Integration on $\mathbb{R}^n$

Here, we briefly discuss the main theorems concerning integration on $\mathbb{R}^n$. Specifically, we will cover the Fubini Theorem and change of variable formula. We will focus on the the case where $(X, \mathcal{M}, \mu) = (\mathbb{R}^n, \mathcal{L}, m)$ for simplicity.

We begin by writing $\mathbb{R}^n$ as a product $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ where $n = n_1 + n_2$ and $n_1, n_2 \geq 1$. We can then write a point in $\mathbb{R}^n$ as (x, y) for $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$. With such a decomposition in mind, the notion of a *slice* becomes natural.

If $E \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, we define the *$x$ and y slices of E* by

$$E_x := \{y \in \mathbb{R}^{n_2} : (x, y) \in E\} \text{ and } E^y := \{x \in \mathbb{R}^{n_1} : (x, y) \in E\}.$$

If $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, we can define the *slice of f* corresponding to $y \in \mathbb{R}^{n_2}$ to be the function $f^y(x) := f(x, y)$ (here, y is fixed). One issue that arises in the proof of the Fubini Theorem is that, even if f is measurable on $\mathbb{R}^n$, it is not necessarily true that the slice f^y is measurable on $\mathbb{R}^{n_1}$ for each y; nor does the corresponding assertion necessarily hold for a measurable set E. To see this, let V be the Vitali non-measurable set in $\mathbb{R}$ and consider $E := V \times \{0\}$. Then E has measure zero in $\mathbb{R}^2$ so that E is measurable. However, the slices E^y are not all measurable since E^0 is V. Fortunately, measurability holds for almost all slices. We can state the Fubini Theorem precisely as follows:

Theorem 1 (Fubini Theorem)**.** *Suppose $f(x, y)$ is integrable on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Then for almost every $x \in \mathbb{R}^{n_2}$:*

(i) The slice f^y is integrable on $\mathbb{R}^{n_1}$.

(ii) The function defined by

$$\int_{\mathbb{R}^{n_1}} f^y(x)\, dx$$

is integrable on $\mathbb{R}^{n_2}$ and

$$\int_{\mathbb{R}^{n_2}} \Big(\int_{\mathbb{R}^{n_2}} f(x, y)\, dx \Big) dy = \int_{\mathbb{R}^n} f.$$

The theorem is symmetric in x and y. Furthermore, the theorem states that the integral of f on $\mathbb{R}^n$ can be computed by iterating lower-dimensional integrals, and that the integrals can be taken in any order, coinciding with the corresponding theorem from multivariable calculus. Since any complex-valued function is of the form $f = g + ih$ where g and h are real-valued, the result extends to complex-valued functions also by applying Theorem 1 to the real and imaginary parts of f. When $f \geq 0$, the assumption of integrability of f can be replaced with mmeasurability of f. This is often referred to as *Tonelli's Theorem.* As an immediate consequence of the Fubini Theorem, we can prove the measure theoretic equivalent to a classic theorem in calculus: If $f : [a, b] \to \mathbb{R}$ is integrable and non-negative, then $\int_a^b f(x)\, dx$ is equal to the area under the graph of f.

Corollary 1. *Suppose $f(x) \geq 0$ is a real-valued function on $\mathbb{R}^n$ and let*

$$\mathcal{A} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq y \leq f(x)\}.$$

Then:

(i) f is measurable on $\mathbb{R}^n$ if and only if $\mathcal{A}$ is measurable in $\mathbb{R}^{n+1}$.

(ii) If the condition (i) holds, then

$$\int_{\mathbb{R}^n} f(x) = m(\mathcal{A}).$$

Remark 1. *It is a good exercise to prove Corollary 1.*

More generally, the Fubini Theorem holds for complete. σ-finite measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. For the general Tonelli Theorem, only the assumption of σ-finiteness is necessary. However, in practice the Fubini Theorem is typically applied on $\mathbb{R}^n$ with the Lebesgue measure. For more information, see section 2.5 in Folland.

We now turn our attention to the *change of variable formula.* Let $G = (g_1, \ldots, g_n) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector-valued function with C^1 component functions g_i. Let DG be the total derivative of G (i.e. the matrix $(\frac{\partial g_i}{\partial x_j})_{i,j=1,\ldots,n}$). We call G a C^1 *diffeomorphism* if G is injective and DG is invertible for all $x \in \Omega$. By the Inverse Function Theorem, the inverse map $G^{-1} : G(\Omega) \to \Omega$ is also a C^1 diffeomorphism and $D(G^{-1})(x) = (DG)^{-1}(G^{-1}(x))$ for all $x \in G(\Omega)$.

Theorem 2 (Change of Variable Formula)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is open and $G : \Omega \to \mathbb{R}^n$ is a C^1 diffeomorphism.*

(i) If f is a Lebesgue measurable function on $G(\Omega)$, then $f \circ G$ is Lebesgue measurable on Ω. Moreover, if $f \geq 0$ or $f \in L^1(G(\Omega))$, then

$$\int_{G(\Omega)} f(x)\,dx = \int_{\Omega} f \circ G(x) |\det DG(x)|\,dx.$$

(ii) If $E \subset \Omega$ and $E \in \mathcal{L}$, then $G(E) \in \mathcal{L}$ and $m(G(E)) = \int_E |\det DG(x)|\,dx$.

Statement (ii) is the most important for intuition. By examining (ii), we see that the change of variable formula quantifies how a C^1 deformation of a measurable set E changes its volume. Notice that, as a direct consequence, the change of variable formula allows us to conclude that the Lebesgue measure is invariant under translation and rotation. To see this, note that if G is a translation map, then $DG = I_{n\times n}$ so its determinant is identically one. If G is a rotation map, then DG is an orthogonal matrix with determinant one. In either case, (ii) shows that $m(G(E)) = m(E)$.

References: *Real Analysis: Measure Theory, Integration, and Hilbert Spaces* by Elias M. Stein and Rami Shakarchi and *Real Analysis: Modern Techniques and Their Applications*, 2nd ed., Gerald B. Folland.