Homework 4 Solutions

1. Let $\epsilon > 0$ and choose $\delta > 0$ so that if $E \subset (0,1)$ is Lebesgue measurable with $m(E) < \delta$ we have
$$\int_E |f_n(x)|\,dx < \epsilon \text{ for each } n = 1, 2, \ldots.$$
Using that $f \in L^1((0,1))$, we may adjust δ if necessary to ensure that
$$\int_E |f(x)|\,dx < \epsilon$$
also. Since $f_n \to f$ a.e. on $(0,1)$ we may choose a Lebesgue measurable $E \subset (0,1)$ such that $f_n \to f$ uniformly on $E^c \cap (0,1)$ and $m(E) < \delta$. Then
$$\begin{aligned}\int_{(0,1)} |f_n(x) - f(x)|\,dx &= \int_E |f_n(x) - f(x)|\,dx + \int_{E^c\cap(0,1)} |f_n(x) - f(x)|\,dx \\ &\le 2\epsilon + \int_{E^c\cap(0,1)} |f_n(x) - f(x)|\,dx.\end{aligned}$$
Taking the limit on each side of the inequality above shows that
$$\lim_{n\to\infty} \int_{(0,1)} |f_n(x) - f(x)|\,dx \le 2\epsilon.$$
Since $\epsilon > 0$ is arbitrary, the limit on the left-hand side is zero. Hence, $f_n \to f$ in $L^1((0,1))$.

2. We have
$$\begin{aligned}\int_0^\infty\int_0^\infty e^{-xy}|\sin x \sin y|\,dxdy &= \int_0^\infty |\sin y|\left(\sum_{k=0}^\infty \int_{2k\pi}^{(2k+1)\pi} e^{-xy}\sin x\,dx - \sum_{k=0}^\infty \int_{(2k+1)\pi}^{(2k+2)\pi} e^{-xy}\sin x\,dx\right)dy \\ \text{(Integration by parts)} &= \int_0^\infty \frac{|\sin y|}{1+y^2}(e^{-2\pi y} + 2e^{-\pi y} + 1)\sum_{k=0}^\infty e^{-2k\pi y}\,dy \\ &= \int_0^\infty \frac{|\sin y|}{1+y^2}\frac{(1+e^{-\pi y})^2}{1-e^{-2\pi y}}\,dy \\ \text{(Difference of squares)} &= \int_0^\infty \frac{|\sin y|}{1+y^2}\frac{1+e^{-\pi y}}{1-e^{-\pi y}}\,dy.\end{aligned}$$
Now, near zero
$$\frac{|\sin y|}{1+y^2}\frac{1+e^{-\pi y}}{1-e^{-\pi y}} \le |y| \cdot \frac{1+e^{-\pi y}}{1-e^{-\pi y}}.$$
The limit of the function on the right-hand side of the inequality as $y \to 0^+$ is π^{-1}. Hence, there is an $\epsilon > 0$ and a constant $C > 0$ such that
$$\frac{|\sin y|}{1+y^2}\frac{1+e^{-\pi y}}{1-e^{-\pi y}} \le \frac{C}{1+y^2} \text{ on } (0,\epsilon).$$
On the other hand,
$$\frac{|\sin y|}{1+y^2}\frac{1+e^{-\pi y}}{1-e^{-\pi y}} \le \frac{1+e^{-\pi y}}{1-e^{-\pi y}}.$$

Since

$$\lim_{y\to\infty} \frac{1+e^{-\pi y}}{1-e^{-\pi y}} = 1,$$

there is another constant $\tilde{C}$ such that

$$\frac{|\sin y|}{1+y^2}\frac{1+e^{-\pi y}}{1-e^{-\pi y}} \le \frac{\tilde{C}}{1+y^2} \text{ on } (\epsilon,\infty).$$

Taking $C_0 := \max\{C, \tilde{C}\}$ shows

$$\frac{|\sin y|}{1+y^2}\frac{1+e^{-\pi y}}{1-e^{-\pi y}} \le \frac{C_0}{1+y^2} \text{ on } (0,\infty).$$

Since the function on the right-hand side is in $L^1((0,\infty))$, we conclude

$$\int_0^\infty \frac{|\sin y|}{1+y^2}\frac{1+e^{-\pi y}}{1-e^{-\pi y}}\,dy < \infty,$$

as desired.

3. We first prove the result when $f \in C_0(\mathbb{R})$. Fix $x_0 \in \mathbb{R}$ and let $\epsilon > 0$. Choose $\delta > 0$ such that $|x-x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. If $|h| < \delta$, then

$$\Big|\frac{1}{2h}\int_{x_0-h}^{x_0+h} f(x)\,dx - f(x_0)\Big| = \Big|\frac{1}{2h}\int_{x_0-h}^{x_0+h} (f(x)-f(x_0))\,dx\Big| \le \epsilon.$$

It follows that

$$\lim_{h\to 0^+} \frac{1}{2h}\int_{x_0-h}^{x_0+h} f(x)\,dx = f(x_0) \text{ for each } x_0 \in \mathbb{R}$$

whenever $f \in C_0(\mathbb{R})$. Note also that $f \in C_0(\mathbb{R})$ is uniformly continuous[1] as well so the δ above can be chosen to be independent of x_0. It follows that $f_h \to f$ uniformly as $h \to 0^+$. Let $\epsilon > 0$ and choose $|h| < \delta$ so small that $|f_h(x) - f(x)| < \epsilon$ for each $x \in \mathbb{R}$. Choose $-\infty < a < b < \infty$ so that $\operatorname{supp} f \pm h \subset [a,b]$ for $|h| < 1$. Then

$$\int_{\mathbb{R}} \Big|\frac{1}{2h}\Big(\int_{x-h}^{x+h} f(t)\,dt - f(x)\Big)\Big|\,dx \le \int_{a-1}^{b+1} \Big|\frac{1}{2h}\Big(\int_{x-h}^{x+h} f(t)\,dt - f(x)\Big)\Big|\,dx < \epsilon\int_{a-1}^{b+1} dx = \epsilon(b-a+2).$$

Since $\epsilon > 0$ is arbitrary, $f_h \to f$ in $L^1(\mathbb{R})$ as $h \to 0^+$.

Suppose now that $f \in L^1(\mathbb{R})$ and let $\{f_n\}_1^\infty \subset C_0(\mathbb{R})$ be such that $f_n \to f$ in $L^1(\mathbb{R})$. We have

$$\begin{aligned}\int_{\mathbb{R}} \Big|\frac{1}{2h}\Big(\int_{x-h}^{x+h} f(t)\,dt - f(x)\Big)\Big|\,dx &\le \int_{\mathbb{R}} \Big|\frac{1}{2h}\Big(\int_{x-h}^{x+h} f(t) - f_n(t))\,dt\Big)\Big|\,dx + \int_{\mathbb{R}} \Big|\frac{1}{2h}\Big(\int_{x-h}^{x+h} f_n(t)\,dt - f_n(x)\Big)\Big|\,dx\\ &\quad + \int_{\mathbb{R}} |f_n(x) - f(x)|\,dx\\ &= I + II + III.\end{aligned}$$

The integrals II and III above can be small by choosing $|h|$ small and n large, respectively. On the other hand, the L^1 convergence of f_n to f implies that $(f_n)_h(x) \to f_h(x)$ pointwise in x as $n \to \infty$. For I, notice that by a change of variable and the Tonelli Theorem

$$I \le \int_{\mathbb{R}} \frac{1}{2h}\int_{x-h}^{x+h} |f(t) - f_n(t)|\,dtdx = \frac{1}{2h}\int_{-h}^{h}\int_{\mathbb{R}} |f(x+t) - f_n(x+t)|\,dxdt = \int_{\mathbb{R}} |f(x) - f_n(x)|\,dx$$

Thus, I tends to zero as $n \to \infty$. Combining the estimates for I, II, and III shows that $f_h \to f$ in $L^1(\mathbb{R})$ as $h \to 0^+$.

[1]Functions in $C_0(\mathbb{R})$ are always uniformly continuous.

4. The trick is to show that $g(x) := \sum_{n=1}^\infty n \int_n^{n+\frac{1}{n}} |f(x+y)|\,dy$ is locally L^1 since this implies $g(x) < \infty$ a.e. which implies a.e. absolute convergence of the original series. Observe that by Tonelli's theorem applied twice

$$\begin{aligned}\int_a^b \sum_{n=1}^\infty n \int_n^{n+\frac{1}{n}} |f(x+y)\,dydx &= \sum_{n=1}^\infty n \int_n^{n+\frac{1}{n}} \int_a^b |f(x+y)|\,dx\,dy \\ &= \sum_{n=1}^\infty n \int_n^{n+\frac{1}{n}} \int_{a+y}^{b+y} |f(t)|\,dt\,dy \\ &\le \sum_{n=1}^\infty n \int_n^{n+\frac{1}{n}} \int_{a+n}^{b+n+\frac{1}{n}} |f(t)|\,dt\,dy \\ &\le \sum_{n=1}^\infty \int_{a+n}^{b+n+1} |f(t)|\,dt.\end{aligned}$$

Setting $b = a+1$ gives

$$\int_a^b \sum_{n=1}^\infty n \int_n^{n+\frac{1}{n}} |f(x+y)\,dydx \le \sum_{n=1}^\infty \int_{a+n}^{a+n+2} |f(t)|\,dt.$$

Moreover,

$$\sum_{n=1}^\infty \int_{a+n}^{a+n+2} |f(t)|\,dt = \int_{a+1}^{a+2} |f(t)|\,dt + 2\sum_{n=2}^\infty \int_{a+n}^{a+n+1} |f(t)|\,dt = \int_{a+1}^{a+2} |f(t)|\,dt + 2\int_{a+2}^\infty |f(t)|\,dt < \infty$$

since $f \in L^1(\mathbb{R})$. It follows that

$$\int_a^{a+1} g(x)\,dx < \infty \text{ for every } a \in \mathbb{R},$$

so for each a we see that $g(x) < \infty$ a.e. on $(a, a+1)$. Since each $x \in \mathbb{R}$ lies in an interval of the form $(a, a+1)$ for some a, the proof is complete.

5. Suppose $f, g \in L^1_{\text{loc}}(\mathbb{R})$ and $a > 0$. Then

$$\begin{aligned}\int_0^a F(x)g(x)\,dx &= \int_0^a \int_0^x f(t)g(x)\,dtdx \\ &= \int_0^a \int_0^a f(t)g(x)\,dx - \int_0^a \int_x^a f(t)g(x)\,dtdx \\ \text{(Fubini)} \quad &= F(a)G(a) - \int_0^a \int_0^t g(x)f(t)\,dxdt \\ &= F(a)G(a) - \int_0^a G(x)f(x)\,dx.\end{aligned}$$

6. Define

$$\chi_{\{|f(x)|>t\}}(x,t) := \begin{cases} 1, & \text{if } (x,t) \in \{|f(x)| > t\} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\int_0^\infty m(t)\,dt = \int_0^\infty \int_{\mathbb{R}} \chi_{\{|f(x)|>t\}}(x,t)\,dx\,dt.$$

By the Tonelli Theorem, we can switch the order of integration[2]:

$$\int_0^\infty \int_{\mathbb{R}} \chi_{\{|f(x)|>t\}}(x,t)\,dx\,dt = \int_{\mathbb{R}} \int_0^\infty \chi_{\{|f(x)|>t\}}(x,t)\,dtdx = \int_{\mathbb{R}} \int_0^{|f(x)|} dtdx = \int_{\mathbb{R}} |f(x)|\,dx.$$

[2]To be precise, you should justify why we can use Tonelli.

Let $\epsilon > 0$. Then

$$\int_0^\infty m(t)\,dt = \epsilon \int_0^\infty m(\epsilon t)\,dt = \epsilon \sum_{k=0}^\infty \int_k^{k+1} m(\epsilon t)\,dt \ge \epsilon \sum_{k=1}^\infty m(\epsilon k),$$

where we have made the change of variable $t \mapsto \epsilon t$ in the first equality. It remains to show

$$\lim_{\epsilon \to 0^+} \epsilon \sum_{k=1}^\infty m(\epsilon k) = \int_{\mathbb{R}} |f(x)|\,dx.$$

We begin with the case $f \in C_0(\mathbb{R})$. For each k, ϵ set

$$E_{k,\epsilon} := \{x : \epsilon k < |f(x)|\} \text{ and } F_{k,\epsilon} := \{x : k\epsilon < |f(x)| \le (k+1)\epsilon\}.$$

Then

$$\begin{aligned}
\int_{\mathbb{R}} \frac{|f(x)|}{\epsilon}\,dx &= \int_{\{0<|f(x)|\le\epsilon\}} \frac{|f(x)|}{\epsilon}\,dx + \int_{E_{1,\epsilon}} \frac{|f(x)|}{\epsilon}\,dx \\
&= \int_{\{0<|f(x)|\le\epsilon\}} \frac{|f(x)|}{\epsilon}\,dx + \sum_{k=1}^\infty \int_{F_k} \frac{|f(x)|}{\epsilon}\,dx \\
&\le \int_{\{0<|f(x)|\le\epsilon\}} \frac{|f(x)|}{\epsilon}\,dx + \sum_{k=1}^\infty k m(F_{k,\epsilon}) + \sum_{k=1}^\infty m(F_{k,\epsilon}).
\end{aligned}$$

Multiplying by ϵ on each side of the inequality gives

$$\int_{\mathbb{R}} |f(x)|\,dx = \int_{\{0<|f(x)|\le\epsilon\}} |f(x)|\,dx + \epsilon \sum_{k=1}^\infty k m(F_{k,\epsilon}) + \epsilon \sum_{k=1}^\infty m(F_{k,\epsilon}) = I_\epsilon + II_\epsilon + III_\epsilon.$$

We handle each term in the sum individually. Notice that $|f(x)|\chi_{\{0<|f(x)|\le\epsilon\}}(x) \le |f(x)|$ for each x Furthermore, $|f(x)|\chi_{\{0<|f(x)|\le\epsilon\}}(x) \to 0$ as $\epsilon \to 0^+$ so we may apply the dominated convergence theorem to conclude that $I_\epsilon \to 0$ as $\epsilon \to 0^+$. For the second term, we have

$$II_\epsilon = \epsilon \sum_{k=1}^\infty k m(F_{k,\epsilon}) = \epsilon \sum_{k=1}^\infty m(E_{k,\epsilon}) = \epsilon \sum_{k=1}^\infty m(\epsilon k).$$

For the last term, note that the compact support of f shows $m(\{x : |f(x)| > 0\}) \le m(\operatorname{supp} f) < \infty$ and

$$III_\epsilon = \epsilon m(E_{1,\epsilon}) \le \epsilon m(\{x : |f(x)| > 0\}) \to 0 \text{ as } \epsilon \to 0^+.$$

Using the estimates above, we conclude

$$\int_{\mathbb{R}} |f(x)|\,dx \le \lim_{\epsilon \to 0^+} \epsilon \sum_{k=1}^\infty m(\epsilon k) \le \int_{\mathbb{R}} |f(x)|\,dx$$

so the result holds for $f \in C_0(\mathbb{R})$. Suppose now that $f \in L^1(\mathbb{R})$ and $\delta > 0$ is given. Choose $\phi \in C_0(\mathbb{R})$ such that $\|f - \phi\|_{L^1} < \delta$ and set $\tilde{m}(k\epsilon) := \{x : |\phi(x)| > \epsilon k\}$. We may also choose ϕ so that $|\phi(x)| \le |f(x)|$ for each $x \in \mathbb{R}$. In this case, $\tilde{m}(k\epsilon) \le m(k\epsilon)$ for each k, ϵ. We have:

$$\left|\epsilon \sum_{k=1}^\infty m(k\epsilon) - \|f\|_{L^1}\right| \le \left|\epsilon \sum_{k=1}^\infty (m(k\epsilon) - \tilde{m}(k\epsilon))\right| + \left|\epsilon \sum_{k=1}^\infty \tilde{m}(\epsilon k) - \|\phi\|_{L^1}\right| + \|\phi - f\|_{L^1}.$$

The last term is $< \delta$ and choosing $\epsilon > 0$ small ensures the middle term is $< \delta$. For the first term, notice that

$$\left|\epsilon \sum_{k=1}^\infty (m(k\epsilon) - \tilde{m}(k\epsilon))\right| = \epsilon \sum_{k=1}^\infty m(k\epsilon) - \epsilon \sum_{k=1}^\infty \tilde{m}(\epsilon k) \le \int_{\mathbb{R}} |f(x)|\,dx - \epsilon \sum_{k=1}^\infty \tilde{m}(\epsilon k).$$

Hence,

$$\left|\epsilon \sum_{k=1}^{\infty} m(k\epsilon) - \|f\|_{L^1}\right| \leq 2\delta + \int_{\mathbb{R}} |f(x)|\, dx - \epsilon \sum_{k=1}^{\infty} \tilde{m}(\epsilon k).$$

Taking the limit as $\epsilon \to 0^+$ on each side of the inequality and using the reverse triangle inequality then gives

$$\lim_{\epsilon \to 0^+} \left|\epsilon \sum_{k=1}^{\infty} m(k\epsilon) - \|f\|_{L^1}\right| \leq 2\delta + \|f - \phi\|_{L^1} \leq 3\delta.$$

Since $\delta > 0$ is arbitrary, the limit on the left-hand side of the expression above is zero and the proof is complete.

7. We first assume $f \in C_0(\mathbb{R})$. By writing $g = g^+ - g^-$, we may assume $g \geq 0$ also[3] We have

$$\int_{\mathbb{R}} f(x) g(nx)\, dx = \sum_{k \in \mathbb{Z}} \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} f(x) g(nx),$$

where the sum is finite since f has compact support. Since $g \geq 0$, we have

$$\Big(\min_{[\frac{k}{n}T, \frac{k+1}{n}T]} f(x) \Big) g(nx) \leq f(x) g(nx) \leq \Big(\max_{[\frac{k}{n}T, \frac{k+1}{n}T]} f(x) \Big) g(nx) f.$$

Set

$$m := \min_{[\frac{k}{n}T, \frac{k+1}{n}T]} f(x) \text{ and } M := \max_{[\frac{k}{n}T, \frac{k+1}{n}T]} f(x).$$

Integrating the inequalities above, we get

$$m \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} g(nx)\, dx \leq \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} f(x) g(nx)\, dx \leq M \int_{\frac{k}{n}T}^{\frac{k+1}{n}T} g(nx)\, dx.$$

Making the change of variable $y = nx$ and using that g is periodic with period T, we see

$$\int_{\frac{k}{n}T}^{\frac{k+1}{n}T} g(nx)\, dx = \frac{1}{n} \int_{kT}^{(k+1)T} g(y)\, dy = \frac{1}{n} \int_0^T g(y)\, dy.$$

Hence, (assuming $g \neq 0$ since the problem is trivial otherwise)

$$m \leq \frac{\int_{\frac{k}{n}T}^{\frac{k+1}{n}T} f(x) g(nx)\, dx}{\frac{1}{n} \int_0^T g(x)\, dx} \leq M.$$

Using the continuity of f, we can thereby apply the intermediate value theorem to find $x_k \in [\frac{k}{n}T, \frac{(k+1)}{n}T]$ such that

$$f(x_k) = \frac{\int_{\frac{k}{n}T}^{\frac{k+1}{n}T} f(x) g(nx)\, dx}{\frac{1}{n} \int_0^T g(x)\, dx} \text{ for each } k.$$

Then

$$\begin{aligned} \int_{\mathbb{R}} f(x) g(nx)\, dx &= \sum_{k \in \mathbb{Z}} \Big(\frac{1}{n} \int_0^T g(x)\, dx \Big) f(x_k) \\ &= \Big(\sum_{k \in \mathbb{Z}} f(x_k) \frac{T}{n} \Big) \Big(\frac{1}{T} \int_0^T g(x)\, dx \Big). \end{aligned}$$

The first term in the product is just a Riemann sum for f, so letting $n \to \infty$ gives the result when $f \in C_0(\mathbb{R})$. For the general case when $f \in L^1(\mathbb{R})$, just argue by approximation in $C_0(\mathbb{R})$ as usual.[4]

[3]Both $g^{\pm} \in C(\mathbb{R})$ are periodic with period T.

[4]I'll leave this part for the reader because I'm sleep deprived.