Homework 5 Solutions

1. By the Beppo-Levi theorem, we have

$$\begin{aligned}
\int_a^b \sum_{n=1}^{\infty} \sqrt{n}\Big|\int_{\sqrt{n}}^{\sqrt{n}+n^{-1}} f(x+y)\,dy\Big|\,dx &\le \sum_{n=1}^{\infty} \sqrt{n}\int_{\sqrt{n}}^{\sqrt{n}+n^{-1}} \int_a^b |f(x+y)|\,dxdy \\
(t = x+y) \quad &= \sum_{n=1}^{\infty} \sqrt{n}\int_{\sqrt{n}}^{\sqrt{n}+n^{-1}} \int_{a+y}^{b+y} |f(t)|\,dtdy \\
&\le \sum_{n=1}^{\infty} \sqrt{n}\int_{\sqrt{n}}^{\sqrt{n}+n^{-1}} \int_{a+\sqrt{n}}^{b+\sqrt{n}+n^{-1}} |f(t)|\,dtdy \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\int_{a+\sqrt{n}}^{b+\sqrt{n}+n^{-1}} |f(t)|\,dt \\
&\le \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\int_{a+\sqrt{n}}^{b+\sqrt{n}+1} |f(t)|\,dt.
\end{aligned}$$

   Now, run the same argument as in Problem 2 below with $b+1$ replacing b.

2. Let $a, b \in \mathbb{R}$ with $-\infty < a < b < \infty$. By the Beppo-Levi theorem, we have

$$\begin{aligned}
\int_a^b \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}|f(x-\sqrt{n})|\,dx &\le \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\int_a^b |f(x-\sqrt{n})|\,dx \\
(t = x-\sqrt{n}) \quad &\le \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\int_{a-\sqrt{n}}^{b-\sqrt{n}} |f(t)|\,dt \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\int_{\mathbb{R}} |f(t)|\chi_{[a-\sqrt{n},b-\sqrt{n}]}(t)\,dt \\
(\text{Monotone Convergence}) \quad &= \int_{\mathbb{R}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\chi_{[a-\sqrt{n},b-\sqrt{n}]}(t)|f(t)|\,dt.
\end{aligned}$$

   Set $\phi(t) := \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\chi_{[a-\sqrt{n},b-\sqrt{n}]}(t)$. We will show that $\phi \in L^\infty(\mathbb{R})$. Intuitively, the idea is to view each of the functions $\chi_{[a-\sqrt{n},b-\sqrt{n}]}(t)$ as carts moving toward $-\infty$ of length $b-a$ and, for a given t, to estimate the first n and last n[1] for which t is in the cart defined by $\chi_{[a-\sqrt{n},b-\sqrt{n}]}(t)$. We consider several cases:

   (a) $t \le a-1$: In this case, we can be sure that t is in a cart after at least $(a-t)^2-1$ steps and is no longer in a cart after $(b-t)^2+1$ steps. To see this, first suppose t is an integer. Then $a-t$ and $b-t$ are integers and the first step for which t is in a cart will be $(a-t)^2$ and the last will be $(b-t)^2$. By subtracting and adding one, we can handle the case when $a-t$ and $b-t$ are not integers, since each individual step takes us a distance less than one from where we stood at the previous step. Hence, for each t we only need to sum over $n \in \mathbb{N}$ lying between $A(t) := (a-t)^2-1$

[1] I will call these the number of "steps" we have taken with our cart.

and $B(t) := (b-t)^2 + 1$. In other words, in this case we can write

$$\begin{aligned}
\phi(t) &= \sum_{A(t)\le n\le B(t)} \frac{1}{\sqrt{n}} \\
&= \sum_{n=\lfloor A(t)\rfloor}^{\lceil B(t)\rceil} \frac{1}{\sqrt{n}} \\
&\le \sum_{n=\lfloor A(t)\rfloor}^{\lceil B(t)\rceil} \int_n^{n+1} \frac{1}{\sqrt{s}}\, ds \\
&= \int_{\lfloor A(t)\rfloor}^{\lceil B(t)\rceil+1} \frac{1}{\sqrt{s}}\, ds \\
&\le \int_{A(t)}^{B(t)+1} \frac{1}{\sqrt{s}}\, ds \\
&= 2\Big((B(t)+1)^{\frac{1}{2}} - A(t)^{\frac{1}{2}}\Big) \\
&\to 0 \text{ as } t \to -\infty.
\end{aligned}$$

Since $2\Big((B(t)+1)^{\frac{1}{2}} - A(t)^{\frac{1}{2}}\Big)$ is continuous for $t < a-1$, the inequalities above imply $\phi(t)$ is uniformly bounded on $(-\infty, a-1)$.

(b) If $a - 1 \le t \le b$, then a similar argument shows that

$$\phi(t) = \sum_{n=1}^{(b-a+1)^2} \frac{1}{\sqrt{n}} < \infty.$$

(c) If $t > b$, then $\phi(t) = 0$ by definition.

Combining (a),(b), and (c) implies $\phi \in L^\infty(\mathbb{R})$. Thus, combining this with the first string of inequalities yields

$$\int_a^b \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} |f(x-\sqrt{n})|\, dx \le \|\phi\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |f(t)|\, dt < \infty$$

since $f \in L^1(\mathbb{R})$. To conclude, note that a, b are arbitrary and apply the usual argument.

3. For $n, m \in \mathbb{N}$ set

$$E := L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \text{ and } E_{n,m} := \{f \in L^2(\mathbb{R}) : \int_{-m}^{m} |f(x)|\, dx < n\}.$$

Then $E = \cap_{m\in\mathbb{N}} \cup_{n\in\mathbb{N}} E_{n,m}$. We will show that each of the sets $E_{n,m}$ are open in $L^2(\mathbb{R})$ which will show that E is Borel. Let $\epsilon > 0$ and fix $f \in E_{n,m}$. We need to show that for ϵ small the L^2-ball $B_\epsilon(f)$ is contained in $E_{n,m}$. Suppose $g \in B_\epsilon(f)$. Then

$$\begin{aligned}
\int_{-m}^{m} |g(x)|\, dx &= \int_{-m}^{m} |g(x) - f(x)|\, dx + \int_{-m}^{m} |f(x)|\, dx \\
\text{(Cauchy-Schwarz)} \quad &\le (2m)^{\frac{1}{2}} \Big(\int_{-m}^{m} |g(x)-f(x)|^2\, dx\Big)^{\frac{1}{2}} + \int_{-m}^{m} |f(x)|\, dx \\
&\le (2m)^{\frac{1}{2}} \epsilon + \int_{-m}^{m} |f(x)|\, dx.
\end{aligned}$$

Hence, if $\epsilon < (2m)^{-\frac{1}{2}}(n - \int_{-m}^{m} |f(x)|\, dx)$, then $\int_{-m}^{m} |g(x)|\, dx < n$ also. In particular, for this $\epsilon > 0$ the ball $B_\epsilon(f)$ is contained in $E_{n,m}$ implying $E_{n,m}$ is open. This completes the proof.

4. Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ for $1 \le p, q \le \infty$. Define F and G as in the problem statement. We break the proof up into several cases:

(i) Continuity: We first consider when $1 < p < \infty$. Fix $x_0 \in \mathbb{R}$ and let $x_n \to x_0$ be a sequence tending to x_0 as $n \to \infty$. Using the Hölder inequality, we find

$$|f(x_n) - F(x_0)| \le \int_{x_0}^{x_n} |f(t)|\, dt \le \|f\|_{L^p(\mathbb{R})} |x_n - x_0|^{\frac{1}{p}} \to 0 \text{ as } n \to \infty.$$

If $p = \infty$, then

$$|F(x_n) - F(x_0)| \le \|f\|_{L^\infty(\mathbb{R})} |x_n - x_0| \to 0 \text{ as } n \to \infty.$$

When $p = 1$, continuity is immediate by absolute continuity of the integral.

(ii) We now prove that G as defined in the problem statement is in $L^1(\mathbb{R})$ for $a > 2 - \frac{1}{p} - \frac{1}{q}$. Again, we consider several cases:

(a) $p = q = \infty$: In this case, $a > 2$ and

$$|G(x)| \le \|f\|_{L^\infty} \|g\|_{L^\infty} \frac{|x|}{(1+|x|)^a} \le \|f\|_{L^\infty} \|g\|_{L^\infty} \frac{1}{(1+|x|)^{a-1}} \in L^1(\mathbb{R})$$

so $G \in L^1(\mathbb{R})$.

(b) $p = q = 1$: In this case, $a > 1$ and

$$\int_{\mathbb{R}} |G(x)|\, dx \le \|f\|_{L^1} \int_{\mathbb{R}} \frac{|g(x)|}{(1+|x|)^a}\, dx \le \|f\|_{L^1} \|g\|_{L^1}$$

where we have used $(1+|x|)^{-a} < 1$ in the last inequality.

(c) $p = 1$ and $q = \infty$: In this case $a > 1$. We have

$$\int_{\mathbb{R}} |G(x)|\, dx \le \|g\|_{L^\infty} \|f\|_{L^1} \int_{\mathbb{R}} \frac{1}{(1+|x|)^a}\, dx < \infty.$$

(d) $p = \infty$ and $q = 1$: We have

$$\int_{\mathbb{R}} |G(x)|\, dx \le \|f\|_{L^\infty} \int_{\mathbb{R}} \frac{|x|}{(1+|x|)^a} |g(x)|\, dx \le \|f\|_{L^\infty} \|g\|_{L^1}.$$

(e) $p \in (1, \infty)$ and $q = \infty$: By the Hölder inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}} |G(x)|\, dx &\le \|g\|_{L^\infty} \int_{\mathbb{R}} (1+|x|)^{-a} \int_0^x |f(t)|\, dt dx \\
&\le \|g\|_{L^\infty} \|f\|_{L^p} \int_{\mathbb{R}} \frac{|x|^{1-\frac{1}{p}}}{(1+|x|)^a}\, dx \\
&\le \|g\|_{L^\infty} \|f\|_{L^p} \int_{\mathbb{R}} \frac{1}{(1+|x|)^{a-1+\frac{1}{p}}}\, dx < \infty
\end{aligned}$$

since $a - 1 + \frac{1}{p} > 1$.

(f) $p = \infty$ and $q \in (1, \infty)$: By the Hölder inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}} |G(x)|\, dx &\le \|f\|_{L^\infty} \int_{\mathbb{R}} \frac{|x|}{(1+|x|)^a} |g(x)|\, dx \\
&\le \|f\|_{L^\infty} \int_{\mathbb{R}} \frac{|g(x)|}{(1+|x|)^{a-1}}\, dx \\
&\le \|f\|_{L^\infty} \|g\|_{L^q} \Big(\int_{\mathbb{R}} \frac{1}{(1+|x|)^{\frac{q}{q-1}(a-1)}}\, dx \Big)^{1-\frac{1}{q}} < \infty
\end{aligned}$$

since $a - 1 > 1 - \frac{1}{q} = \frac{q}{q-1}$.

(g) $1 < p, q < \infty$: In this case, we simply apply the Hölder inequality twice. We have

$$\begin{aligned}\int_{\mathbb{R}} |G(x)|\,dx &\leq \|f\|_{L^p} \int_{\mathbb{R}} \frac{|x|^{1-\frac{1}{p}}}{(1+|x|)^a} |g(x)|\,dx \\ &\leq \|f\|_{L^p} \int_{\mathbb{R}} \frac{1}{(1+|x|)^{a-1+\frac{1}{p}}} |g(x)|\,dx \\ &\leq \|f\|_{L^p} \|g\|_{L^q} \Big(\int_{\mathbb{R}} \frac{1}{(1+|x|)^{\frac{q}{q-1}(a-1+\frac{1}{p})}}\,dx \Big)^{1-\frac{1}{q}} < \infty\end{aligned}$$

since $a - 1 + \frac{1}{p} > 1 - \frac{1}{q}$.

Combining all cases completes the proof.

5. Applying the change of variable $t = x^2$, we find

$$\begin{aligned}\int_0^\infty |f(x^2)x^{-a}\sin x|\,dx &= \frac{1}{2}\int_0^\infty |f(t)t^{-\frac{a+1}{2}} \sin\Big(t^{\frac{1}{2}}\Big)|\,dt \\ &= \frac{1}{2}\int_0^1 |f(t)t^{-\frac{a+1}{2}} \sin\Big(t^{\frac{1}{2}}\Big)|\,dt + \frac{1}{2}\int_1^\infty |f(t)t^{-\frac{a+1}{2}} \sin\Big(t^{\frac{1}{2}}\Big)|\,dt \\ &:= I_1 + I_2.\end{aligned}$$

We estimate each of the integrals separately starting with I_1. Using that $|\sin\Big(t^{\frac{1}{2}}\Big)| \leq t^{\frac{1}{2}}$ we find

$$\begin{aligned}I_1 &\leq \frac{1}{2}\int_0^1 |f(t)|t^{-\frac{a}{2}}\,dt \\ \text{(Hölder)} \quad &\leq \frac{1}{2}\Big(\int_0^1 |f(t)|^3\,dt\Big)^{\frac{1}{3}} \Big(\int_0^1 t^{-\frac{3a}{4}}\,dt\Big)^{\frac{2}{3}} \\ &\leq \frac{1}{2}\Big(\int_0^\infty |f(t)|^3\,dt\Big)^{\frac{1}{3}} \Big(\int_0^1 t^{-\frac{3a}{4}}\,dt\Big)^{\frac{2}{3}}\end{aligned}$$

Since $a < \frac{4}{3}$, we have $\frac{3a}{4} < 1$ so the second integral in the product is finite. Hence, $I_1 < \infty$. A similar computation works for I_2:

$$\begin{aligned}I_2 &= \frac{1}{2}\int_1^\infty |f(t)t^{-\frac{a+1}{2}} \sin\Big(t^{\frac{1}{2}}\Big)|\,dt \\ &\leq \frac{1}{2}\int_1^\infty |f(t)|t^{-\frac{a+1}{2}}\,dt \\ \text{(Hölder)} \quad &\leq \frac{1}{2}\Big(\int_1^\infty |f(t)|^3\,dt\Big)^{\frac{1}{3}} \Big(\int_1^\infty t^{-\frac{3(a+1)}{4}}\,dt\Big)^{\frac{2}{3}} \\ &\leq \frac{1}{2}\Big(\int_0^\infty |f(t)|^3\,dt\Big)^{\frac{1}{3}} \Big(\int_1^\infty t^{-\frac{3(a+1)}{4}}\,dt\Big)^{\frac{2}{3}}.\end{aligned}$$

Since $\frac{3(a+1)}{4} > 1$, the second integral in the product is finite so $I_2 < \infty$ also. This completes the proof.

6. Write

$$\int_{\mathbb{R}} |f(y)|\,dy = \int_{|y|\leq 1} |f(y)|\,dy + \int_{|y|>1} |f(y)|\,dy =: I_1 + I_2.$$

We estimate each of the integrals separately. The easiest one is I_1. If $p = 1$, then I_1 is finite immediately. If $p > 1$, then applying Hölder's inequality with $g = \chi_{[-1,1]}$ shows that $I_1 < \infty$. Hence, we focus on I_2.

Notice that

$$|y - x| \leq \frac{|x|}{2} \Leftrightarrow x - \frac{|x|}{2} \leq y \leq \frac{|x|}{2} + x.$$

For each x such that $|x| \geq 1$, define the interval J_x by

$$J_x := \left\{ y : x - \frac{|x|}{2} \leq y \leq x + \frac{|x|}{2} \right\}$$

and note that $|J_x| = |x|$ for each x. Applying the Hölder inequality and using the hypothesis, we find

$$\begin{aligned} \int_{J_x} |f(y)|\, dy &\leq |x|^{\frac{1}{q}} \left(\int_{J_x} |f(y)|^p\, dx \right)^{\frac{1}{p}} \\ &\leq |x|^{\frac{1}{q}} |x|^{-\frac{a}{p}} \\ &= |x|^{\alpha}, \end{aligned}$$

where $\alpha = \frac{a+1}{p} - 1 > 0$ since $a+1 > p$ and $\frac{1}{p} + \frac{1}{q} = 1$. To conclude, observe that since $(1, \infty) \subset \cup_{j=0}^{\infty} J_{2^j}$ we have

$$\begin{aligned} \int_1^{\infty} |f(y)|\, dy &\leq \sum_0^{\infty} \int_{J_{2^j}} |f(y)|\, dy \\ &\leq \sum_{j=0}^{\infty} \left(\frac{1}{2^{\alpha}} \right)^j < \infty. \end{aligned}$$

An identical argument can be used to estimate $\int_{-\infty}^{1} |f(y)|\, dy$, and the case when $p = 1$ is identical except that we do not apply the Hölder inequality in the first estimate. It follows that $I_2 < \infty$ so $f \in L^1(\mathbb{R})$.