Modes of Convergence

1 Notes

We now have several notions of convergence at our disposal. Recall that uniform convergence implies pointwise convergence, which implies pointwise a.e. convergence. However, the implications cannot be reversed. Another mode of convergence that is useful is *convergence in measure*.

Definition 1. *(a) We say that a sequence $\{f_n\}$ of measurable real-valued functions on $(X, \mathcal{M}, \mu)$ is Cauchy in measure if for every $\epsilon > 0$,*

$$\mu(\{x : |f_n(x) - f_m(x)| \geq \epsilon\}) \to 0 \text{ as } m, n \to \infty.$$

(b) We say that hat $\{f_n\}$ converges in measure to f if for every $\epsilon > 0$

$$\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \to 0 \text{ as } n \to \infty.$$

It is not hard to show that if $f_n \to f$ in L^1, then $f_n \to f$ in measure. However, the converse is false. To see this, consider $f_n := n^{-1}\chi_{(0,n)}$, for example.

The key theorems are as follows:

Theorem 1. *If $f_n \to f$ in L^1, there is a subsequence $\{f_{n_j}\}$ such that $f_{n_j} \to f$ pointwise almost everywhere.*

Theorem 1 also holds in the case $f_n \to f$ in measure.

Theorem 2 (Egoroff's Theorem)**.** *Suppose that $\mu(X) < \infty$ and $f_1, f_2, \ldots$ and f are measurable real-valued functions on X such that $f_n \to f$ a.e. Then for every $\epsilon > 0$ there exists a $E \subset X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.*

The convergence in Theorem 2 is called *almost uniform convergence*. Thus, Egoroff's Theorem says that pointwise a.e. convergence of a sequence of measurable functions implies almost uniform convergence. Furthermore, it is not difficult to show that almost uniform convergence implies a.e. convergence and convergence in measure (see Problem 3).

2 Problems

1. $f_n \to f$ in measure iff for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) < \epsilon$ for every $n \geq N$.

   Solution. If $f_n \to f$ in measure, then for every $\epsilon > 0$ we find $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$ as $n \to \infty$. In particular, given $\epsilon > 0$ we may find $N \in \mathbb{N}$ such that $n \geq N$ implies $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) < \epsilon$. Now, suppose that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) < \epsilon$. Note that for every δ such that $0 < \delta < \epsilon$, there exists $K \in \mathbb{N}$ such that $n \geq K$ implies $\mu(\{x : |f_n(x) - f(x)| \geq \delta\}) < \delta < \epsilon$. In addition, $\{x : |f_n(x) - f(x)| \geq \epsilon\} \subset \{x : |f_n(x) - f(x)| \geq \delta\}$ for each $n \geq \max\{N, K\}$ since $\delta < \epsilon$, so $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) < \delta$ for each $n \geq \max\{N, K\}$. Since $\epsilon > 0$ is arbitrary and δ is any number satisfying $0 < \delta < \epsilon$, we conclude $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$ as $n \to \infty$ for every $\epsilon > 0$. That is, $f_n \to f$ in measure. □

2. If $\mu(E_n) < \infty$ for each $n \in \mathbb{N}$ and $\chi_{E_n} \to f$ in L^1, then f is a.e. equal to the characteristic function of a measurable set.

Solution. Since $f_n := \chi_{E_n} \to f$ in L^1 we can extract a subsequence of the f_n, $\{f_{n_k}\}$, converging to f pointwise a.e. Choose x so that $f_{n_k}(x) \to f(x)$. We claim that either $f(x) = 1$ or $f(x) = 0$. Indeed, if $f(x) \notin \{0, 1\}$ then there is an $\epsilon > 0$ such that

$$\min\{|f(x) - 1|, |f(x)|\} > \epsilon.$$

Since the range of f_{n_k} is contained in $\{0, 1\}$ for each n_k, this implies $|f_{n_k} - f(x)| > \epsilon$ for each n_k contradicting that $f_{n_k}(x) \to f(x)$. Furthermore, f is measurable so that the set $E := \{f \neq 0\}$ is measurable. By what was just proved, we conclude that f is a.e. equal to χ_E. □

3. Show that if $f_n \to f$ almost uniformly, then $f_n \to f$ almost everywhere and in measure.

Solution. Suppose $f_n \to f$ almost uniformly. Then for every $\epsilon > 0$ there is $E \subset X$ with $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c. Choose E_n so that this condition is satisfied for each $n \in \mathbb{N}$ with $\epsilon_n = \frac{1}{n}$. Set $E = \bigcup_{n\in\mathbb{N}} E_n^c$. Then $f_n \to f$ on E pointwise and $\mu(E^c) = \mu\big(\bigcap_{n\in\mathbb{N}} E_n\big) = 0$. Thus, $f_n \to f$ a.e. Let $\delta, \epsilon > 0$. Choose E so that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c. Choose $n \in \mathbb{N}$ so that $n \geq N$ implies $|f_n - f| < \delta$ on E^c. Then $\{x : |f_n - f| \geq \delta\} \subset E$ so

$$\mu(\{x : |f_n - f| \geq \delta\}) < \epsilon$$

and this holds for each $n \geq N$. Since $\epsilon, \delta > 0$ are arbitrary, it follows that $f_n \to f$ in measure. □

4. (**Fatou's Lemma**) If $\{f_n\} \subset L^+$, $f_n \geq 0$, and $f_n \to f$ in measure, then $\int f \leq \liminf \int f_n$.

Solution. Suppose $f_n \to f$ in measure. Let $\delta, \epsilon > 0$ and choose $N \in \mathbb{N}$ so that $n \geq N$ implies

$$\mu(\{x : |f_n - f| \geq \frac{\delta}{2}\}) < \epsilon.$$

Observe that if $m > n \geq N$, we have

$$|f_n - f_m| \leq |f_n - f| + |f - f_m|.$$

Then, if $|f_n(x) - f_m(x)| \geq \delta$, either

$$|f_n(x) - f(x)| \geq \frac{\delta}{2} \text{ or } |f_m(x) - f(x)| \geq \frac{\delta}{2}.$$

In either case, we find

$$\mu(\{x : |f_n(x) - f_m(x)| \geq \delta\}) < \epsilon \text{ for all } m > n \geq N$$

so $\{f_n\}$ is Cauchy in measure. Furthermore, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \int f_{n_k} = \liminf_{n\to\infty} \int f_n.$$

It is clear that $f_{n_k} \to f$ in measure also. Thus, By Theorem 2.30, there exists a subsequence $\{f_{n_{k_j}}\}$ such that $f_{n_{k_j}} \to f$ as $j \to \infty$ a.e. Applying Fatou's lemma, we find

$$\int f \leq \lim_{j\to\infty} \int f_{n_{k_j}} = \lim_{k\to\infty} \int f_{n_k} = \liminf_{n\to\infty} \int f_n,$$

as desired. □

5. (**Dominated Convergence Theorem**) Suppose $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure.

(a) $\int f = \lim \int f_n$.

(b) $f_n \to f$ in L^1.

Solution. Since $f_n \to f$ in measure, $\{f_n\}$ is Cauchy in measure so by Theorem 2.30 there is a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k} \to f$ a.e. Since $|f_{n_k}| \le g$ for each k, $|f| \le g$ a.e. so $f \in L^1$ also. Since

$$|f_n - f| = |f_n + g - (f + g)| = |(g - f_n) - (g - f)| \text{ for each } n \in \mathbb{N},$$

it is clear that if $f_n \to f$ in measure, both $f_n + g \to f + g$ in measure and $g - f_n \to g - f$ in measure. Since $g + f_n \ge 0$ a.e. and $g - f_n \ge 0$ a.e., we may apply Problem 2 to find

$$\int g + \int f \le \liminf \int (g + f_n) = \int g + \liminf \int f_n,$$
$$\int g - \int f \le \liminf \int (g - f_n) = \int g - \limsup \int f_n.$$

Thus, $\liminf \int f_n \ge \int f \ge \limsup \int f_n$ so $\int f = \lim \int f_n$. To see that $f_n \to f$ in L^1 also, simply note that if $f_n \to f$ in measure, then $|f_n - f| \to 0$ in measure. Furthermore, it holds that $|f_n - f| \le 2|g|$ a.e. By what was just proved, $|f_n - f| \to 0$ in L^1 which holds iff $f_n \to f$ in L^1. $\square$