Homework 1 Solutions

1. Recall that, since $m(A) > 0$, the set $A - A$ contains an interval I centered at the origin. We know that $2^{-n} \in I$ for large $n \in \mathbb{N}$. Since $I \subset A - A$, we can thereby choose $x, y \in A$ with $x - y = 2^{-n}$. In particular, $|x - y| = 2^{-n}$ so the proof is complete.

2. (a) We first prove the right-to-left direction. Suppose $A \in \mathcal{A}$ is μ-null. For each $n \in \mathbb{N}$, choose δ_n such that if $B \in \mathcal{A}$ satisfies $\mu(B) < \delta_n$, then $\nu(B) < n^{-1}$. Since $\mu(A) < \delta_n$ for each n we must have $\nu(A) < n^{-1}$ for each n implying $\nu(A) = 0$. That is, ν is absolutely continuous with respect to μ.
   Suppose now that ν is absolutely continuous with respect to μ. If the $\epsilon - \delta$ condition is not satisfied, there is an $\epsilon > 0$ such that for each $n \in \mathbb{N}$ we can find $E_n \in \mathcal{A}$ with $\mu(E_n) < 2^{-n}$ and $\nu(E_n) \geq \epsilon$. Let $F_k := \cup_k^\infty E_n$ and $F = \cap_1^\infty F_k$. Then
   $$\mu(F_k) < \sum_k^\infty 2^{-n} = 2^{1-k},$$
   so $\mu(F) = 0$. On the other hand, $\nu(F_k) \geq \epsilon$ for each k and, since ν is finite, $\nu(F) = \lim \nu(F_k) \geq \epsilon$ contradicting that ν is absolutely continuous with respect to μ.

   (b) Let $(X, \mathcal{M}, \mu) = (\mathbb{R}, \mathcal{L}, m)$ and define $\nu(A) := \int_A x^{-2}\, dm$ for Lebesgue measurable A. Then ν is a measure that is absolutely continuous with respect to m and $\nu([-\epsilon, \epsilon]) = \infty$ for every $\epsilon > 0$ so the left-to-right implication in part (a) does not hold.

   Remark: This is closely related to the *Radon-Nikodym Theorem*, which we will study later in the quarter. The intuition is that if ν is absolutely continuous with respect to μ and finite, then we may write
   $$\nu(A) = \int_A f\, d\mu \text{ for some nonnegative } f \in L^1(\mu).$$
   Notice that this expression for ν implies the left-to-right implication immediately.

3. By Fatou's Lemma, we have
   $$\int_{\mathbb{R}} |f(x)|\, dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}} |f_n(x)|\, dx < \infty$$
   since the f_n are bounded in $L^1(\mathbb{R})$. Hence, $f \in L^1(\mathbb{R})$. Applying Fatou's Lemma once more, we find
   $$\int_{\mathbb{R}} |f(x)|\, dx \leq \liminf_{n\to\infty} \Big(\int_{\mathbb{R}} |f_n(x)|\, dx - \int_{\mathbb{R}} |f_n(x) - f(x)|\, dx \Big).$$
   On the other hand, the reverse triangle inequality applied to the second term on the right-hand side of the inequality above shows
   $$\begin{aligned}\int_{\mathbb{R}} |f_n(x)|\, dx - \int_{\mathbb{R}} |f_n(x) - f(x)|\, dx &\leq \int_{\mathbb{R}} |f_n(x)|\, dx - \Big(\int_{\mathbb{R}} |f_n(x)|\, dx - \int_{\mathbb{R}} |f(x)|\, dx \Big) \\ &= \int_{\mathbb{R}} |f(x)|\, dx.\end{aligned}$$
   Hence,
   $$\limsup_{n\to\infty} \Big(\int_{\mathbb{R}} |f_n(x)|\, dx - \int_{\mathbb{R}} |f_n(x) - f(x)|\, dx \Big) \leq \int_{\mathbb{R}} |f(x)|\, dx.$$
   Combining this with the inequality for the $\liminf$ above concludes the proof.

4. If f is continuous on $[0,1]$, then there is an $M>0$ such that $|f(x)|\le M$ for all $x\in[0,1]$. Since $x^n f(x)\to 0$ almost everywhere on $[0,1]$ and $f\in L^1([0,1])$ by continuity, the dominated convergence theorem shows that

$$\lim_{n\to\infty}\int_0^1 x^n f(x)\,dx = 0.$$

Now, for each $n,k\in\mathbb{N}$ we may write

$$n\int_0^1 x^n f(x)\,dx = n\int_0^{1-\frac{1}{k}} x^n f(x)\,dx + n\int_{1-\frac{1}{k}}^1 x^n f(x)\,dx.$$

Let $\epsilon>0$ be given and choose $k\in\mathbb{N}$ so large that $|1-x|<\frac{1}{k}$ implies

$$-\epsilon < f(x)-f(1) < \epsilon.$$

We estimate the terms in the sum above individually. We have

$$n\int_0^{1-\frac{1}{k}} x^n f(x)\,dx \le n\Big(1-\frac{1}{k}\Big)^n M \to 0 \text{ as } n\to\infty.$$

On the other hand,

$$\begin{aligned} n\int_{1-\frac{1}{k}}^1 x^n f(x)\,dx &\le n\int_{1-\frac{1}{k}}^1 x^n(f(x)-f(1)) + n\int_{1-\frac{1}{k}}^1 x^n f(1)\,dx\,dx \\ &\le (\epsilon+f(1))n\int_{1-\frac{1}{k}}^1 x^n\,dx \\ &= (\epsilon+f(1))\frac{n}{n+1}\Big(1-\Big(1-\frac{1}{k}\Big)^{n+1}\Big). \end{aligned}$$

Taking the lim sup as $n\to\infty$ and combining with the previous inequalities gives

$$\limsup_{n\to\infty}\Big(n\int_{1-\frac{1}{k}}^1 x^n f(x)\,dx\Big)\le \epsilon+f(1).$$

Similarly,

$$\liminf_{n\to\infty}\Big(n\int_{1-\frac{1}{k}}^1 x^n f(x)\,dx\Big)\ge f(1)-\epsilon.$$

Since this holds for each $\epsilon>0$, combining with the previous inequalities shows

$$f(1)\le \liminf_{n\to\infty}\Big(n\int_0^1 x^n f(x)\,dx\Big)\le \limsup_{n\to\infty}\Big(n\int_0^1 x^n f(x)\,dx\Big)\le f(1).$$

Hence, the limit exists and is equal to $f(1)$.

5. Fix $t\in\mathbb{R}\setminus\{0\}$ and $a\in\mathbb{R}$. The idea is to control the growth of the integral in question on annuli emanating from the origin. We have

$$\int_{\mathbb{R}}|f(x)||g(tx)|\,dx = \sum_1^\infty\int_{n-1\le|x|\le n}|f(x)||g(xt)|\,dx \le \sum_1^\infty e^{-|t|(n-1)}\int_{n-1\le|x|\le n}|f(x)|\,dx.$$

In addition, the growth assumption on the integral of f gives

$$e^{-|t|(n-1)}\int_{n-1\le|x|\le n}|f(x)|\,dx \le e^{-|t|(n-1)}\int_{|x|\le n}|f(x)|\,dx \le e^{-|t|(n-1)}n^a.$$

To conclude, simply note that the series

$$\sum_1^\infty e^{-|t|(n-1)}n^a$$

converges by the ratio test. Hence, $f(x)g(tx)\in L^1(\mathbb{R})$.