Homework 2 (Due Friday, January 19, 2024, 11:59 pm)

Problem 1. Let $f \in L^1(\mathbb{R})$. Show that

$$\int_{\mathbb{R}} e^{ix\xi} f(x)dx \to 0 \quad \text{as} \quad |\xi| \to \infty.$$

Here $\xi \in \mathbb{R}$.

Problem 2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be non-negative integrable functions with $\|f_n\|_{L^1} = 1$. Suppose that $f_n \to f$ almost everywhere with $\|f\|_{L^1} = 1$. Show that

$$\int_E f_n(x)dx \to \int_E f(x)dx$$

uniformly in the choice of a measurable set $E \subset \mathbb{R}$.

Problem 3. Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{|h|\to\infty} \int_{\mathbb{R}} |f(x+h) - f(x)|dx$$

exists and compute it.

Problem 4. Show that the dominated convergence theorem follows from Egoroff's theorem in the case of the Lebesgue measure on $\mathbb{R}$.

Problem 5. Let $f \in L^1(\mathbb{R})$, and let E_j be a sequence of Lebesgue measurable sets such that $\mu(E_j) \to 0$ as $j \to \infty$. Show that

$$\int_{E_j} f(x)dx \to 0.$$

Problem 6. Let $f \in L^1(\mathbb{R})$, and let $a_1, \ldots, a_k \in \mathbb{R}$ and $b_1, \ldots, b_k \in \mathbb{R} \setminus \{0\}$. Assume that the numbers $\frac{a_j}{b_j}$ are all distinct. Determine

$$\lim_{t\to\infty} \int \left| \sum_{j=1}^{k} f(b_j x + t a_j) \right| dx.$$

Problem 7. Let (f_n) be a sequence of measurable functions on $\mathbb{R}$ such that $|f_n(x)| \le 1$ for all x and n and assume that

$$f_n \to f \quad \text{a.e.,} \quad \text{as} \quad n \to \infty.$$

Show that $g * f_n \to g * f$ uniformly on each compact set, if $g \in L^1(\mathbb{R})$. Here $*$ is the convolution, i.e.

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Hint: Use Egoroff's theorem.