Homework 6 Solutions

Remark: Some of the proofs are sketched, but you should be able to recover all the details with just a little bit more work.

1. We first consider when $p \neq \infty$ and $q = \infty$. Take

$$f(x) := \begin{cases} \frac{1}{|x|^{\frac{1}{2p}+\frac{d-1}{p}}} \text{ for } x \in B_1(0) \setminus \{0\} \\ 0 \text{ otherwise.} \end{cases}$$

Using polar coordinates,[1] we find

$$\int_{\mathbb{R}^d} |f(x)|^p \, dx = C(n) \int_0^1 r^{-\frac{1}{2}-d+1} r^{d-1} \, dr = C(d) \int_0^1 r^{-\frac{1}{2}} \, dr < \infty$$

so $f \in L^1(\mathbb{R}^d)$. Since $f \notin L^\infty(\mathbb{R}^d)$, this concludes this case. On the other hand, constant functions are L^∞ but not L^1 on $\mathbb{R}^d$.

Suppose now that $1 \leq p < q < \infty$ and choose $a > 1$ such that $ap < q$. Then arguing as above we find

$$g(x) := |x|^{-\frac{1}{ap}-\frac{d-1}{p}} \cdot \chi_{(0,1]}(x) \in L^p(\mathbb{R}^d)$$

since $|g(x)|^p = |x|^{-\frac{1}{a}-d+1} \chi_{B_1(0)\setminus\{0\}}(x)$ and $a^{-1} < 1$. However, $|g(x)|^q = |x|^{-\frac{q}{ap}-\frac{q(d-1)}{p}} \chi_{B_1(0)\setminus\{0\}}(x)$ so $g \notin L^q(\mathbb{R}^d)$ since $\frac{q}{ap} > 1$ and $\frac{q(d-1)}{p} - (d-1) > 0$. An analogous argument shows

$$h(x) = |x|^{-\frac{a}{q}-\frac{d-1}{q}} \cdot \chi_{\mathbb{R}^d \setminus B_1(0)}(x) \in L^q(\mathbb{R}^d)$$

but $h \notin L^p(\mathbb{R}^d)$.

2. First, note that

$$\frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q} \Leftrightarrow 1 = \frac{\lambda r}{p} + \frac{(1-\lambda) r}{q}.$$

Suppose first that $\lambda \in (0,1)$. If $q = \infty$, then the expression above gives $p = \lambda r$. Then

$$\Big(\int_{\mathbb{R}} |f(x)|^r \, dx \Big)^{\frac{1}{r}} = \Big(\int_{\mathbb{R}} |f(x)|^{\lambda r} |f(x)|^{(1-\lambda) r} \, dx \Big)^{\frac{1}{r}} \leq \|f\|_{L^\infty}^{(1-\lambda)} \Big(\int_{\mathbb{R}} |f(x)|^{\lambda r} \, dx \Big)^{\frac{\lambda}{\lambda r}}.$$

Suppose now that $q < \infty$. By Hölder's inequality

$$\begin{aligned} \Big(\int_{\mathbb{R}} |f(x)|^r \, dx \Big)^{\frac{1}{r}} &= \Big(\int_{\mathbb{R}} |f(x)|^{\lambda r} |f(x)|^{(1-\lambda) r} \, dx \Big)^{\frac{1}{r}} \\ &\leq \Big(\int_{\mathbb{R}} |f(x)|^p \, dx \Big)^{\frac{\lambda r}{p}} \Big(\int_{\mathbb{R}} |f(x)|^q \, dx \Big)^{\frac{(1-\lambda) r}{q}}. \end{aligned}$$

Taking the r-th root gives the result. The desired result is immediate if either $\lambda = 1$ or $\lambda = 0$, since $p = r$ and $q = r$ in the second case.

[1] If you are uncomfortable with working in polar coordinates in high dimensions, take a look at Section 2.7 in Folland.

3. We will show that $\|f - f_n\|_{L^p}^p \to 0$ as $n \to \infty$. Note that

$$|f - f_n|^p \leq \big(2\max(|f|, |f_n|)\big)^p \leq 2^p(|f|^p + |f_n|^p).$$

Then $|f - f_n|^p \to 0$ pointwise a.e. with

$$|f - f_n|^p \leq 2^p g_n \text{ where } g_n := 2^p(|f|^p + |f_n|^p).$$

Since $g_n \to 2\|f\|_{L^p}^p$ in L^1 by assumption, we may apply the generalized dominated convergence theorem to conclude $\|f - f_n\|_{L^p}^p \to 0$.

4. We first assume $p_j \in (1, \infty)$ for each $j = 1, \dots, n$. We have:

$$\sum_{j=1}^n \frac{1}{p_j} = \frac{1}{q} \Leftrightarrow \sum_{j=1}^n \frac{q}{p_j} = 1.$$

We proceed by induction. When $n = 2$, by Hölder's inequality

$$\int_{\mathbb{R}^d} |f_1(x)f_2(x)|^q \, dx \leq \Big(\int_{\mathbb{R}^d} |f_1(x)|^{p_1} \, dx\Big)^{\frac{q}{p_1}} \Big(\int_{\mathbb{R}^d} |f_2(x)|^{p_2} \, dx\Big)^{\frac{q}{p_2}}.$$

Taking the q-th root on each side of the inequality above gives

$$\|f_1 f_2\|_{L^q(\mathbb{R}^d)} \leq \|f_1\|_{L^{p_1}(\mathbb{R}^d)} \|f_2\|_{L^{p_2}(\mathbb{R}^d)}.$$

Suppose now that the desired conclusion holds for some $n \in \mathbb{N}$. By Hölder's inequality

$$\int_{\mathbb{R}^d} |f_1(x) \cdots f_n(x) f_{n+1}(x)|^q \, dx \leq \Big(\int_{\mathbb{R}^d} |f_1(x) \cdots f_n(x)|^{r'} \, dx\Big)^{\frac{q}{r'}} \Big(\int_{\mathbb{R}^d} |f_{n+1}(x)|^{p_{n+1}} \, dx\Big)^{\frac{q}{p_{n+1}}},$$

where

$$\frac{1}{r'} = \sum_{k=2}^n \frac{1}{p_k}.$$

Then

$$\frac{q}{p_{n+1}} + \frac{q}{r'} = 1$$

so we may apply the $n = 2$ case to the first term in the product on the right-hand side to conclude

$$\Big(\int_{\mathbb{R}^d} |f_1(x) \cdots f_n(x)|^{r'} \, dx\Big)^{\frac{q}{r'}} \Big(\int_{\mathbb{R}^d} |f_{n+1}(x)|^{p_{n+1}} \, dx\Big)^{\frac{q}{p_{n+1}}} \leq \Pi_{k=1}^{n+1} \|f_k\|_{L^{p_k}(\mathbb{R}^d)}.$$

This completes the proof when $p_j \in (1, \infty)$. If $p_j = 1$ for any j, then we must have $p_k = \infty$ for each $k \neq j$ and $q = 1$. Then

$$\int_{\mathbb{R}^d} |f_1(x) \cdots f_n(x)| \, dx \leq \Pi_{k \neq j} \|f_k\|_{L^\infty(\mathbb{R}^d)} \|f_j\|_{L^1(\mathbb{R}^d)}.$$

Suppose now that $p_k = \infty$ for $k = j+1, \dots, n$. Then

$$\int_{\mathbb{R}^d} |f_1(x) \cdots f_n(x)|^q \, dx \leq \Pi_{k=j+1}^n \|f_k\|_{L^\infty(\mathbb{R}^d)}^q \int_{\mathbb{R}^d} |f_1(x) \cdots f_j(x)|^q \, dx.$$

Since

$$\frac{1}{q} = \sum_{k=1}^j \frac{1}{p_k},$$

we can apply the first case to the second term in the product above to conclude the proof.

5. First, we show
$$\lim_{h\to 0}\int_{\mathbb{R}^n} |f(x+h)-f(x)|^p\,dx = 0.$$
Suppose that $f \in C_0(\mathbb{R}^n)$. Then
$$\lim_{h\to 0} |f(x+h)-f(x)|^p = 0.$$
Let K be the support of f. Then there is a $\delta_0 > 0$ and a compact set $\tilde{K}$ such that for all $|h| < \delta$ the support of $f(x+h)$ is contained in $\tilde{K}$. Since $f \in C_0(\mathbb{R}^n)$, we know $\|f\|_\infty < \infty$. Hence,
$$|f(x+h)-f(x)|^p \le 2^p\|f\|_\infty^p \chi_{\tilde{K}}(x) \text{ for each } x \in \mathbb{R}^n.$$
Since the right-hand side of the inequality is in L^1, we may apply the dominated convergence theorem to conclude
$$\lim_{h\to 0}\int_{\mathbb{R}^n} |f(x+h)-f(x)|^p\,dx = 0 \text{ when } f \in C_0(\mathbb{R}^n).$$
Suppose now that $f \in L^p(\mathbb{R}^n)$ and choose $\phi \in C_0(\mathbb{R}^n)$ with $\|f-\phi\|_{L^p}^p < 4^{-p}\epsilon$. Then
$$\begin{aligned}\int_{\mathbb{R}^n} |f(x+h)-f(x)|^p\,dx \le& 4^p\int_{\mathbb{R}^n} |f(x+h)-\phi(x+h)|^p\,dx + 4^p\int_{\mathbb{R}^n} |\phi(x+h)-\phi(x)|^p\,dx \\ &+ 4^p\int_{\mathbb{R}^n} |\phi(x)-f(x)|^p\,dx \\ &\le 2\epsilon + 4^p\int_{\mathbb{R}^n} |\phi(x+h)-\phi(x)|^p\,dx.\end{aligned}$$
Taking the limit as $h \to 0$ and using the first part shows
$$\lim_{h\to 0}\int_{\mathbb{R}^n} |f(x+h)-f(x)|^p\,dx \le 2\epsilon.$$
Since $\epsilon > 0$, the claim is proved.

We now show that
$$\lim_{|h|\to\infty}\int_{\mathbb{R}^n} |f(x+h)-f(x)|^p\,dx = 2\int_{\mathbb{R}^n} |f(x)|^p\,dx.$$
We first suppose $f = \chi_{B_r(x_0)}$ for some $x_0 \in \mathbb{R}^n$ and some $r > 0$. Then there is an $\alpha > 0$ such that $|h| > \alpha$ implies $B_r(x_0-h) \cap B_r(x_0) = \emptyset$. Then for all such h we have:
$$\begin{aligned}\int_{\mathbb{R}^n} |f(x+h)-f(x)|^p\,dx &= \int_{\mathbb{R}^n} |\chi_{B_r(x_0-h)}(x) - \chi_{B_r(x_0)}(x)|^p\,dx \\ &= \int_{\mathbb{R}^n} |\chi_{B_r(x_0-h)}(x)|^p\,dx + \int_{\mathbb{R}^n} |\chi_{B_r(x_0)}(x)|^p\,dx \\ &= 2m(B_r(x_0)) \\ &= 2\int_{\mathbb{R}^n} |\chi_{B_r(x_0)}(x)|^p\,dx \\ &= 2\int_{\mathbb{R}^n} |f(x)|^p\,dx.\end{aligned}$$
This proves the claim for characteristic functions of open balls. By linearity, we can extend the result to simple functions of the form
$$f(x) = \sum_{i=1}^k \chi_{B_{r_i}(x_i)}(x).$$
Arguing as in the first part, we can then extend the result to arbitrary $f \in L^p(\mathbb{R}^n)$ by approximating with simple functions of the above form.

6. Set

$$h(x) := \int_{\mathbb{R}^d} f(x-y)g(y)\,dy$$

and fix $x_0 \in \mathbb{R}^d$ and suppose $x_n \to x_0$. Then

$$\begin{aligned}
|h(x_n) - h(x_0)| &= \Big| \int_{\mathbb{R}^d} (f(x_n - y) - f(x_0 - y))g(y)\,dy \Big| \\
&\le \int_{\mathbb{R}^d} |f(x_n - y) - f(x_0 - y)||g(y)|\,dy \\
&\le \Big(\int_{\mathbb{R}^d} |f(x_n - y) - f(x_0 - y)|^p\,dy \Big)^{\frac{1}{p}} \Big(\int_{\mathbb{R}^d} |g(y)|^q\,dy \Big)^{\frac{1}{q}}.
\end{aligned}$$

The second term in the product is finite by assumption and by Problem 5 the first term in the product tends to zero as $n \to \infty$. Hence,

$$|h(x_n) - h(x_0)| \to 0 \text{ as } n \to \infty.$$

Since x_0 is arbitrary and $\{x_n\}$ is an arbitrary sequence converging to x_0, h is a continuous function.

7. By symmetry of the argument, it suffices to show $fg \in L^1((0,\infty))$. Since $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$, we may apply Problem 4 to obtain

$$\int_{[0,1]} |f(x)||g(x)|\,dx \le \Big(\int_{[0,1]} |f(x)|^2\,dx \Big)^{\frac{1}{2}} \Big(\int_{[0,1]} |g(x)|^3\,dx \Big)^{\frac{1}{3}} \Big(\int_{[0,1]} dx \Big)^{\frac{1}{6}} < \infty. \tag{1}$$

Using (1), we find that for $r \ge 1$

$$\begin{aligned}
\int_{[r,2r]} |f(x)||g(x)|\,dx &\le \Big(\int_{[r,2r]} |f(x)|^2\,dx \Big)^{\frac{1}{2}} \Big(\int_{[r,2r]} |g(x)|^3\,dx \Big)^{\frac{1}{3}} \Big(\int_{[r,2r]} dx \Big)^{\frac{1}{6}} \\
&\le r^{\frac{a}{2}} \cdot r^{\frac{b}{3}} \cdot r^{\frac{1}{6}} \\
&= r^{\frac{3a+2b+1}{6}}.
\end{aligned}$$

Next, notice that

$$\begin{aligned}
\int_0^\infty |f(x)g(x)|\,dx &= \int_0^1 |f(x)g(x)|\,dx + \int_1^2 |f(x)g(x)|\,dx + \int_2^4 |f(x)g(x)|\,dx + \cdots \\
&= \int_0^1 |f(x)g(x)|\,dx + \sum_{n=0}^\infty \int_{2^n}^{2^{n+1}} |f(x)g(x)|\,dx.
\end{aligned}$$

Combining the two previous estimates gives:

$$\begin{aligned}
\int_0^\infty |f(x)g(x)|\,dx &\le \int_0^1 |f(x)g(x)|\,dx + \sum_{n=0}^\infty (2^n)^{\frac{3a+2b+1}{6}} \\
&= \int_0^1 |f(x)g(x)|\,dx + \sum_{n=0}^\infty (2^{\frac{3a+2b+1}{6}})^n,
\end{aligned} \tag{2}$$

where the second term in the sum on the right-hand side is finite since it is a convergent geometric sum. Also, the generalized Hölder inequality gives

$$\int_0^1 |f(x)g(x)|\,dx \le \Big(\int_0^1 |f(x)|^2\,dx \Big)^{\frac{1}{2}} \Big(\int_0^1 |g(x)|^3\,dx \Big)^{\frac{1}{3}} < \infty. \tag{3}$$

Combining (1), (2), and (3) completes the proof.