.77) and Morrey's inequality, we see that $y^a \overline{w}_y$ is H\'older continuous up to $\partial \mathbb{R}^2_+$. Since $L_a \overline{w} \leq 0$ by (4.79) and $\overline{w}$ is not identically a constant, we have $\overline{w} > \overline{w}(0,0)$ in $\mathbb{R}^2_+$ by the Proposition D.2.1 (maximum principle). Thus, by Lemma D.2.1 (Hopf Lemma), we see that

$$\begin{aligned} 0 &> -\lim_{y \to 0^+} y^a \overline{w}_y(0,y) \\ &= \lim_{y \to 0^+} \frac{y^{2a}}{2} [v_x^2(0,y) - v_y^2(0,y)] \\ \text{(Neumann Condition)} \quad &= \lim_{y \to 0^+} \frac{y^{2a}}{2} v_x^2(0,y) \geq 0, \end{aligned}$$

a contradiction. Now, suppose $a < 0$. Since $(0,0)$ is a global minimum for $\overline{w}(x,y)$, we find

$$\begin{aligned} 0 &\geq \liminf_{y \to 0^+} -y^{-a} \overline{w}_y(0,y) \\ &= \liminf_{y \to 0^+} \frac{1}{2} [v_x^2(0,y) - v_y^2(0,y)] \\ &= \frac{1}{2} v_x^2(0,0) > 0, \end{aligned}$$

which is also a contradiction. Here, we have used (D.2).

- (Case 2: $y_0 > 0$). By (4.82), we have $v_x > 0$ in $\overline{\mathbb{R}^2_+}$. Using (4.78) and (4.80), we obtain

$$
\begin{aligned}
0 &= L_{-a}\overline{w} + ay^{-1}v_y^2 \\
&= L_{-a}\overline{w} + (ay^{-1-a}\frac{v_y}{v_x})\overline{w}_x \\
&= \operatorname{div}(y^{-a}\nabla\overline{w}) + b(x,y)\overline{w}_x \text{ in } \mathbb{R}^2_+,
\end{aligned} \tag{4.83}
$$

where $b(x,y) := ay^{-1-a}v_y v_x^{-1}$. However, the operator in (4.83) is uniformly elliptic with continuous coefficients in compact sets of $\{y > 0\}$. From the maximum principle for uniformly elliptic operators, it follows that $\overline{w}$ cannot achieve its minimum at (x_0, y_0), since $y_0 > 0$ and we have shown that $\overline{w}$ is not identically constant. Therefore, we know $\overline{w}$ cannot achieve its infimum at a point in $\overline{\mathbb{R}^2_+}$.

To conclude the proof, assume

$$\inf_{\overline{\mathbb{R}^2_+}} \overline{w} < 0.$$

By Lemma 4.3.2, we see that $\overline{w}(x,y) \to 0$ as $y \to \infty$ locally uniformly in x. By Lemma 4.3.1, we have $\overline{w}(x,y) \to 0$ as $|x| \to \infty$ uniformly in y. Indeed, for the first statement, it is clear that $w(x,y) \to \frac{1}{1+a}[F(v(x,0)) - F(1)]$ as $y \to \infty$ by Lemma 4.3.2, so that $\overline{w}(x,y) \to 0$ as $y \to \infty$. To establish local uniform convergence in x, observe that the function $f^*(x) = F(v(x,0)) - F(1)$ is uniformly continuous on compact subsets of $\mathbb{R}$ by continuity of v and F. Hence, for a given compact subset $K \subset \mathbb{R}$ and $\epsilon > 0$, there exists $R := R_{K,\epsilon}$ such that $|x_1 - x_2| < R$ implies $|f^*(x_1) - f^*(x_2)| < (1+a)\cdot\epsilon$, provided $x_1, x_2 \in K$. Then, we find

$$
\begin{aligned}
\lim_{y\to\infty} |w(x_1,y) - w(x_2,y)| &= |f^*(x_1) - f^*(x_2)| \\
&< (1+a)\epsilon,
\end{aligned}
$$

so that there exists $Y := Y_{K,\epsilon}$ such that $y \geq Y$ implies $|w(x_1,y) - w_(x_2,y)| < (1+a)\epsilon$. Then, if

$y \geq Y$, we see that

$$
\begin{aligned}
|\overline{w}(x_1,y) - \overline{w}(x_2,y)| &= \left| \frac{1}{1+a}[f^*(x_1) - f^*(x_2)] + w(x_1,y) - w(x_2,y) \right| \\
&\leq \frac{1}{1+a}|f^*(x_1) - f^*(x_2)| + |w(x_1,y) - w(x_2,y)| \\
&\leq \epsilon + |w(x_1,y) - w(x_2,y)| \\
&\leq (2+a)\epsilon,
\end{aligned}
$$

so that $\overline{w}$ is Cauchy in x as $y \to \infty$ on compact subsets K of $\mathbb{R}$. The second statement is a direct application of Lemma 4.3.1. Therefore, since $\overline{w} < 0$, it should be achieved at a point in $\overline{\mathbb{R}^2_+}$. However, this contradicts what has been proven. It follows that

$$
\inf_{\mathbb{R}^2_+} \overline{w} \geq 0,
$$

so that $\overline{w} \geq 0$. In fact, if $\overline{w}$ vanished at a point in $\overline{\mathbb{R}^2_+}$, this point would be the infimum of $\overline{w}$, a contradiction. Hence $\overline{w} > 0$ in $\overline{\mathbb{R}^2_+}$. This completes the proof. □

We have thereby established a Modica-type estimate for layer solutions of the problem (4.18) in dimension $n = 1$. Unfortunately, this is the only known result for the stationary (i.e. time independent) case. Thus, for dimensions $n \geq 2$, establishing a Modica-type estimate similar to above would constitute an exciting new result. We point out, however, that for $s = \frac{1}{2}$ in the non-stationary case, a corresponding Hamiltonian identity has been established by Caffarelli, Mellet, and Sire in [12].

4.3.2 A Nonlocal Pohozaev Identity and Monotonicity Formula

The final theorem to be proved is the nonlocal version of the Pohozaev-type monotonicity formula derived in Chapter 3. Similar to the local case, we will obtain the monotonicity formula by making use of a nonlocal Pohozaev identity for the fractional Laplacian.

Lemma 4.3.3 (Nonlocal Pohozaev Identity; Lemma 3.1 in [8]). *Let $s \in (0,1)$ and $f \in C^{1,\alpha}(\mathbb{R})$ with $\alpha > \max\{0, 1-2s\}$, and suppose that v is a bounded solution of the problem* (4.5). *Then, for every $R > 0$*

$$\frac{n-2s}{2}\int_{B_R^+} y^a|\nabla v|^2\,dx\,dy + n\int_{B_R\times\{0\}} d_s^{-1}F(v(x,0))\,dx =$$
$$\frac{R}{2}\int_{\partial^+B_R^+} y^a|\nabla v|^2\,dS^n - R\int_{\partial^+B_R^+} y^a(\partial_\nu v)^2\,dS^n + R\int_{\partial B_R\times\{0\}} d_s^{-1}F(v)\,dS^{n-1}, \tag{4.84}$$

where $\partial_\nu v$ denotes the outer normal derivative of v on $\partial^+B_R^+$.

Proof. Set $z = (x,y)$. Multiplying the equation

$$\operatorname{div}(y^a\nabla v) = 0$$

by $\langle z, \nabla v\rangle$, we see that

$$\begin{aligned} 0 &= \operatorname{div}(y^a\nabla v)\langle z,\nabla v\rangle \\ &= \operatorname{div}(y^a\nabla v\langle z,\nabla v\rangle) - \nabla\Big(\langle z,\nabla v\rangle\Big)\cdot y^a\nabla v \\ &= \operatorname{div}(y^a\nabla v\langle z,\nabla v\rangle) - y^a\left(|\nabla v|^2 + \left\langle z, \nabla\left(\frac{|\nabla v|^2}{2}\right)\right\rangle\right). \end{aligned}$$

Computing, we find

$$\begin{aligned} y^a\left\langle z,\nabla\left(\frac{|\nabla v|^2}{2}\right)\right\rangle &= \operatorname{div}\left(y^a z\left(\frac{|\nabla v|^2}{2}\right)\right) - \\ &\qquad y^a(n+1)\frac{|\nabla v|^2}{2} - (1-2s)y^a\frac{|\nabla v|^2}{2} \\ &= \operatorname{div}\left(y^a z\left(\frac{|\nabla v|^2}{2}\right)\right) - \frac{n+2-2s}{2}y^a|\nabla v|^2. \end{aligned}$$

Hence,

$$\operatorname{div}\left(y^a\left(\nabla v\langle z,\nabla v\rangle - z\frac{|\nabla v|^2}{2}\right)\right) + \frac{n-2s}{2}y^a|\nabla v|^2 = 0.$$

Integrating by parts in B_R^+, we obtain

$$\int_{\partial^+ B_R^+} y^a \langle \nu, \nabla v\rangle\langle z, \nabla v\rangle \, dS^n - \lim_{y\to 0^+} \int_{B_R\times\{0\}} y^a v_y \langle x, \nabla_x v\rangle \, dx - \\ \frac{1}{2}\int_{\partial^+ B_R^+} y^a |\nabla v|^2 \langle z, \nu\rangle \, dS^n + \frac{n-2s}{2}\int_{B_R^+} y^a |\nabla v|^2 \, dx\, dy = 0,$$

where ν is the outer unit normal to $\partial^+ B_R^+$. Note that $z = R\nu$ on $\partial^+ B_R^+$ and $-d_s \lim_{y\to 0^+} y^a v_y = f(v)$ on $B_R \times \{0\}$. Therefore,

$$R\int_{\partial^+ B_R^+} y^a (\partial_\nu v)^2 \, dS^n + \int_{B_R\times\{0\}} d_s^{-1} f(v(x,0))\langle x, \nabla_x v\rangle \, dx - \\ \frac{R}{2}\int_{\partial^+ B_R^+} y^a |\nabla v|^2 \, dS^n + \frac{n-2s}{2}\int_{B_R^+} y^a |\nabla v|^2 \, dx\, dy = 0. \tag{4.85}$$

Furthermore, on $\{y = 0\}$ we have

$$\begin{aligned}\int_{B_R\times\{0\}} f(v(x,0))\langle x, \nabla_x v\rangle \, dx &= -\int_{B_R\times\{0\}} \big\langle x, \nabla_x F(v(x,0))\big\rangle \, dx \\ &= \int_{B_R\times\{0\}} -\operatorname{div}\big(xF(v(x,0))\big) + nF(v(x,0)) \, dx \\ &= n\int_{B_R\times\{0\}} F(v(x,0) \, dx - R\int_{\partial B_R\times\{0\}} F(v) \, dS^{n-1}.\end{aligned}$$

Replacing in (4.85) and rearranging terms gives the result. □

With Lemma 4.3.3 at hand, the proof of the nonlocal monotonicity formula is now a simple differentiation and integration by parts.

Theorem 4.3.2 (Nonlocal Monotonicity Formula; Proposition 3.2 in [8])**.** *Let $s \in (0,1)$ and $f \in C^{1,\alpha}(\mathbb{R})$ with $\alpha > \max\{0, 1-2s\}$, and suppose that v is a bounded solution of the problem* (4.5) *with $F(t) \geq 0$ for every $t \in \mathbb{R}$. Then, the function*

$$I_s(R) = \frac{1}{R^{n-2s}}\left(\frac{d_s}{2}\int_{B_R^+} y^a |\nabla v|^2 \, dx\, dy + \int_{B_R\times\{0\}} F(v(x,0)) \, dx\right) \tag{4.86}$$

is a nondecreasing function of $R > 0$*.*

Proof. Differentiating $I_s(R)$ with respect to R, we find

$$I_s'(R) = \frac{(n-2s)d_s}{2R^{n-2s+1}} \int_{B_R^+} y^a |\nabla v|^2 \, dx\, dy - \frac{n-2s}{R^{n-2s+1}} \int_{B_R \times \{0\}} F(v(x,0))\, dx + \\ \frac{d_s}{2R^{n-2s}} \int_{\partial^+ B_R^+} y^a |\nabla v|^2 \, dS^n + \frac{1}{R^{n-2s}} \int_{\partial B_R \times \{0\}} F(v)\, dS^{n-1}.$$

Then, applying the Lemma 4.3.3 we obtain

$$\begin{aligned}
I_s'(R) = &\frac{d_s}{R^{n-2s-1}} \left(\frac{R}{2} \int_{\partial^+ B_R^+} y^a |\nabla v|^2 \, dS^n + R \int_{\partial B_R \times \{0\}} d_s^{-1} F(v)\, dS^{n-1} \right) - \\
&\frac{(n-2s)d_s}{2R^{n-2s+1}} \int_{B_R^+} y^a |\nabla v|^2 \, dx\, dy - \frac{n-2s}{R^{n-2s+1}} \int_{B_R \times \{0\}} F(v(x,0))\, dx \\
= &\frac{d_s}{R^{n-2s}} \left(\frac{n-2s}{2R} \int_{B_R^+} y^a |\nabla v|^2 \, dx\, dy + \frac{n}{R} \int_{B_R \times \{0\}} d_s^{-1} F(v)\, dx + \int_{\partial^+ B_R^+} y^a (\partial_\nu v)^2 \, dS^n \right) - \\
&\frac{(n-2s)d_s}{2R \cdot R^{n-2s+1}} \int_{B_R^+} y^a |\nabla v|^2 \, dx\, dy - \frac{n-2s}{R^{n-2s+1}} \int_{B_R \times \{0\}} F(v(x,0))\, dx \\
= &\frac{d_s}{R^{n-2s}} \int_{\partial^+ B_R^+} y^a (\partial_\nu v)^2 \, dS^n + \frac{2s}{R^{n-2s+1}} \int_{B_R \times \{0\}} F(v(x,0))\, dx \geq 0,
\end{aligned}$$

as claimed. □

Comparing with $I(R)$ in (3.28), the quantity $I_s(R)$ is nearly identical aside from a missing factor of R^{-1}. In the local case, we were able to prove the monotonicity formula for $I(R)$ by using the Modica estimate Proposition 3.1.1. However, in the nonlocal case we have no such estimate (at least for dimension $n \geq 2$) allowing us to only obtain (4.86).

Remark 4.3.2. Though not the subject of this project, we feel it is worthwhile to point out that other monotonicity formula exist for the fractional Laplacian. For example, in [9] Cabré and Sire prove a monotonicity formula for radial solutions of (4.18). In addition, Caffarelli and Silvestre prove a nonlocal version of *Almgren's frequency formula* in their 2007 paper introducing the extension problem (see [13]). Fazly and Shahgholian in [25] have also worked out a monotonicity formula similar to (4.86) for solutions of the coupled elliptic system

$$(-\Delta)^s u_i = |u|^{p-1} u_i \text{ in } \mathbb{R}^n.$$

4.4 Current Directions and Further Study

In addition to the cases $n \leq 3$ we have discussed prior, the conjecture has been proven by Savin in [51] for dimensions $4 \leq n \leq 8$ and $s \in (\frac{1}{2}, 1)$ for the fractional Allen-Cahn equation assuming the solution is a layer solution in $\mathbb{R}^n$. Thus, the fractional De Giorgi conjecture remains completely open for dimensions $4 \leq n \leq 8$ with $s \in (0, \frac{1}{2}]$, along with various other extended De Giorgi conjectures in both lower and higher dimensions. We also note that in the systems case De Giorgi's conjecture is discussed by Fazly et al. in [23, 26] in lower dimensions.

Of immediate interest to the author for future study is the stability conjecture for the fractional Allen-Cahn equation, in particular, the recent work of Figalli and Serra in dimension $n = 3$ and $s = \frac{1}{2}$ (see [28]). Note that when $n = 2$, for general s, it is considered by Sire and Valdinocci in [57] and Gui and Li in [34]. The stability conjecture for a more general kernel it is discussed by Hamel et al. in [37] and Fazly et al. in [24, 27]. Aside from this and the results presented prior, the stability conjecture is wide open. The author is further interested in studying the geometric perspectives of De Giorgi's conjecture for the classical and fractional Laplacians, such as the work in dimensions $4 \leq n \leq 8$ of Savin (see [49, 51]) and the work of del Pino et al. and Chan et al. in higher dimensions (see [15, 16, 47]) and finite Morse index solutions (see [36, 59]). More broadly, the author is keenly interested in pursuing study of related subjects in geometry, beginning with Schoen's and Uhlenbeck's earlier work in harmonic maps (see [54, 56], for example), as well as topics in local and nonlocal minimal surface theory having applications in mathematical physics (e.g. phase transitions, general relativity).

APPENDIX A: FUNCTION SPACES AND INEQUALITIES

In this section, we present some standard definitions, identities, and theorems that will be used freely throughout this paper. We have assumed familiarity with techniques in analysis at the introductory graduate level, as well as a working knowledge of the classical theory of PDE. Unless otherwise stated, we will assume that $\Omega \subset \mathbb{R}^n$ is open and that $u : \Omega \to \mathbb{R}$. All of the following material can be found in [21], [32], or [48].

A.1 Notation

Let α be a vector of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each component α_i is a nonnegative integer. We say that α is a multiindex of order $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Given a multiindex α, we may define

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u(x).$$

In particular, if k is a nonnegative integer, we define

$$D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\},$$

that is, $D^k u(x)$ is the set of all partial derivatives of order k. By assigning an ordering to the various partial derivatives, we me can regard $D^k u(x)$ as a point in $\mathbb{R}^{n^k}$. We define

$$|D^k u| = \Big(\sum_{|\alpha|=k} |D^\alpha u|^2 \Big)^{\frac{1}{2}}.$$

For example, when $k = 1$, we regard the elements of Du as being arranged in a vector

$$Du := (u_{x_1}, \ldots, u_{x_n}) = \nabla u,$$

so that $|Du| = |\nabla u|$. Throughout this work, we will always regard Du as ∇u. We write

$$u_r := \frac{x}{|x|} \cdot \nabla u$$

to denote the radial derivative of u.

When $k = 2$, we regard elements of $D^2 u$ as being arranged as in the Hessian matrix[1]. Thus, we find that $D^2 u \in \mathbb{S}^n$, the space of real symmetric $n \times n$ matrices (assuming appropriate regularity on u).

In the body of this note, several constants will be introduced. In general, we let C denote an arbitrary constant, letting it absorb extraneous factors when necessary. When the dependence of C on parameters is important, or C is known explicitly, we have made it clear.

A.2 Function Spaces

Here, we discuss some fundamental function spaces encountered in this project. First, we consider the Banach space of k times continuously differentiable functions on Ω

$$C^k(\Omega) := \{u : \Omega \to \mathbb{R}^n : u \text{ is continuous}\}$$

with their accompanied norms

$$\|u\|_{C^k(\Omega)} := \sum_{n=0}^{k} \sup_{\Omega} |D^k u(x)|.$$

We can similarly define the space

$$C^k(\overline{\Omega}) := \{u \in C^k(\Omega) : D^\alpha u \text{ is uniformly continuous on bounded subsets of } \Omega, \text{ for all } |\alpha| \leq k\}.$$

It follows that if $u \in C^k(\overline{\Omega})$, then $D^\alpha u$ extends continuously to $\overline{\Omega}$ for each multiindex α satisfying

[1] Recall that the Hessian matrix of u is the matrix of second order partial derivatives of u.

$|\alpha| \leq k$. Finally, we define

$$C^\infty(\Omega) := \bigcap_{k=0}^{\infty} C^k(\Omega).$$

The space $u \in C^\infty(\overline{\Omega})$ is defined analogously. The space $C_0^k(\Omega)$ denotes those functions in $C^k(\Omega)$ with compact support (i.e. vanishing outside of a compact set).

The reader must necessarily be familiar with the Lebesgue spaces

$$L^p(\Omega) := \{u : \Omega \to \mathbb{R} : u \text{ is Lebesgue measurable, } \|u\|_{L^p(\Omega)} < \infty\}$$

where

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$. We note that the case $p = 2$ is of fundamental importance, since in this case we obtain a Hilbert space with inner product

$$\langle u, v \rangle_{L^2(\Omega)} := \int_\Omega uv \, dx.$$

We will also consider the space

$$L^\infty(\Omega) := \{u : \Omega \to \mathbb{R} : u \text{ is Lebesgue measurable, } \|u\|_{L^\infty(\Omega)} < \infty\},$$

where

$$\|u\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_\Omega |u|.$$

Sometimes, the integrability of a function u on the entirety of Ω is not necessary. As such, we will concern ourselves with the spaces

$$L^p_{loc}(\Omega) := \{u : \Omega \to \mathbb{R} : u \in L^p(U) \text{ for each } U \text{ open such that } U \subset\subset \Omega\}.$$

Here, $U \subset\subset \Omega$ signifies the fact that U is open in Ω with compact closure entirely contained in Ω.

The aforementioned material is all in the hopes of defining the Hölder Spaces and, most importantly, the Sobolev spaces. We say that functions $u : \Omega \to \mathbb{R}$ satisfying

$$|u(x) - u(y)| \leq C|x-y|^{\alpha}$$

for each $x, y \in \Omega$ and $0 < \alpha \leq 1$, C constant, are *Hölder continuous with exponent* α. The Hölder space $C^{k,\alpha}(\overline{\Omega})$ consists of all functions $u \in C^k(\overline{\Omega})$ for which the norm

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} := \sum_{|\beta| \leq k} \left\|D^{\beta}u\right\|_{C(\overline{\Omega})} + \sum_{\beta = k} [D^{\beta}u]_{C^{0,\alpha}(\overline{\Omega})} \tag{A.1}$$

is finite and the α-th Hölder seminorm of $u : \Omega \to \mathbb{R}$ is given by

$$[u]_{C^{0,\alpha}(\overline{\Omega})} := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \left(\frac{|u(x) - u(y)|}{|x-y|^{\alpha}} \right).$$

In short, the space $C^{k,\alpha}(\overline{\Omega})$ consists of those functions $u : \Omega \to \mathbb{R}$ that are k-times continuously differentiable and whose k-th partial derivatives are bounded and Hölder continuous with exponent α. Equipped with the norm (A.1), the space $C^{k,\alpha}(\overline{\Omega})$ is a Banach space.

Of particular importance in the theory of PDE are the Sobolev spaces $W^{k,p}$. To define these spaces, we first need to define the notion of a weak derivative.

Definition A.2.1 (Weak Derivative)**.** Suppose $u, v \in L^1_{loc}(\Omega)$ and α is a multiindex. We say that $v \in L^1_{loc}(\Omega)$ is the *α-th weak partial derivative of* u, written

$$D^{\alpha}u = v,$$

provided

$$\int_{\Omega} uD^{\alpha}\phi \, dx = (-1)^{|\alpha|} \int_{\Omega} v\phi \, dx$$

for all test functions $\phi \in C_0^{\infty}(\Omega)$.

This definition is motivated by the integration by parts formula for a function $u \in C^k(\Omega)$. Indeed, for such a u and a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$ of order k, we find

$$\int_\Omega u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u \phi \, dx$$

by the integrattion by parts formula, since $\phi \in C_0^\infty(\Omega)$ has compact support.

Alas, we define the Sobolev spaces $W^{k,p}$. Fix $1 \leq p \leq \infty$ and let k be a nonnegative integer.

Definition A.2.2. The *Sobolev space* $W^{k,p}(\Omega)$ consists of all locally integrable functions $u : \Omega \to \mathbb{R}$ such that, for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$.

Note that elements of $W^{k,p}(\Omega)$ need not be continuous or bounded in Ω. Moreover, when $n = 1$ and Ω is an open interval in $\mathbb{R}$, we see that $u \in W^{1,p}(\Omega)$ if and only if u equals a.e. an absolutely continuous function whose ordinary derivative (which exists a.e.) belongs to $L^p(\Omega)$.

When coupled with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \, dx \right)^{\frac{1}{p}}, \ 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} \operatorname{ess\,sup}_\Omega |D^\alpha u|, \ p = \infty, \end{cases}$$

the space $W^{k,p}(\Omega)$ is a Banach space. As an explicit example, it can be shown that the $W^{1,p}(\Omega)$ norm is equivalent to

$$\|u\|_{W^{1,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)}^p + \|\nabla u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}},$$

the gradient being defined in the weak sense.

When $p = 2$ we obtain a Hilbert space and write $H^k(\Omega) = W^{k,2}(\Omega)$. We denote by $W_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ and identify $W_0^{k,p}(\Omega)$ as those functions $u \in W^{k,p}(\Omega)$ such that $D^\alpha u = 0$ on $\partial\Omega$ for all $|\alpha| \leq k - 1$. Naturally, issues arise since we identify functions in $W^{k,p}(\Omega)$ which agree a.e. and $\partial\Omega$ may have Lebesgue measure zero. It is therefore vital for us to consider the trace operator (see [21] for the basic trace theory).

It is worthwhile to note that, in a general domain Ω, we are only guaranteed approximation by smooth functions in $W^{k,p}_{loc}(\Omega)$. However, imposing the condition that Ω is bounded allows for approximation by smooth functions in $W^{k,p}(\Omega)$. If we further require that $\partial\Omega$ is C^1, then we can, in addition, assume our approximating functions are of class $C^\infty(\overline{\Omega})$.

We have denoted the Schwarz space of rapidly decaying C^∞ functions in $\mathbb{R}^n$ by $\mathscr{S}(\mathbb{R}^n)$.

Definition A.2.3. The *Schwarz space* of rapidly decaying C^∞ functions $u : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\mathscr{S}(\mathbb{R}^n) := \{u \in C^\infty(\mathbb{R}^n) \ : \ \|u\|_{\alpha,\beta} < \infty\}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n)$ are multiindices. The norm $\|u\|_{\alpha,\beta}$ is given by

$$\|u\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta u(x)|. \tag{A.2}$$

Here, x^α is defined as the product $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

We may then define the set of all tempered distributions $\mathscr{S}'(\mathbb{R}^n)$, the topological dual of $\mathscr{S}(\mathbb{R}^n)$, to be the set of all distributions[2] T such that $T(u_k) \to 0$ whenever $\lim_{k\to\infty} \|u_k\|_{\alpha,\beta} = 0$, where $\{u_k\}_{k\in\mathbb{N}}$ is a sequence defined on $\mathscr{S}(\mathbb{R}^n)$.

The relationship between the Sobolev space H^k and the Fourier transform is established by the following theorem:

Theorem A.2.1 (Characterization of H^k via Fourier Transform)**.** *Let k be a nonnegative integer.*

1. *A function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if*

$$(1 + |\xi|^k)\hat{u} \in L^2(\mathbb{R}^n).$$

2. *In addition, there exists a positive constant C such that*

$$\frac{1}{C}\|u\|_{H^k(\mathbb{R}^n)} \le \left\|(1 + |\xi|^k)\hat{u}\right\|_{L^2(\mathbb{R}^n)} \le C\|u\|_{H^k(\mathbb{R}^n)}$$

[2] For more on the general theory of distributions and its applications in PDE, see [21] or [48].

for each $u \in H^k(\mathbb{R}^n)$.

In the body of this thesis, we present an analogous result for the fractional Sobolev space $W^{s,p}(\Omega)$, with $0 < s < 1$.

A.3 Some Useful Inequalities

For continuity, we include some fundamental inequalities from real analysis that we employ throughout the project.

We begin with *Cauchy's Inequality*:

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2} \tag{A.3}$$

for each $a, b \in \mathbb{R}$. This follows immediately from the inequality $(a-b)^2 \geq 0$. We now suppose $1 < p$ and $q < \infty$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. Then *Young's inequality* asserts

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{A.4}$$

whenever $a, b > 0$. This can be seen by noting that the mapping $x \mapsto e^x$ is convex.

The reader may be familiar with *Hölder's inequality*, which says that, for $1 \leq p$ and $q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, if $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, we have

$$\int_\Omega |uv|\, dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}. \tag{A.5}$$

Indeed, by homogeneity, we may assume that u and v are normalized so that $\|u\|_{L^p(\Omega)} = \|v\|_{L^q(\Omega)} = 1$. Then we may apply Young's inequality for $1 < p$ and $q < \infty$ to $|uv|$ and integrate over Ω to get the result. The case $p = 1$ and $q = \infty$ is a bit more technical and we leave out the details.

For $1 \leq p$ and $u, v \in L^p(\Omega)$, we find

$$\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}. \tag{A.6}$$

This is referred to as *Minkowski's inequality* and, in effect, establishes the triangle inequality for the Banach space $L^p(\Omega)$. This follows from applying Hölder's inequality with $q = \frac{p}{p-1}$. By similar methods, we may obtain the discrete analogous of these inequalities for the sequence spaces $\ell^p(\mathbb{R}^n)$ (or, more generally, $\ell^p(\mathbb{C}^n)$).

The *Sobolev inequalities* are useful tools for establishing additional regularity of functions in certain Sobolev spaces. We first state *Morrey's inequality*:

Theorem A.3.1 (Morrey's Inequality)**.** *Assume $n < p \leq \infty$. Then there exists a constant $C = C(n, p)$ such that*

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$, where

$$\alpha := 1 - \frac{n}{p}.$$

With Morrey's inequality in mind, it is natural to wonder if more general estimates can be obtained which are applicable for more general domains. It turns out that they can.

Theorem A.3.2 (General Sobolev Inequalities)**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$, with a C^1 boundary. Assume $u \in W^{k,p}(\Omega)$.*

1. *If $k < \frac{n}{p}$, then $u \in L^q(\Omega)$, where*

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

   In addition, we have the estimate

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)},$$

   where $C = C(k, p, n, \Omega)$.

2. *If* $k < \frac{n}{p}$, *then* $u \in C^{k-[\frac{n}{p}]-1,\alpha}(\overline{\Omega})$, *where*

$$\alpha = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p}, \text{ if } \frac{n}{p} \notin \mathbb{Z} \\ \text{Any positive number } < 1, \text{ if } \frac{n}{p} \in \mathbb{Z}. \end{cases}$$

In addition, we have the estimate

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\alpha}(\overline{\Omega})} \leq C\|u\|_{W^{k,p}(\Omega)},$$

where $C = C(k, p, n, \alpha, \Omega)$.

In particular, if $u \in W^{k,p}(\Omega)$ for all $p > n$, then $u \in C^{k-1,\alpha}$ for all $\alpha \in (0, 1)$.

APPENDIX B: ELLIPTIC OPERATORS AND NOTIONS OF SOLUTIONS

Here, we define elliptic operators, along other related concepts and conventions. We also define various notions of solutions we have used in this project. For a detailed presentation of these concepts, we highly recommend [21] or [32].

B.1 Elliptic Operators

A second-order partial differential operator takes the form

$$Lu = -\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u \tag{B.1}$$

or

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u \tag{B.2}$$

for given coefficient functions a_{ij}, b_i, c and $i, j = 1, \ldots, n$. The PDE $Lu = f$ is in *divergence form* if L is given by (B.1) and is in *nondivergence form* if it is given by (B.2). If the highest order coefficients a_{ij} are C^1 for each $i, j = 1, \ldots, n$, then an operator given in divergence form can be rewritten into nondivergence form, and vice versa. The divergence form is more natural for energy methods, due to integration by parts, and the nondivergence form is more appropriate for maximum principle techniques. As is customary, we assume the symmetry condition $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$. The following definition is vital.

Definition B.1.1. We say that the partial differential operator L is (*uniformly) elliptic* if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \theta|\xi|^2 \tag{B.3}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^n$.

In particular, this means that for each $x \in \Omega$, the symmetric $n \times n$ matrix $A(x) = (a_{ij}(x))$ is positive definite, with smallest eigenvalue greater than or equal to θ. The classical example of such an operator is the Laplacian[1] operator $L = -\Delta$. As one might expect, solutions of general second-order elliptic PDE $Lu = 0$ are similar to harmonic functions.

B.2 Notions of Solutions

In PDE, it is common to consider many notions of solutions since it is not always the case that a C^2 solution exists to a given problem. In addition, it is not even necessary in applications to require our solutions be C^2. The abstract definitions that follow further allow us to take advantage of the Banach and Hilbert space theory.

We consider the boundary-value problem

$$\begin{cases} Lu = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{B.4}$$

Classical solutions are C^2 solutions satisfying the PDE and the boundary condition in the usual pointwise sense. *Strong solutions* are solutions belonging to the Sobolev space $H^2(\Omega)$. The PDE is satisfied in the pointwise sense, a.e. with respect to the Lebesgue measure in Ω, while the boundary condition is satisfied in the sense of traces. *Distributional solutions* are solutions belonging only $L^1_{loc}(\Omega)$ and the equation holds in the distributional sense. More delicate is the notion of a *weak solution*. Suppose that $a_{ij}, b_i, c \in L^\infty(\Omega)$ for each $i, j = 1, \ldots, n$ and $f \in L^2(\Omega)$. Then the *bilinear form* $B : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ associated with the divergence form of the elliptic operator L is

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij} u_{x_i} v_{x_j} + \sum_{n=1}^{n} b_i u_{x_i} v + cuv \, dx$$

for $u, v \in H^1_0(\Omega)$. We say that $u \in H^1_0(\Omega)$ is a *weak solution* of the boundary value problem (B.4) if

$$B[u, v] = \langle f, v \rangle_{L^2(\Omega)}$$

[1] To see this, simply note that we may write $a_{ij} = \delta_{ij}$ for each $i, j = 1, \ldots, n$.

for all $v \in H_0^1(\Omega)$. This is also referred to as the *variational formulation* of (B.4). Note that if we require $u = g$ on $\partial\Omega$ in the sense of traces and with $\partial\Omega$ of class C^1, we may recast the problem with zero boundary conditions. The existence and uniqueness of weak solutions is given by a result from functional analysis, the *Lax-Milgram theorem*. There are other existence and uniqueness theorems for when the hypotheses of the Lax-Milgram theorem are not met, yet we mention only this one due to its significance and applicability.

APPENDIX C: SPECIAL FUNCTIONS AND INTEGRAL IDENTITIES

The material presented here has been adapted from [30]. For a more thorough treatment of the proceeding topics, we refer the reader to the text of Gerald B. Folland ([29]),

We begin by recalling Euler's gamma function.

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt,$$

defined for $x > 0$. One can very that the identity $\Gamma(\frac{1}{2})$ is simply a reformulation of the famous identity

$$\int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{\pi}.$$

$\Gamma(z)$ can be similarly defined as a holomorphic function for each $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$. For such z, one has

$$\Gamma(z+1) = z\Gamma(z). \tag{C.1}$$

Formula (C.1) and its iterations can be used to meromorphically extend $\Gamma(z)$ to the entire complex plane having simple poles at $z = -k$, for $k \in \mathbb{N} \cup \{0\}$, with residues $(-1)^k$. Fix $0 < s < 1$. Of particular importance for our considerations is the identity

$$\Gamma(1-s) = -s\Gamma(-s) \tag{C.2}$$

which can be obtained from (C.1). Furthermore, we have

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \tag{C.3}$$

and

$$2^{2z-1}\Gamma(z)\Gamma\Big(z + \frac{1}{2}\Big) = \sqrt{\pi}\Gamma(2z). \tag{C.4}$$

Part of the significance of the gamma function is its relationship to the $(n-1)$-dimensional

Hausdorff measure of the unit sphere $S^{n-1} \subset \mathbb{R}^n$, and the n-dimensional volume of the unit ball

$$\sigma_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \tag{C.5}$$

and

$$\omega_n = \frac{\sigma_{n-1}}{n} = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}, \tag{C.6}$$

respectively.

Deeply connected to the gamma function is *Euler's beta function*, denoted $B = B(x, y)$ and defined for $z = x + iy$ in the first quadrant of the complex plane by

$$B(x,y) = 2\int_0^{\frac{\pi}{2}} (\cos(v))^{2x-1}(\sin(v))^{2y-1}\,dv. \tag{C.7}$$

Applying the change of variable $s \mapsto (\sin(v))^2$, we may alternatively write

$$B(x,y) = \int_0^1 (1-s)^{x-1}s^{y-1}\,ds. \tag{C.8}$$

The beta function is related to the gamma function by the formula

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{C.9}$$

Moreover, the following useful integral identity holds:

Proposition C.0.1 (see [30])**.** *Let $b > -n$ and $a > n + b$, then*

$$\int_{\mathbb{R}^n} \frac{|x|^b}{(1+|x|^2)^{\frac{a}{2}}}\,dx = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\frac{\Gamma(\frac{b+n}{2})\Gamma(\frac{a-b-n}{2})}{\Gamma(\frac{a+b}{2})}. \tag{C.10}$$

In particular, if $b = 0$ and $a = n + 1$, then

$$\int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{\frac{n+1}{2}}} = \frac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}. \tag{C.11}$$

For every $\nu \in \mathbb{C}$ satisfying $\mathrm{Re}(\nu) > -\frac{1}{2}$, we define the *Bessel function of the first kind and of complex order* ν by the formula

$$J_\nu(z) = \frac{1}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}\left(\frac{z}{2}\right)^\nu \int_{-1}^{1} e^{izt}(1-t^2)^{\frac{2\nu-1}{2}}\,dt. \tag{C.12}$$

We may also write J_ν as a power series by

$$J_\nu(z) = \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{z}{2})^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)} \tag{C.13}$$

where $|z| < \infty$ and $|\arg(z)| < \pi$.

J_ν arises as the solution to the *Bessel equation of order* ν:

$$z^2\frac{\partial^2}{\partial z^2}J + \frac{\partial}{\partial z}J + (z^2-\nu^2)J = 0. \tag{C.14}$$

When $\nu \notin \mathbb{Z}$, we obtain a second linearly independent solution to (C.14), $J_{-\nu}$. Note that

$$\begin{aligned} z^{-\nu}J_\nu(z) &\to \frac{2^{-\nu+1}}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}\int_0^1 (1-s^2)^{\frac{2\nu-1}{2}}\,ds \\ &= \frac{2^{-\nu+1}}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}B\left(\nu+\frac{1}{2},\frac{1}{2}\right), \end{aligned} \tag{C.15}$$

as $z \to 0$. Furthermore, we have the asymptotic estimate for the Bessel function:

$$J_\nu(z) \cong \frac{2^{-\nu}}{\Gamma(\nu+1)}z^\nu \tag{C.16}$$

One may also consider the *generalized Bessel equation*:

$$y^2u''(y) + (1-2\alpha)yu'(y) + [\beta^2\gamma^2y^{2\gamma} + (\alpha^2-\nu^2\gamma^2)]u(y) = 0. \tag{C.17}$$

Let $\Phi(z)$ be a solution to the Bessel equation (C.14) and consider the function defined by the transformation

$$u(y) = y^{\alpha}\Phi(\beta y^{\gamma}). \tag{C.18}$$

Then, by direct computation, one may verify that $u(y)$ solves the generalized Bessel equation (C.17). These facts will be used in the computation for the Poisson kernel for the harmonic extension problem in $\mathbb{R}^{n+1+a}$.

Now, consider the *modified Bessel equation of order* ν, $\nu \in \mathbb{C}$, given by

$$z^2 \frac{\partial^2}{\partial z^2}\Phi + z\frac{\partial}{\partial z}\Phi - (z^2 + \nu^2)\Phi = 0, \tag{C.19}$$

with linearly independent solutions given by the *modified Bessel function of the first kind*

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(\frac{z}{2})^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \tag{C.20}$$

($|z| < \infty$ and $|\arg(z)| < \pi$) and the *modified Bessel function of the third kind*

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi\nu}, \tag{C.21}$$

with $|\arg(z)| < \pi$, $\nu \notin \mathbb{Z}$. As before, we also have the *generalized modified Bessel equation*

$$y^2 u''(y) + (1-2\alpha)yu'(y) + [(\alpha^2 - \nu^2\gamma^2) - \beta^2\gamma^2 y^{2\gamma}]u(y) = 0, \tag{C.22}$$

with solutions determined by $u(y)$ in (C.18). Of course, in this case we assume that $\Phi(z)$ solves the modified Bessel equation (C.19). We will need the following asymptotics for $I_\nu(z)$ and $I_{-\nu}(z)$, found on page 108 of [39]: As $z \to 0$ we have

$$I_\nu(z) \cong \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^{\nu} \tag{C.23}$$

and

$$I_{-\nu}(z) \cong \frac{1}{\Gamma(1-\nu)}\left(\frac{z}{2}\right)^{-\nu}. \tag{C.24}$$

In computing the Poisson kernel for the extension problem, we will make use of the following *Fourier-Bessel representation*.

Theorem C.0.1 (Fourier-Bessel Representation)**.** *Let $u(x) = f(|x|)$, and suppose that*

$$t \mapsto t^{\frac{n}{2}} J_{\frac{n}{2}-1}(t) \in L^1(\mathbb{R}^+),$$

where we have denoted by $J_{\frac{n}{2}-1}$ the Bessel function of order $\nu = \frac{n}{2} - 1$ defined by (C.12)*. Then,*

$$\hat{u}(\xi) = 2\pi|\xi|^{-\frac{n}{2}+1} \int_0^{+\infty} t^{\frac{n}{2}} f(t) J_{\frac{n}{2}-1}(2\pi|\xi|t)\, dt. \tag{C.25}$$

The *hypergeometric functions* will also be of use to us. In order to introduce them, we recall the definition of *Pochamer's symbols*:

$$\alpha_k := \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+k+1),$$

for $k \in \mathbb{N}$ and $\alpha_0 = 1$. Notice that, since the gamma function has a pole at $z = 0$, we have

$$0_k = \begin{cases} 1 \text{ if } k = 0, \\ 0, \text{ if } k \geq 1. \end{cases}$$

We now let $p, q \in \mathbb{N}_0$ be such that $p \leq q + 1$, and let $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ be given parameters such that $-\beta_j \notin \mathbb{N}_0$ for $j = 1, \ldots, q$. Given a number $z \in \mathbb{C}$, the power series

$${}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_p)_k}{(\beta_1)_k \ldots (\beta_q)_k} \frac{z^k}{k!} \tag{C.26}$$

is called the *generalized hypergeometric function*. When $p = 2$ and $q = 1$, the function ${}_2F_1(\alpha_1, \ldots, \alpha_2; \beta_1; z)$

is *Gauss' hypergeometric function*, typically denoted by $F(\alpha_1, \ldots, \alpha_2; \beta_1; z)$.

By the ratio test, one finds that the radius of convergence of the hypergeometric series is ∞ when $p \leq q$, however, equals 1 when $p = q + 1$. Moreover, we have the following facts:

- $F(\alpha, 0; \beta; z) = F(0, \alpha; \beta; z) = 1$ and
- $F(\alpha, \beta; \beta; -z) = {}_1F_0(\alpha; -z) = (1 + z)^{-\alpha}$.

Interestingly, the hypergeometric function ${}_0F_1$ is essentially a Bessel function, up to powers and rescaling. In fact, we have

$$I_\nu(z) = \frac{1}{\Gamma(\nu+1)} \Big(\frac{z}{2}\Big)^\nu {}_0F_1\left(\nu + 1; \Big(\frac{z}{2}\Big)^2\right). \tag{C.27}$$

Finally, we have the following integral identities:

1. For $\nu - \lambda + 1 > |\mu|$

$$\int_0^\infty t^{-\lambda} K_\mu(at) J_\nu(bt)\, dt = \frac{b^\nu \Gamma(\frac{\nu-\lambda+\mu+1}{2}) \Gamma(\frac{\nu-\lambda-\mu+1}{2})}{2^{\lambda+1} a^{\nu-\lambda+1} \Gamma(1+\nu)} \cdot F\Big(\frac{\nu-\lambda+\mu+1}{2}, \frac{\nu-\lambda-\mu+1}{2}; \nu+1; -\frac{b^2}{a^2}\Big). \tag{C.28}$$

2. For any β, γ satisfying $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(\gamma) > 0$ we have

$$\int_0^\infty t^{\nu-1} e^{-(\frac{\beta}{t}+\gamma t)}\, dt = 2\Big(\frac{\beta}{\gamma}\Big)^{\frac{\nu}{2}} K_\nu(2\sqrt{\beta\gamma}). \tag{C.29}$$

These can be found on page 693 and page 340 of [33], respectively.

APPENDIX D: REGULARITY AND MAXIMUM PRINCIPLES FOR THE EXTENSION PROBLEM

In this chapter, we list some essential regularity results along with some maximum principles and related theorems for the extension problem. All of these results can be found in [9]. As a result, the following results are stated for the problem (4.18). However, each result has an analogous statement for the problem (4.5) due to the equivalency of problems (4.5) and (4.18) discussed in the introduction of Chapter 4.

D.1 Regularity

We list some regularity results we have used for the extension problem introduced in Chapter 2.

Proposition D.1.1 (Lemma 4.4 in [9]). *Let $f \in C^{1,\alpha}(\mathbb{R})$ with $\alpha > \max(0, 1-2s)$. Suppose u is a bounded solution of*

$$(-\Delta)^s u = f(u) \text{ in } \mathbb{R}^n.$$

Then, $u \in C^{2,\beta}(\mathbb{R}^n)$ for some $\beta \in (0,1)$, where β depends only on s and α.

Furthermore, given $s_0 > \frac{1}{2}$, there exists $\beta \in (0,1)$ depending only on n, s_0, and α (and hence independent of s) such that for every $s > s_0$,

$$\|u\|_{C^{2,\beta}(\mathbb{R}^n)} \leq C$$

for some constant C depending only on n, s_0, $\|f\|_{C^{1,\alpha}}$, and $\|u\|_{L^\infty(\mathbb{R}^n)}$ (and hence independent of $s \in (s_0, 1)$).

In addition, the function defined by $v = P_s * u$, where P_s is the Poisson kernel given by (2.38)*, satisfies the estimate for every $s > s_0$*

$$\|v\|_{C^\beta(\overline{\mathbb{R}^{n+1}_+})} + \|\nabla_x v\|_{C^\beta(\overline{\mathbb{R}^{n+1}_+})} + \left\|D^2_x v\right\|_{C^\beta(\overline{\mathbb{R}^{n+1}_+})} \leq C$$

for some constant C independent of $s \in (s_0, 1)$. In fact, C depends only on n, s_0, $\|f\|_{C^{1,\alpha}}$, and $\|u\|_{L^\infty(\mathbb{R}^n)}$.

We also have the following gradient estimates for the extension problem.

Proposition D.1.2. *Let $f \in C^{1,\alpha}(\mathbb{R})$ with $\alpha > \max\{0, 1-2s\}$ and suppose $v \in L^\infty(\mathbb{R}^{n+1}_+)$ is a weak solution of problem* (4.18)*. Then, $\nabla_x v$ and $y^a v_y$ belong to $L^\infty(\mathbb{R}^{n+1}_+)$. In addition, given $s_0 > \frac{1}{2}$, there exists a constant C_1 depending only on n, s_0, $\|f\|_{C^{1,\alpha}}$, and $\|v\|_{L^\infty(\mathbb{R}^{n+1}_+)}$ such that, for every $s > s_0$, we have*

$$\|\nabla_x v\|_{L^\infty(\mathbb{R}^{n+1}_+)} + (1+a)\|y^a v_y\|_{L^\infty(\mathbb{R}^{n+1}_+)} \leq C_1. \tag{D.1}$$

Furthermore, we have

$$|\nabla v(x,y)| \leq \frac{C_2}{y} \text{ for } y > 0, \tag{D.2}$$

where C_2 is uniformly bounded for $a \in (-1,1)$ (i.e. $s \in (0,1)$). As a consequence of (D.1) *and* (D.2)*, we have*

$$y^a |\nabla v|^2 \in L^1_{loc}(\overline{\mathbb{R}^{n+1}_+}). \tag{D.3}$$

The next result is concerned with solutions of (4.18) having limits in one variable (e.g. layer solutions).

Proposition D.1.3. *Let v be a bounded solution of* (4.18) *such that*

$$\lim_{x_n \to \pm\infty} v(x,0) = L^\pm \tag{D.4}$$

for every $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and some constants $L^\pm$. Then,

$$f(L^+) = f(L^-) = 0 \tag{D.5}$$

and

$$\lim_{x_n \to \pm\infty} v(x,y) = L^\pm \tag{D.6}$$

for every $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ *and* $y \geq 0$*. Moreover, for every fixed* $R > 0$ *and* $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$*, we have*

$$\left\| v - L^{\pm} \right\|_{L^\infty(B_R^+(x,0))} \to 0 \textit{ as } x_n \to \pm\infty, \tag{D.7}$$

$$\|\nabla_x v\|_{L^\infty(B_R^+(x,0))} \to 0 \textit{ as } x_n \to \pm\infty, \textit{ and} \tag{D.8}$$

$$\|y^a v_y\|_{L^\infty(B_R^+(x,0))} \to 0 \textit{ as } x_n \to \pm\infty. \tag{D.9}$$

D.2 Maximum Principles

Proposition D.2.1 (Maximum Principle for L_a; Remark 4.2 in [9])**.** *The following weak and strong maximum principles hold for the operator* L_a*:*

1. *(Weak Maximum Principle). The standard weak maximum principle holds for weak solutions of (5.49). More generally, if* u *weakly solves*

$$\begin{cases} -L_a u \geq 0 \textit{ in } B_r^+, \\ -y^a u_y \geq 0 \textit{ on } \Gamma_r^0, \\ u \geq 0 \textit{ on } \Gamma_r^+, \end{cases} \tag{D.10}$$

   then $u \geq 0$ *in* B_r^+*.*

2. *(Strong Maximum Principle). Moreover, either* $u \equiv 0$ *or* $u > 0$ *in* $B_r^+ \cup \Gamma_r^0$*.*

Remark D.2.1. The same weak and strong maximum principles hold in other bounded domains of $\mathbb{R}^{n+1}_+$ other than B_r^+. In fact, their proofs are essentially the same, requiring only minor adjustments. Moreover, the maximum principles also hold for the Dirichlet problem in B_r^+, obtained by replacing the Neumann condition in (5.52) with $u \geq$ on Γ_r^0.

The next result is a *Hopf boundary lemma* for the operator L_a.

Lemma D.2.1 (Hopf Lemma; Proposition 4.11 in [9])**.** *Let* $a \in (-1, 1)$ *and consider the cylinder* $\mathcal{C}_{r,1} := \Gamma_r^0 \times (0, 1) \subset \mathbb{R}^{n+1}_+$ *where* Γ_r^0 *is the ball centered at the origin of radius* r *in* $\mathbb{R}^n$*. Let*

$u \in C(\overline{\mathcal{C}_{r,1}}) \cap H^1(\mathcal{C}_{r,1}, y^a)$ *satisfy*

$$\begin{cases} L_a u \leq 0 \textit{ in } \mathcal{C}_{r,1}, \\ u > 0 \textit{ in } \mathcal{C}_{r,1}, \\ u(0,0) = 0. \end{cases} \tag{D.11}$$

Then,

$$\limsup_{y \to 0^+} -y^a \frac{u(0,y)}{y} < 0. \tag{D.12}$$

In addition, if $y^a u_y \in C(\overline{\mathcal{C}_{r,1}})$*, then*

$$\partial_{\nu^a} u(0,0) < 0. \tag{D.13}$$

We note that, since we have assumed $a = 1 - 2s$ with $s \in (0,1)$, the hypothesis $a \in (-1,1)$ is met. Furthermore, the space $H^1(\mathcal{C}_{r,1}, y^a)$ denotes the Sobolev space $H^1(\Omega)$, where $\Omega := \mathcal{C}_{r,1}$ and integration is respect to the measure $y^a\, dx\, dy$, with $dx\, dy$ being the standard Lebesgue measure on $\mathbb{R}^n$.

We also have a Harnack inequality.

Proposition D.2.2 (Harnack Inequality; Theorem 3.4 in [9])**.** *Let* u *be a positive solution of* $L_a u = 0$ *in* $B_{4r}(x_0) \subset \mathbb{R}^{n+1}$*. Then,* $\sup_{B_r(x_0)} u \leq C \inf_{B_r(x_0)} u$ *for some constant* $C = C(n,a)$ *depending only on* n *and* a*. As a consequence, bounded solutions of* $L_a u = 0$ *in all of* $\mathbb{R}^{n+1}$ *are constant.*

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VITA

Bryan Dimler was born and raised in San Antonio, TX. After initially studying psychology, Bryan returned to the University of Texas at San Antonio to study mathematics. Soon after returning to school, Bryan earned a Bachelor of Science in Mathematics and, subsequently, entered the Master of Science in Mathematics program at the University of Texas at San Antonio. As a graduate student, Bryan has focused his studies in analysis, partial differential equations, and, more recently, geometric analysis. Moreover, Bryan has recently accepted an offer of admission from the University of California Irvine doctoral program in mathematics where he intends to focus his studies in partial differential equations, differential geometry, and geometric analysis.