Homework 7 (Due Saturday, March 2, 2024, 11:59 pm)

Problem 1. Let f be a locally integrable function in $\mathbb{R}$ such that for some constant A one has

$$\int_I f(x)dx = A$$

for all intervals I of length 1. Prove that $f(x+1) = f(x)$ for almost all x.

Problem 2. Show that the function

$$f(x) = \frac{1}{\log(2+|x|)}, \quad x \in \mathbb{R},$$

can not be written as a sum of finitely many functions in $\bigcup_{1\le p<\infty} L^p(\mathbb{R})$.

Problem 3. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and let $g : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable bounded function vanishing outside a compact set. Define $g_\delta(x) = \delta^{-n} g(x/\delta)$, $\delta > 0$. Show that

$$\lim_{\delta\to 0} \int f(x-y) g_\delta(y)\, dy = f(x) \int g(y)\, dy$$

for almost all $x \in \mathbb{R}^n$.

Problem 4. Let $f \in L^1_{loc}(\mathbb{R})$. Assume that for each integer $n > 0$, we have

$$f(x + \frac{1}{n}) \ge f(x)$$

for a.a. $x \in \mathbb{R}$. Show that for each real $a \ge 0$, we have

$$f(x+a) \ge f(x)$$

for a.a. $x \in \mathbb{R}$.

Hint: Consider the function

$$f_s(x) = \int_x^{x+s} f(y)dy, \quad s > 0.$$

Problem 5. Let E be a measurable subset of $[0,1]$ such that

$$m\left(E \cap \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]\right) = \frac{m(E)}{2^n}$$

for $k = 1, 2, \ldots, 2^n$ and for all $n = 1, 2, \ldots$. Show that $m(E) = 0$ or $m(E) = 1$.

Hint: Use the Lebesgue differentiation theorem.

Problem 6. Consider the 3D Weierstrass kernel

$$K_t(x) = (4\pi t)^{-3/2} e^{-|x|^2/(4t)}, \quad t > 0,$$

where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^3$. Prove that, if $f \in L^3(\mathbb{R}^3)$, then $t^{1/2}\|K_t * f\|_{L^\infty} \to 0$ as $t \to 0$. Here,

$$K_t * f(x) = \int_{\mathbb{R}^3} K_t(x-y)f(y)dy$$

is the convolution.