Homework 3 Solutions

1. Set
$$A^+ := \{f \geq 0\} \text{ and } A^- := \{f < 0\}.$$
Then A^+ and A^- are in $\mathcal{A}$ and
$$\int_{A^+} f\, d\mu = \int_{A^-} f\, d\mu = 0.$$
By assumption, both integrals on the right-hand side are zero. Since f does not change sign on A^+ nor on A^-, $f = 0$ a.e. on each of these sets. Since $X = A^+ \cup A^-$, it follows that $f = 0$ a.e. on X.

2. We first prove the result for $f \in C_0(\mathbb{R})$. Since f has compact support, we know that $\operatorname{supp} f \subset [a, b]$ for some $-\infty < a < b < \infty$. Furthermore, by u-substitution we have
$$\int_{-T}^{T} f(s+t)\, dt = \int_{s-T}^{s+T} f(t)\, dt.$$
The integral above is zero if $s > b + T$ or $s < a - T$. Hence, for T fixed we find
$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{2T}\Big|\int_{-T}^{T} f(s+t)\, dt\Big|\, ds &= \frac{1}{2T}\int_{a-T}^{b+T} \Big|\int_{s-T}^{s+T} f(t)\, dt\Big|\, ds \\ &= \frac{1}{2T}\Big(\int_{a-T}^{a-T+\sqrt{T}} + \int_{a-T+\sqrt{T}}^{b+T-\sqrt{T}} + \int_{b+T-\sqrt{T}}^{b+T}\Big)\Big|\int_{s-T}^{s+T} f(t)\, dt\Big|\, ds.\end{aligned}$$
Notice that
$$\begin{aligned}\frac{1}{2T}\Big(\int_{a-T}^{a-T+\sqrt{T}} + \int_{b+T-\sqrt{T}}^{b+T}\Big)\Big|\int_{s-T}^{s+T} f(t)\, dt\Big|\, ds &\leq \frac{1}{2T}\|f\|_{L^1}\Big(\int_{a-T}^{a-T+\sqrt{T}} dt + \int_{b+T-\sqrt{T}}^{b+T} dt\Big) \\ &= \frac{\|f\|_{L^1}}{\sqrt{T}} \to 0 \text{ as } T \to \infty.\end{aligned}$$
Hence, we only need to focus on the middle integral in the sum. For $s \in (a - T + \sqrt{T}, b + T - \sqrt{T})$, we have $s + T \geq a + \sqrt{T} > b$ for T large and $s - T \leq b - \sqrt{T} < a$ for T large. Thus, for large T
$$\int_{s-T}^{s+T} f(t)\, dt = \int_{-\infty}^{\infty} f(t)\, dt.$$
Then for large T
$$\begin{aligned}\frac{!}{2T}\int_{a-T+\sqrt{T}}^{b+T-\sqrt{T}} \Big|\int_{s-T}^{s+T} f(t)\, dt\Big|\, ds = \Big|\int_{-\infty}^{\infty} f(t)\, dt\Big| &= \Big|\int_{-\infty}^{\infty} f(t)\, dt\Big| \cdot \Big(1 + \frac{b-a}{2} - \frac{1}{\sqrt{T}}\Big) \\ &\to \Big|\int_{-\infty}^{\infty} f(t)\, dt\Big| \text{ as } T \to \infty.\end{aligned}$$
This proves the result when $f \in C_0(\mathbb{R})$.
Suppose now that $f \in L^1(\mathbb{R})$ and let $\{\phi_n\}_1^\infty \subset C_0(\mathbb{R})$ such that $\phi_n \to f$ in L^1. Suppose further that $|\phi_n(t)| \leq |f(t)|$ for all t. For each n, T set
$$g_n(s,T) := \frac{1}{2T}\Big|\int_{-T}^{T} \phi_n(s+t)\, dt\Big| \text{ and } g(s,T) := \frac{1}{2T}\Big|\int_{-T}^{T} f(s+t)\, dt\Big|.$$

The assumption that $\phi_n \to f$ in L^1 implies $g_n(s,T) \to g(s,T)$ pointwise for each s, T fixed. Furthermore,

$$|g_n(s,T)| \le \frac{1}{2T}\int_{-T}^{T} |\phi_n(s+t)|\,dt \le \frac{1}{2T}\int_{-T}^{T} |f(s+t)|\,dt \in L^1(\mathbb{R}; ds).$$

We may therefore apply the dominated convergence theorem to conclude that

$$\int_{-\infty}^{\infty} |g_n(s,T) - g(s,T)|\,ds \to 0 \text{ as } n \to \infty$$

for each fixed T. Now,

$$\begin{aligned}\Big|\int_{-\infty}^{\infty} \frac{1}{2T}\Big|\int_{-T}^{T} f(s+t)\,dt\Big|\,ds - \Big|\int_{-\infty}^{\infty} f(t)\,dt\Big|\Big| \le{}& \int_{-\infty}^{\infty} |g(s,T) - g_n(s,T)|\,ds \\ &+ \Big|g_n(s,T) - \Big|\int_{-\infty}^{\infty} \phi_n(t)\,dt\Big|\Big| \\ &+ \int_{-\infty}^{\infty} |\phi_n(t) - f(t)|\,dt.\end{aligned}$$

Let $\epsilon > 0$ be given. Choosing n large, the first and last terms in the sum on the right-hand side above can be made $< \epsilon$. Similarly, by choosing T large, the middle term can be made $< \epsilon$. It follows that, for large T, the quantity on the left-hand side of the inequality above is less than 3ϵ. Since $\epsilon > 0$ is arbitrary, the proof is complete.

3. This is just Theorem 2.30 on page 61 in Folland.

4. **Proof 1:** Since $f_n \to f$ in measure, $\{f_n\}$ is Cauchy in measure so by Theorem 2.30 there is a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k} \to f$ a.e. Since $|f_{n_k}| \le g$ for each k, $|f| \le g$ a.e. so $f \in L^1$ also. Since

$$|f_n - f| = |f_n + g - (f+g)| = |(g - f_n) - (g - f)| \text{ for each } n \in \mathbb{N},$$

it is clear that if $f_n \to f$ in measure, both $f_n + g \to f + g$ in measure and $g - f_n \to g - f$ in measure. Since $g + f_n \ge 0$ a.e. and $g - f_n \ge 0$ a.e., we may apply Problem 4 on the week 2 discussion worksheet to find

$$\begin{aligned}\int g + \int f &\le \liminf \int (g + f_n) = \int g + \liminf \int f_n, \\ \int g - \int f &\le \liminf \int (g - f_n) = \int g - \limsup \int f_n.\end{aligned}$$

Thus, $\liminf \int f_n \ge \int f \ge \limsup \int f_n$ so $\int f = \lim \int f_n$. To see that $f_n \to f$ in L^1 also, simply note that if $f_n \to f$ in measure, then $|f_n - f| \to 0$ in measure. Furthermore, it holds that $|f_n - f| \le 2|g|$ a.e. By what was just proved, $|f_n - f| \to 0$ in L^1 which holds iff $f_n \to f$ in L^1.

Remark: Notice that I did not use σ-finiteness.

Quick Proof: Since $f_n \to f$ in measure, any subsequence of $\{f_n\}$ converges to f in measure also. Let $\{f_{n_k}\}$ be any subsequence of $\{f_n\}$. Since $f_{n_k} \to f$ in measure, we can extract a further subsequence $\{f_{n_{k_j}}\}$ that converges to f pointwise a.e. By assumption, $|f_{n_{k_j}}(x)| \le g(x)$ for a.e. x and each n_{k_j}. Hence, $f_{n_{k_j}} \to f$ in L^1 by the dominated convergence theorem. In particular, every subsequence of $\{f_n\}$ has a subsequence converging to f in L^1. Since L^1 is a metric space, $f_n \to f$ in L^1 also.

5. Clearly, $d(f,g) \ge 0$ and $d(f,g) = d(g,f)$. Furthermore,

$$d(f,g) = \int \frac{|f-g|}{1+|f-g|}d\mu \le \int_X d\mu = \mu(X) < \infty$$

so d is well-defined. Since the integrand is nonnegative, $d(f,g)=0$ iff $|f-g|=0$ a.e., which holds iff $f=g$ a.e. Set $h(t)=\frac{t}{1+t}$ where $t\in[0,\infty)$. Then

$$h'(t)=\frac{1}{(1+t)^2}\geq 0$$

for all $t\geq 0$. Thus, h is non-decreasing. Since $|f-g|\leq|f-\tilde{g}|+|\tilde{g}-g|$,

$$d(f,g)\leq\int\Big(\frac{|f-\tilde{g}|}{1+|f-\tilde{g}|}+\frac{|\tilde{g}-g|}{1+|\tilde{g}-g|}\Big)d\mu=\int\frac{|f-\tilde{g}|}{1+|f-\tilde{g}|}d\mu+\int\frac{|\tilde{g}-g|}{1+|\tilde{g}-g|}d\mu=d(f,\tilde{g})+d(\tilde{g},f)$$

for any measurable complex-valued function $\tilde{g}$. We conclude that d is a metric on the space of measurable function.

Suppose $f_n\to f$ with respect to d and define $h_n:=\frac{|f_n-f|}{1+|f_n-f|}$. Since $\rho(f_n,f)\to 0$ as $n\to\infty$, $h_n\to 0$ in L^1, hence, in measure by Proposition 2.29. Notice that, if $0<\epsilon<1$, then $h(t)\geq\epsilon$ if and only if $t\geq\epsilon(1-\epsilon)^{-1}$. Hence,

$$\left\{x:\frac{|f_n-f|}{1+|f_n-f|}\geq\epsilon\right\}=\Big\{x:|f_n-f|\geq\frac{\epsilon}{1-\epsilon}\Big\}.$$

In addition, the function $t\mapsto t(1-t)^{-1}$ is surjective as map $(0,1)\to(0,\infty)$. The preceding observations show that the convergence of $\frac{|f_n-f|}{1+|f_n-f|}$ to zero in measure implies the convergence of f_n to f in measure. Suppose now that $f_n\to f$ in measure. Let $\epsilon>0$ be given. Then

$$\begin{aligned}\int\frac{|f_n-f|}{1+|f_n-f|}d\mu&=\int_{\{x:|f_n-f|\geq\epsilon\}}\frac{|f_n-f|}{1+|f_n-f|}d\mu+\int_{\{x:|f_n-f|<\epsilon\}}\frac{|f_n-f|}{1+|f_n-f|}d\mu\\&\leq\int_{\{x:|f_n-f|\geq\epsilon\}}\frac{|f_n-f|}{1+|f_n-f|}d\mu+\epsilon\mu(X)\\&\leq\mu(\{x:|f_n-f|\geq\epsilon\})+\epsilon\mu(x)\\&\to\epsilon\mu(X)\text{ as }n\to\infty.\end{aligned}$$

Since $\epsilon>0$ is arbitrary, $d(f_n,f)\to 0$ as $n\to\infty$ so $f_n\to f$ with respect to d.

6. We first prove the result when $f\in C_0(\mathbb{R})$. Set

$$m_{k,n}:=\min_{x\in[\frac{k}{n},\frac{k+1}{n}]}f(x)\text{ and }M_{k,n}:=\max_{x\in[\frac{k}{n},\frac{k+1}{n}]}f(x).$$

Then

$$m_{k,n}\leq n\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(x)\,dx\leq M_{k,n}.$$

Since f is continuous, we may apply the mean value theorem for integrals to find $x_k\in[\frac{k}{n},\frac{k+1}{n}]$ such that

$$f(x_k)=n\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(x)\,dx.$$

Thus,

$$\lim_{n\to\infty}\sum_{k=-n^2}^{n^2}\Big|\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(x)\,dx\Big|=\lim_{n\to\infty}\sum_{k=-n^2}^{n^2}\frac{1}{n}|f(x_k)|$$

For each fixed n, let $I_n:=[n,n+\frac{1}{n}]$. Choosing n large, we can be sure that $\operatorname{supp}f\subset I_n$. Hence, for large n the sum on the right-hand side above is simply the Riemann sum of $|f(x)|$ with intervals of length $\frac{1}{n}$ with a point x_k in each interval. By calculus,

$$\lim_{n\to\infty}\sum_{k=-n^2}^{n^2}\frac{1}{n}|f(x_k)|=\int_{\mathbb{R}}|f(x)|\,dx.$$

Suppose now that $f \in L^1(\mathbb{R})$ and choose $g \in C_0(\mathbb{R})$ with $\|f - g\|_{L^1} < \epsilon$ for $\epsilon > 0$ given. We have:

$$\begin{aligned}
\Big| \sum_{k=-n^2}^{n^2} \Big| \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\,dx \Big| - \|f\|_{L^1} \Big| &\le \Big| \sum_{k=-n^2}^{n^2} \Big| \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\,dx \Big| - \Big| \sum_{k=-n^2}^{n^2} \Big| \int_{\frac{k}{n}}^{\frac{k+1}{n}} g(x)\,dx \Big|\Big| \\
&\quad + \Big| \sum_{k=-n^2}^{n^2} \Big| \int_{\frac{k}{n}}^{\frac{k+1}{n}} g(x)\,dx \Big| - \|g\|_{L^1} \Big| + |\|g\|_{L^1} - \|f\|_{L^1}| \\
&\le \sum_{k=-n^2}^{n^2} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(x) - g(x)|\,dx + \Big| \sum_{k=-n^2}^{n^2} \Big| \int_{\frac{k}{n}}^{\frac{k+1}{n}} g(x)\,dx \Big| - \|g\|_{L^1} \Big| + \epsilon \\
&\le \int_{-n^2}^{n^2} |f(x) - g(x)| + \Big| \sum_{k=-n^2}^{n^2} \Big| \int_{\frac{k}{n}}^{\frac{k+1}{n}} g(x)\,dx \Big| - \|g\|_{L^1} \Big| + \epsilon \\
&\le \Big| \sum_{k=-n^2}^{n^2} \Big| \int_{\frac{k}{n}}^{\frac{k+1}{n}} g(x)\,dx \Big| - \|g\|_{L^1} \Big| + 2\epsilon.
\end{aligned}$$

Letting $n \to \infty$ and using the first part, we conclude

$$\lim_{n\to\infty} \Big| \sum_{k=-n^2}^{n^2} \Big| \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\,dx \Big| - \|f\|_{L^1} \Big| \le 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we are done.

7. Fix $t \in \mathbb{R}$ and notice that

$$h(t,x) = \frac{|f(x-t) - f(x)|}{1 + g(x)^t} \le |f(x-t)| + |f(x)| \in L^1(\mathbb{R}).$$

Hence, $x \mapsto h(t,x) \in L^1(\mathbb{R})$. We first prove the claim for $f \in C_0(\mathbb{R})$. In this case, $h(t,\mathbb{R}) \in C_0(\mathbb{R}^2)$ so if $t_n \to t_0$ we have $h(t_n, x) \to h(t_0, x)$ for each fixed $x \in \mathbb{R}$. Since $f \in C_0(\mathbb{R})$, there is a constant $M > 0$ such that $|f(x)| \le M$ for each x. Since $t_n \to t_0$ the sequence $\{t_n\}$ is bounded. Hence, there is a compact set $\tilde{K}$ containing $K \pm t_n$ for each $n \in \mathbb{N} \cup \{0\}$. Set $s(x) := M\chi_{\tilde{K}}$. Then $s \in L^1(\mathbb{R})$ and

$$|f(x - t_n)| + |f(x)| \le 2s(x) \text{ for each } n$$

so we may apply the dominated convergence theorem to conclude $H(t_n) \to H(t_0)$. It follows that H is continuous when $f \in C_0(\mathbb{R})$.

Suppose now that $f \in L^1(\mathbb{R})$ and let $\{\phi_n\}_1^\infty \subset C_0(\mathbb{R})$ be a sequence converging to f in $L^1(\mathbb{R})$. Using the reverse triangle inequality followed by the triangle inequality, we find

$$\int_{\mathbb{R}} \Big| \frac{|f(x-t) - f(x)|}{1 + g(x)^t} - \frac{|\phi_n(x-t) - \phi_n(x)|}{1 + g(x)^t} \Big|\,dx \le \int_{\mathbb{R}} \Big(|f(x-t) - \phi_n(x-t)| + |\phi_n(x) - f(x)| \Big)\,dx.$$

Since the right-hand side tends to zero as $n \to \infty$, we see that

$$\frac{|\phi_n(x-t) - \phi_n(x)|}{1 + g(x)^t} \to \frac{|f(x-t) - f(x)|}{1 + g(x)^t} \text{ in } L^1(\mathbb{R}).$$

Fix $t_0 \in \mathbb{R}$ and suppose $t_k \to t_0$ as $k \to \infty$. Then

$$\begin{aligned}
|H(t_k) - H(t_0)| &\le \int_{\mathbb{R}} \Big| \frac{|f(x-t_k) - f(x)|}{1 + g(x)^{t_k}} - \frac{|f(x-t_0) - f(x)|}{1 + g(x)^{t_0}} \Big|\,dx \\
&\le \int_{\mathbb{R}} \Big| \frac{|f(x-t_k) - f(x)|}{1 + g(x)^{t_k}} - \frac{|\phi_n(x-t_k) - \phi_n(x)|}{1 + g(x)^{t_k}} \Big|\,dx \\
&\quad + \int_{\mathbb{R}} \Big| \frac{|\phi_n(x-t_k) - \phi_n(x)|}{1 + g(x)^{t_k}} - \frac{|\phi_n(x-t_0) - \phi_n(x)|}{1 + g(x)^{t_0}} \Big|\,dx \\
&\quad + \int_{\mathbb{R}} \Big| \frac{|\phi_n(x-t_0) - \phi_n(x)|}{1 + g(x)^{t_0}} - \frac{|f(x-t_0) - f(x)|}{1 + g(x)^{t_0}} \Big|\,dx.
\end{aligned}$$

Let $\epsilon > 0$. By choosing n large, the first and last term in the sum on the right-hand side can be made less than ϵ. Hence, for n large,

$$|H(t_k) - H(t_0)| \leq 2\epsilon + \int_{\mathbb{R}} \left| \frac{|\phi_n(x - t_k) - \phi_n(x)|}{1 + g(x)^{t_k}} - \frac{|\phi_n(x - t_0) - \phi_n(x)|}{1 + g(x)^{t_0}} \right| dx.$$

Letting $k \to \infty$ in the inequality above and using what was proved first for $C_0(\mathbb{R})$ functions, we conclude that

$$\lim_{k\to\infty} |H(t_k) - H(t_0)| \leq 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, the limit is zero. Moreover, this holds for any $t_0 \in \mathbb{R}$ so H is continuous.