Homework 2 Solutions

1. We first prove the desired limit when $f(x) = \chi_{(a,b)}(x)$ for some $a, b \in \mathbb{R}$ with $a < b$. We have

$$\Big|\int_{\mathbb{R}} e^{ix\xi} f(x)\, dx\Big| = \Big|\int_a^b e^{ix\xi}\, dx\Big| = \frac{|e^{ib\xi} - e^{ia\xi}|}{\xi} \leq \frac{2}{\xi} \to 0 \text{ as } |\xi| \to \infty.$$

Hence, the result holds for characteristic functions of open intervals. Suppose now that $f \in L^1(\mathbb{R})$ and let $\{\phi_n\}_1^\infty$ be a sequence of simple functions of the form

$$\phi_n = \sum_1^k \chi_{(a_n^j, b_n^j)}$$

approximating f in $L^1(\mathbb{R})$. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ so large that $\|f - \phi_n\|_1 < \epsilon$ for all $n \geq N$. Fix $n \geq N$. Then

$$\begin{aligned}\Big|\int_{\mathbb{R}} e^{ix\xi} f(x)\, dx\Big| &\leq \Big|\int_{\mathbb{R}} e^{ix\xi}(f(x) - \phi_n(x))\, dx\Big| + \Big|\int_{\mathbb{R}} e^{ix\xi}\phi_n\, dx\Big| \\ &\leq \|f - \phi_n\|_1 + \Big|\int_{\mathbb{R}} e^{ix\xi}\phi_n\, dx\Big| \\ &\leq \epsilon + \Big|\int_{\mathbb{R}} e^{ix\xi}\phi_n\, dx\Big|.\end{aligned}$$

The second term in the sum tends to zero as $|\xi| \to \infty$ by what was proved first. Hence,

$$\lim_{|\xi|\to 0}\Big|\int_{\mathbb{R}} e^{ix\xi} f(x)\, dx\Big| \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the proof is complete.

Remark: This is called the *Riemann-Lebesgue Lemma*. It is a fundamental tool in Fourier analysis, which is a super cool subject you should totally learn if you are interested in analysis.

2. This is a consequence of the *generalized dominated convergence theorem*, which was proved in the homework last quarter (see Problem 20 on pg. 59 of the text). To apply the theorem, take $g_n := |f_n|$ and $g := |f|$. We can then conclude that $\|f_n - f\|_{L^1} \to 0$ as $n \to \infty$ by dominated convergence. Since

$$\int_E |f_n(x) - f(x)|\, dx \leq \|f_n - f\|_{L^1}$$

for every measurable $E \subset \mathbb{R}$, the L^1 convergence of the f_n to f implies the convergence is uniform in the choice of measurable E.

3. We first consider the case when $f = \chi_{(a,b)}$. The general case will be obtained by approximation with simple functions in L^1. Suppose without loss of generality that $|h| > 2(b-a)$. Direct computation shows

$$|\chi_{(a,b)}(x+h) - \chi_{(a,b)}(x)| = \chi_{(a-h,b-h)\cup(a,b)}(x).$$

Since $(a-h, b-h)$ and (a,b) are disjoint, we get

$$\int_{\mathbb{R}} |\chi_{(a,b)}(x+h) - \chi_{(a,b)}(x)|\, dx = 2(b-a) = 2\int_{\mathbb{R}} |\chi_{(a,b)}(x)|\, dx.$$

It follows that, if ϕ is a simple function of the form

$$\phi = \sum_1^n \chi_{(a_n,b_n)}$$

and h is chosen so that $|h|$ is very large (depending on the a_j, b_j), that

$$\int_{\mathbb{R}} |\phi(x+h) - \phi(x)|\, dx = 2\int_{\mathbb{R}} |\phi(x)|\, dx.$$

Based on the above computation, we expect

$$\lim_{|h|\to\infty} \int_{\mathbb{R}} |f(x+h) - f(x)|\, dx = 2\int_{\mathbb{R}} |f(x)|\, dx.$$

Let $\epsilon > 0$, let $f \in L^1(\mathbb{R})$, and choose an L^1 simple function ϕ such that $\|\phi - f\|_{L^1} < \epsilon$. We have:

$$\begin{aligned}
\Big|\int_{\mathbb{R}} (|f(x+h) - f(x)| - 2|f(x)|)\, dx\Big| &\le \Big|\int_{\mathbb{R}} (|f(x+h) - f(x)| - |\phi(x+h) - \phi(x)|)\, dx\Big| \\
&\quad + \Big|\int_{\mathbb{R}} (|\phi(x+h) - \phi(x)| - 2|f(x)|)\, dx\Big| \\
&\le \int_{\mathbb{R}} |f(x+h) - \phi(x+h)|\, dx + \int_{\mathbb{R}} |\phi(x) - f(x)|\, dx + \\
&\quad + \Big|\int_{\mathbb{R}} (|\phi(x+h) - \phi(x)| - 2|f(x)|)\, dx\Big| \\
&\le 2\epsilon + \Big|\int_{\mathbb{R}} (|\phi(x+h) - \phi(x)| - 2|f(x)|)\, dx\Big|
\end{aligned}$$

Taking the limit as $|h| \to \infty$ on each side of the inequality above gives

$$\begin{aligned}
\lim_{|h|\to\infty} \Big|\int_{\mathbb{R}} (|f(x+h) - f(x)| - 2|f(x)|)\, dx\Big| &\le 2\epsilon + 2\Big|\int_{\mathbb{R}} (|\phi(x)| - |f(x)|)\, dx\Big| \\
&\le 2\epsilon + 2\int_{\mathbb{R}} |\phi(x) - f(x)|\, dx \\
&\le 4\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that the limit on the left-hand side of the inequality above is zero.

Remark: This problem demonstrates a good qual trick. If you need to first determine what the limit should be, try working out what the limit is when f is replaced by the characteristic function of a bounded open interval. Often times, this computation will be easy. To generalize, use approximation in L^1 by simple functions.

4. Suppose that $\{f_n\}_1^\infty$ is a sequence of functions in $L^1(\mathbb{R})$ such that $f_n \to f$ pointwise a.e. on $\mathbb{R}$ with $|f_n| \le g \in L^1(\mathbb{R})$ for each $n \in \mathbb{N}$. Since $g \in L^1(\mathbb{R})$, given $\epsilon > 0$ we may choose $r > 0$ so large that

$$\int_{|x|>r} g(x)\, dx < \delta.$$

By Egoroff's Theorem, there is a Lebesgue measurable subset of $\{|x| \le r\}$ with $m(E) < \delta$ and $f_n \to f$ uniformly on $E^c \cap \{|x| > r\}$. Then

$$\begin{aligned}
\int_{\mathbb{R}} |f_n(x) - f(x)|\, dx &= \int_{|x|>r} |f_n(x) - f(x)|\, dx + \int_{|x|\le r} |f_n(x) - f(x)|\, dx \\
&\le 2\int_{|x|>r} g(x)\, dx + 2\int_E g(x)\, dx + \int_{E^c\cap\{|x|\le r\}} |f_n(x) - f(x)|\, dx.
\end{aligned}$$

Since $g \in L^1(\mathbb{R})$, we can choose δ so small that the second term in the sum is less than ϵ. Hence,

$$\int_{\mathbb{R}} |f_n(x) - f(x)|\,dx \le 3\epsilon + \int_{E^c \cap \{|x| \le r\}} |f_n(x) - f(x)|\,dx.$$

Taking the limit as $n \to \infty$ on each side of the inequality above and using that $f_n \to f$ uniformly on $E^c \cap \{|x| \le r\}$ shows that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x) - f(x)|\,dx \le 3\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $f_n \to f$ in $L^1(\mathbb{R})$ and, consequently, that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\,dx = \int_{\mathbb{R}} f(x)\,dx.$$

5. Observe that

$$\mu(E_j) = \mu(\{x : |\chi_{E_j}(x)| = 1\}) = \mu(\{x : |\chi_{E_j}(x)| \ge \epsilon\}) \text{ for each } \epsilon \in (0,1].$$

If $\epsilon > 1$, the last term in the string of equalities above is zero; hence, letting $j \to \infty$ shows that $\chi_{E_j} \to 0$ in measure. Now, for each $j \in \mathbb{N}$ set $f_j := f \cdot \chi_{E_j}$. Then $f_j \to 0$ in measure with $|f_j| \le |f| \in L^1(\mu)$ for each j. By the Dominated Convergence Theorem (see Problem 5 in the week 2 discussion),

$$\lim_{j\to\infty} \int_{E_j} f(x)\,dx = 0,$$

as desired.

6. We first prove the result for $f \in C_0(\mathbb{R})$. Since $\frac{a_i}{b_i} \ne \frac{a_j}{b_j}$ unless $i = j$, $\operatorname{supp} f(b_i x + t a_i)$ is disjoint for $t >> a$. To see this, let $K \subset \mathbb{R}$ be compact such that $\operatorname{supp} f \subset K$. Assume there are indices $i \ne j$ such that $b_i x + t a_i$, $b_j x + t a_j \in K$. Then $x + \frac{t a_i}{b_i} \in K'$ and $x + \frac{t a_j}{b_j} \in K'$ for some translation of K, K'. Then

$$t\Big(\frac{a_i}{b_i} - \frac{a_j}{b_j}\Big) = x + \frac{t a_i}{b_i} - \Big(x + \frac{t a_j}{b_j}\Big) \in K' - K' \text{ for all } t$$

where $K' - K'$ is compact. Letting $t \to \infty$ contradicts that $K' - K'$ is compact. It follows that for t sufficiently large the supports are disjoint so

$$\int \Big|\sum_{j=1}^{k} f(b_j x + t a_j)\Big|\,dx = \sum_{j=1}^{k} \int |f(b_j x + t a_j)|\,dx.$$

Applying the change of variable $y = b_j x + t a_j$ in each integral we find

$$\sum_{j=1}^{k} \frac{1}{|b_j|} \|f\|_{L^1(\mathbb{R})} = \lim_{t\to\infty} \int \Big|\sum_{j=1}^{k} f(b_j x + t a_j)\Big|\,dx.$$

Let $\epsilon > 0$. Suppose now that $f \in L^1(\mathbb{R})$ and let $\phi \in C_0(\mathbb{R})$ with $\|f - \phi\|_{L^1} < \epsilon$. Then

$$\begin{aligned}
\Big|\int \Big|\sum_{j=1}^{k} f(b_j x + t a_j)\Big|\,dx - \sum_{j=1}^{k} \frac{1}{|b_j|}\|f\|_{L^1(\mathbb{R})}\Big| &\le \Big|\int \Big|\sum_{j=1}^{k} f(b_j x + t a_j)\Big|\,dx - \int \Big|\sum_{j=1}^{k} \phi(b_j x + t a_j)\Big|\,dx\Big| \\
&\quad + \Big|\int \Big|\sum_{j=1}^{k} \phi(b_j x + t a_j)\Big|\,dx - \sum_{j=1}^{k} \frac{1}{|b_j|}\|\phi\|_{L^1(\mathbb{R})}\Big| + \sum_{j=1}^{k} \frac{1}{|b_j|}\big|\|\phi\|_{L^1} - \|f\|_{L^1}\big| \\
&\le C\epsilon + \Big|\int \Big|\sum_{j=1}^{k} \phi(b_j x + t a_j)\Big|\,dx - \sum_{j=1}^{k} \frac{1}{|b_j|}\|\phi\|_{L^1(\mathbb{R})}\Big|,
\end{aligned}$$

where C is a constant depending on k and the constants a_j, b_j. Letting $t \to \infty$ shows that

$$\lim_{t\to\infty} \Big| \int \Big| \sum_{j=1}^{k} f(b_j x + t a_j) \Big| \, dx - \sum_{j=1}^{k} \frac{1}{|b_j|} \|f\|_{L^1(\mathbb{R})} \Big| \le C\epsilon.$$

Since $\epsilon > 0$ is arbitrary, the limit on the left-hand side above is zero, as desired.

7. Since $g \in L^1$, given $\epsilon > 0$ and $x \in \mathbb{R}$ we may choose $r > 0$ so large that

$$\int_{|x|>r} |g(x)| \, dx < \epsilon.$$

A u-substitution then implies

$$\int_{|x-y|>r} |g(x-y)| \, dy < \epsilon \text{ for each } x \in \mathbb{R}.$$

Let $K \subset \mathbb{R}$ be a compact set and fix $x \in K$. We have:

$$\begin{aligned}
|g * f_n(x) - g * f(x)| &\le \int_{\mathbb{R}} |g(x-y)||f_n(y) - f(y)| \, dy \\
&\le \int_{|x-y|\le r} |g(x-y)||f_n(y) - f(y)| \, dy + \int_{|x-y|>r} |g(x-y)||f_n(y) - f(y)| \, dy \\
&\le \int_{|x-y|\le r} |g(x-y)||f_n(y) - f(y)| \, dy + 2\epsilon,
\end{aligned}$$

where we have used the fact that $|f_n(x)| \le 1$ and $|f(x)| \le 1$ for all x and n. Notice that $f_n(y) \to f(y)$ pointwise a.e. and

$$|g(x-y)||f_n(y) - f(y)| \le 2|g(x-y)| \in L^1,$$

so we may apply the dominated convergence theorem to conclude that for each $x \in K$ there is an $N_x \in \mathbb{N}$ such that

$$\int_{|x-y|\le r} |g(x-y)||f_n(y) - f(y)| \, dy < \epsilon \text{ for all } n \ge N_x.$$

To conclude, notice that the collection of intervals $I_x := \{y : |x-y| \le r\}$ form an open cover of K; thus, we may extract a finite sub-cover $I_{x_1}, \ldots, I_{x_k}$. Taking $N := \max\{N_{x_1}, \ldots, N_{x_k}\}$, we conclude that

$$|g * f_n(x) - g * f(x)| \le 3\epsilon \text{ for all } x \in K \text{ and all } n \ge N.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $g * f_n \to g * f$ uniformly on K. Since K is an arbitrary compact set, this concludes the proof.

Remark: Notice that I did not use Egoroff's Theorem.