## Homework 4 (Due Saturday, February 3, 2024, 11:59 pm)

**Problem 1.** Let $f_n \in L^1((0,1))$ be such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E |f_n(x)|dx < \varepsilon,$$

whenever $m(E) < \delta$ and $n = 1, 2, \ldots$. Assume that $f_n \to f$ almost everywhere, where $f \in L^1((0,1))$. Show that

$$\|f_n - f\|_{L^1((0,1))} \to 0 \quad \text{as} \quad n \to \infty.$$

**Problem 2.** Set

$$f(x,y) = e^{-xy} \sin x \sin y, \quad (x,y) \in (0,\infty) \times (0,\infty) := \mathbb{R}_+ \times \mathbb{R}_+.$$

Show that $f \in L^1(\mathbb{R}_+ \times \mathbb{R}_+)$.

**Problem 3.** Let $f \in L^1(\mathbb{R})$ and set

$$f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)dt, \quad h > 0.$$

Show that $f_h \in L^1(\mathbb{R})$ and $f_h \to f$ in $L^1(\mathbb{R})$ as $h \to 0$.

**Problem 4.** Let $f \in L^1(\mathbb{R})$. Show that the series

$$\sum_{n=1}^{\infty} n \int_n^{n+1/n} f(x+y)dy$$

converges absolutely for a.a. $x \in \mathbb{R}$.

**Problem 5.** Let $f, g \in L^1_{\text{loc}}(\mathbb{R})$ and set

$$F(x) = \int_0^x f(t)dt, \quad G(x) = \int_0^x g(t)dt.$$

Show that, if $a > 0$,

$$\int_0^a F(x)g(x)dx + \int_0^a f(x)G(x)dx = F(a)G(a).$$

**Problem 6.** Let $f \in L^1(\mathbb{R})$ and let

$$m(t) = \mu(\{x : |f(x)| > t\}).$$

Show that

$$\int_{\mathbb{R}} |f(x)|dx = \int_0^\infty m(t)dt \geq \sum_{k=1}^{\infty} \varepsilon m(\varepsilon k), \quad \varepsilon > 0.$$

Show also that

$$\sum_{k=1}^{\infty} \varepsilon m(\varepsilon k) \to \int_{\mathbb{R}} |f(x)| dx$$

as $\varepsilon \to 0^+$.

**Problem 7.** Let $f \in L^1(\mathbb{R})$ and let $g \in C(\mathbb{R})$ be continuous periodic with period $T$. Show that

$$\int_{\mathbb{R}} f(x) g(nx) dx \to \left( \int_{\mathbb{R}} f(x) dx \right) \left( \frac{1}{T} \int_0^T g(y) dy \right),$$

as $n \to \infty$.

Hint: Assume first that $f \in C_0(\mathbb{R})$.