Homework 3 (Due Friday, January 26, 2024, 11:59 pm)

Problem 1. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f \in L^1(X, d\mu)$. Show that if

$$\int_E f d\mu = 0,$$

for any $E \in \mathcal{A}$, then $f = 0$ a.e.

Problem 2. Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{1}{2T} \left| \int_{-T}^{T} f(s+t)dt \right| ds = \left| \int_{-\infty}^{\infty} f(t)dt \right|.$$

Hint: Assume first that $f \in C_0(\mathbb{R})$.

Problem 3. Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that the sequence (f_n) is Cauchy in measure if for every $\varepsilon > 0$,

$$\mu(\{x : |f_n(x) - f_m(x)| \geq \varepsilon\}) \to 0 \quad \text{as} \quad n, m \to \infty.$$

Assume that (f_n) is Cauchy in measure. Show that there is a measurable function f such that $f_n \to f$ in measure, and there is a subsequence $\{f_{n_j}\}$ that convergence to f a.e.

Problem 4. Let $(X, \mathcal{A}, \mu)$ be a measure space and assume that μ is σ-finite. Then let $f_n \in L^1(X, \mu)$ and let f be measurable. Assume that $f_n \to f$ in measure and $|f_n| \leq g \in L^1(X, \mu)$. Show that $f_n \to f$ in $L^1(X, \mu)$.

Problem 5. Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose that $\mu(X) < \infty$. Let

$$S = \{(\text{equivalence class}) \text{ of measurable complex functions on } X\}.$$

(Here, as usual, two measurable complex functions are equivalent if they agree a.e.) For $f \in S$, define

$$\rho(f) = \int_X \frac{|f|}{1+|f|} d\mu.$$

Show that

$$d(f, g) = \rho(f - g)$$

is a metric on S, and that $f_n \to f$ in this metric if and only if $f_n \to f$ in measure.

Problem 6. Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)dx \right| = \int_{-\infty}^{\infty} |f(x)|dx.$$

Problem 7. Let $f \in L^1(\mathbb{R})$ and g be measurable on $\mathbb{R}$ and $0 < g(x) < \infty$ for all $x \in \mathbb{R}$. Let

$$h(t,x) = \frac{|f(x-t) - f(x)|}{1 + g(x)^t}.$$

(1) Show that the function $x \mapsto h(t,x)$ is in $L^1(\mathbb{R})$ for every $t \in \mathbb{R}$.

(2) Show that the function

$$H(t) = \int_{\mathbb{R}} h(t,x)dx$$

is continuous on $\mathbb{R}$.