Homework 7 Solutions

1. Set
$$F(x) := \int_x^{x+1} f(t)\,dt = \int_0^{x+1} f(t)\,dt - \int_0^x f(t)\,dt.$$
Then $F(x) \equiv A$, so by the Lebesgue Differentiation Theorem[1] F is differentiable almost everywhere and
$$F'(x) = f(x+1) - f(x) = 0 \text{ for a.e. } x \in \mathbb{R}.$$

2. The proof is by contradiction. Suppose that
$$f(x) = \sum_{j=1}^n g_j(x) \text{ for some } g_j \in L^{p_j} \text{ and each } 1 \le p_j < \infty.$$
Set $E_t := \{x : f(x) > t\}$ for $t > 0$. By the Chebyshev inequality, we have
$$\begin{aligned} m(E_t) &\le \sum_{j=1}^n m\Big(\Big\{x : g_j(x) > \frac{t}{n}\Big\}\Big) \\ &\le \sum_{j=1}^n \frac{n^{p_j}}{t^{p_j}} \|g_j\|_{L^{p_j}}^{p_j} \\ &\le \sum_{j=1}^n \frac{C_j}{t^{p_j}}, \end{aligned}$$
where $C_j := n^{p_j}\|g_j\|_{L^{p_j}}^{p_j}$. We note that the quantity on the right hand side is a polynomial in $y(t) := \frac{1}{t}$. On the other hand, by the definition of f
$$f(x) > t > 0 \Leftrightarrow |x| < e^{\frac{1}{t}} - 2$$
implying
$$m(E_t) = 2(e^{\frac{1}{t}} - 2) = 2(e^y - 2).$$
In particular,
$$2(e^y - 2) \le \sum_{j=1}^n C_j y^{p_j}$$
for all $y > 0$, which cannot be since a polynomial cannot bound an exponential for y large. It follows that f cannot be written as a sum of finitely many functions in $\cup_{1 \le p < \infty} L^p(\mathbb{R})$.

3. We have
$$\int_{\mathbb{R}^n} f(x-y) g_\delta(y)\,dy = \int_{\mathbb{R}^n} f(x-y)\delta^{-n} g(\delta^{-1}y)\,dy.$$
Making the change in variables $z(y) = \delta^{-1}y$. Then $Dz(y) = (\delta^{-1}\delta_{ij})_{i,j=1}^n$ so
$$\det Dz(y) = \delta^{-n},$$

[1] See my notes on the Lebesgue Differentiation Theorem. In short, the Lebesgue Differentiation implies the usual FTC assuming the integrand is only L^1 locally.

and the change of variable formula gives

$$\begin{aligned}\int_{\mathbb{R}^n} f(x-y)g_\delta(y)\,dy &= \int_{z^{-1}(\mathbb{R}^n)} f(x-\delta z)g(z)\,dz\\ &= \int_{\mathbb{R}^n} f(x-\delta z)g(z)\,dz.\end{aligned}$$

Notice that

$$\Big|\int_{\mathbb{R}^n} (f(x-\delta z)-f(x))g(z)\,dz\Big| \le \|g\|_{L^\infty(\mathbb{R}^n)} \int_{\operatorname{supp} g} |f(x-\delta z)-f(x)|\,dz.$$

Thus, using the usual approximation argument by continuous functions with compact support, it is not hard to show that the integral on the right hand side tends to zero in L^1 for any locally integrable f.

4. Fix $a \ge 0$. Then for each $n \in \mathbb{N}$ there is a k_n such that

$$\frac{k_n-1}{n} \le a \le \frac{k_n}{n}.$$

Set

$$f_s(x) := \int_x^{x+s} f(y)\,dy = \int_0^{x+s} f(y)\,dy - \int_0^x f(y)\,dy \text{ for } s>0.$$

Setting $s=\frac{1}{n}$ and applying the Lebesgue Differentiation Theorem, we find

$$f'_{\frac{1}{n}}(x) = f\big(x+\frac{1}{n}\big) - f(x) \ge 0 \text{ for a.e. } x \in \mathbb{R}.$$

It follows that

$$f_{\frac{1}{n}}\Big(x+\frac{k_n-1}{n}\Big) \ge f_{\frac{1}{n}}(x)$$

so that

$$n\int_{x+\frac{k_n-1}{n}}^{x+\frac{k_n}{n}} f(t)\,dt \ge n\int_x^{x+\frac{1}{n}} f(t)\,dt.$$

Since the sets

$$\Big[x+\frac{k_n-1}{n}, x+\frac{k_n}{n}\Big] \text{ and } \Big[x, x+\frac{1}{n}\Big]$$

shrink nicely to $x+a$ and x, respectively, we can apply the Lebesgue Differentiation Theorem once more to conclude that for every $a \ge 0$ and a.e. $x \in \mathbb{R}$ we have

$$f(x+a) \ge f(x) \text{ for a.e. } x.$$

5. Let $E \subset [0,1]$ be measurable and fix $x \in [0,1]$. Note that, for each $n \in \mathbb{N}$, there is a $k_n \in \{1,\dots,2^n\}$ such that $x \in [\frac{k_n-1}{2^n}, \frac{k_n}{2^n}]$. In addition, the sets $[\frac{k_n-1}{2^n}, \frac{k_n}{2^n}]$ shrink nicely to x and

$$2^n m\Big(E \cap \Big[\frac{k_n-1}{2^n}, \frac{k_n}{2^n}\Big]\Big) = 2^n \int_{\frac{k_n-1}{2^n}}^{\frac{k_n}{2^n}} \chi_E(t)\,dt.$$

By the Lebesgue Differentiation Theorem, for almost all $x \in [0,1]$

$$\lim_{n\to\infty} 2^n m\Big(E \cap \Big[\frac{k_n-1}{2^n}, \frac{k_n}{2^n}\Big]\Big) = \chi_E(x) \in \{0,1\}.$$

Since the left-hand side of the expression above is equal to $m(E)$, we are done.

6. We first prove the result when $f \in C_0(\mathbb{R}^3)$. Set

$$z(y) := \frac{x-y}{t^{\frac{1}{2}}}.$$

Then $\det Dz(y) = -t^{-\frac{3}{2}}\, dy$. Thus,

$$\begin{aligned} t^{\frac{1}{2}} \|K_t * f\|_{L^\infty} &\leq t^{\frac{1}{2}} \Big| \int_{\mathbb{R}^3} (4\pi t)^{-\frac{3}{2}} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy \Big| \\ &= t^{\frac{1}{2}} \Big| \int_{\mathbb{R}^3} (4\pi)^{-\frac{3}{2}} e^{-\frac{|z|^2}{4}} f\big(x - t^{\frac{1}{2}} z\big)\, dz \Big| \\ &\leq (4\pi)^{-\frac{3}{2}} \|f\|_{L^\infty} t^{\frac{1}{2}} \int_{\mathbb{R}^3} e^{-\frac{|z|^2}{4}}\, dz \\ &\to 0 \text{ as } t \to 0 \end{aligned}$$

where we have applied the change of variable formula in the second line. It follows that the desired result holds for all $f \in C_0(\mathbb{R}^3)$.

We now suppose that $f \in L^3(\mathbb{R}^3)$ is arbitrary. Let $\epsilon > 0$ and choose $\phi \in C_0(\mathbb{R}^3)$ such that $\|f - \phi\|_{L^3} < \epsilon$. By the triangle inequality, we have

$$\begin{aligned} t^{\frac{1}{2}} \|K_t * f\|_{L^\infty} &\leq t^{\frac{1}{2}} \|K_t * (f-\phi)\|_{L^\infty} + t^{\frac{1}{2}} \|K_t * \phi\|_{L^\infty} \\ &:= I + II. \end{aligned}$$

By the first part, $II \to 0$ as $t \to 0$. Using the change of variable z above and the Hölder inequality, we can bound I:

$$\begin{aligned} I &\leq t^{-1} (4\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4t}} |f(y) - \phi(y)|\, dy \\ &\leq t^{-1} (4\pi)^{-\frac{3}{2}} \Big(\int_{\mathbb{R}^3} e^{-\frac{3|x-y|^2}{8t}}\, dy \Big)^{\frac{2}{3}} \|f - \phi\|_{L^3} \\ &\leq t^{-1} (4\pi)^{-\frac{3}{2}} \Big(\int_{\mathbb{R}^3} -t^{\frac{3}{2}} e^{-\frac{|z|^2}{8}}\, dz \Big)^{\frac{2}{3}} \epsilon \\ &\leq C\epsilon \end{aligned}$$

for some constant $C > 0$ independent of ϵ. It follows that I can be made arbitrarily small depending on the choice of ϕ. Combining the estimates for both I and II shows that

$$t^{\frac{1}{2}} \|K_t * f\|_{L^\infty} \to 0 \text{ as } t \to 0,$$

as desired.