Homework 5 (Due Saturday, February 17, 2024, 11:59 pm)

Problem 1. Let $f \in L^1(\mathbb{R})$. Show that the series

$$\sum_{n=1}^{\infty} \sqrt{n} \int_{\sqrt{n}}^{\sqrt{n}+n^{-1}} f(x+y)dy$$

converges absolutely for a.a. $x \in \mathbb{R}$.

Problem 2. Prove that if $f \in L^1(\mathbb{R})$ then the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} f(x - \sqrt{n})$$

is absolutely convergent for almost all x.

Problem 3. Show that the set

$$\left\{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |f(x)| dx < \infty \right\} \in \mathcal{B}_{L^2(\mathbb{R})},$$

i.e. it is a Borel subset of $L^2(\mathbb{R})$.

Problem 4. Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ with $1 \leq p, q \leq \infty$. Show that the function $F(x) = \int_0^x f(y)dy$ is continuous and the function

$$G(x) = (|x|+1)^{-a} F(x) g(x)$$

is in $L^1(\mathbb{R})$ if $a > 2 - \frac{1}{p} - \frac{1}{q}$.

Problem 5. Let $f \in L^3((0,\infty))$ and let $a \in \mathbb{R}$ be such that $\frac{1}{3} < a < \frac{4}{3}$. Show that the function

$$x \mapsto f(x^2) x^{-a} \sin x$$

is in $L^1((0,\infty))$.

Problem 6. Suppose that $f \in L^p_{loc}(\mathbb{R})$ for some p, $1 \leq p < \infty$, and that for some constant $a > p - 1$,

$$\int_{2|y-x| \leq |x|} |f(y)|^p dy \leq |x|^{-a}, \quad \text{for} \quad |x| \geq 1.$$

Prove that $f \in L^1(\mathbb{R})$.

We say that $f \in L^p_{loc}(\mathbb{R})$, $1 \leq p < \infty$, if

$$\int_a^b |f(x)|^p dx < \infty,$$

for any $-\infty < a < b < \infty$.