Homework 8 (Due Tuesday, March 12, 2024, 11:59 pm)

Problem 1. Show if ν is a signed measure, then E is ν-null iff $|\nu|(E) = 0$. Also, prove that if ν and μ are signed measures, $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Problem 2. Let ν be a signed measure, and let μ be a positive measure. Show that $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Problem 3. Suppose that $\{\nu_j\}$ is a sequence of positive measures.

(1) Assume that $\nu_j \perp \mu$ for all j. Show that $\sum_{j=1}^{\infty} \nu_j \perp \mu$.
(2) Assume that $\nu_j \ll \mu$ for all j. Show that $\sum_{j=1}^{\infty} \nu_j \ll \mu$.

Problem 4. Let E be a measurable subset of $\mathbb{R}^d$ with positive measure and let $x_1, x_2, \dots$ be a sequence of points which is dense in $\mathbb{R}^d$, i.e. for each $x \in \mathbb{R}^d$ and each $\varepsilon > 0$, there is an n such that $|x - x_n| < \varepsilon$. Prove that the union of the sets $x_n + E$ is almost all of $\mathbb{R}^d$.

Problem 5. Suppose μ and ν are positive measures on a measurable space $(X, \mathcal{A})$ such that μ is σ-finite, ν is finite, $\nu \leq \mu$ and $\nu \ll \mu - \nu$. Prove that

$$\mu(\{x \in X : \frac{d\nu}{d\mu} = 1\}) = 0.$$

Problem 6. Let μ and ν be σ-finite positive measures on $(X, \mathcal{A})$ such that $\nu \ll \mu$. Set $\lambda = \mu + \nu$. Show that if f is the Radon–Nikodym derivative of ν with respect to λ, $f = \frac{d\nu}{d\lambda}$, then $0 \leq f < 1$ μ-a.e. and

$$\frac{d\nu}{d\mu} = \frac{f}{1-f}$$

μ-a.e.

Problem 7. Show that if μ_1, ν_1 are σ-finite positive measures on $(X, \mathcal{A})$ and μ_2, ν_2 are σ-finite positive measures on $(Y, \mathcal{B})$, then $\nu_1 \ll \mu_1$ and $\nu_2 \ll \mu_2$ implies

$$\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$$

and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x, y) = \frac{d\nu_1}{d\mu_1}(x)\frac{d\nu_2}{d\mu_2}(y)$$

for $\mu_1 \times \mu_2$-a.e. (x, y).