HOMEWORK 6 (DUE SATURDAY, FEBRUARY 24, 2024, 11:59 PM)

Problem 1. Show that for $1 \le p \neq q \le \infty$, $L^p(\mathbb{R}^d)$ is not contained in $L^q(\mathbb{R}^d)$.

Problem 2. Show that if $f \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$, $1 \le p < q \le \infty$, then $f \in L^r(\mathbb{R})$ for $r \in [p, q]$, and show that

$$\|f\|_{L^r(\mathbb{R})} \le \|f\|_{L^p(\mathbb{R})}^{\lambda} \|f\|_{L^q(\mathbb{R})}^{1-\lambda}, \quad \frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q}, \quad \lambda \in [0, 1].$$

Problem 3. Let $f_n \in L^p(\mathbb{R}^d)$, $1 \le p < \infty$, and assume that $f_n(x) \to f(x)$ for almost all x and that $\|f_n\|_{L^p} \to \|f\|_{L^p} < \infty$, $n \to \infty$. Show that

$$\|f - f_n\|_{L^p} \to 0, \quad n \to \infty.$$

Problem 4. Let $f_j \in L^{p_j}(\mathbb{R}^d)$, $j = 1, \dots, n$, where $p_j \in [1, \infty]$ and

$$\sum_{j=1}^{n} \frac{1}{p_j} = \frac{1}{q}$$

with $q \in [1, \infty]$. Show that $f_1 \dots f_n \in L^q(\mathbb{R}^d)$ and

$$\|f_1 \dots f_n\|_{L^q} \le \Pi_{j=1}^{n} \|f_j\|_{L^{p_j}}.$$

Problem 5. Let $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Compute the limits

$$\lim_{h \to 0} \int |f(x+h) - f(x)|^p dx$$

and

$$\lim_{h \to \infty} \int |f(x+h) - f(x)|^p dx.$$

Problem 6. Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, $1 \le p, q \le \infty$ with

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Show that

$$x \mapsto \int f(x-y) g(y) dy$$

is a continuous function.

Problem 7. Let $f \in L^2_{loc}(\mathbb{R})$ and $g \in L^3_{loc}(\mathbb{R})$. Let a, b be real numbers such that

$$3a + 2b + 1 < 0.$$

Assume that for all real $r \geq 1$, we have

$$\int_{r \leq |x| \leq 2r} |f(x)|^2 dx \leq r^a \quad \text{and} \quad \int_{r \leq |x| \leq 2r} |g(x)|^3 dx \leq r^b.$$

Show that $fg \in L^1(\mathbb{R})$.