Homework 8 Solutions

1. Suppose E is ν-null and write $\nu = \nu^+ - \nu^-$. Write $X = P \cup N$ where the union is disjoint, P is a positive set, and N is a negative set. Then $E = (P \cap E) \cup (N \cap E)$. Furthermore, $|\nu| = \nu^+ + \nu^-$. Since $E \cap P$ and $E \cap N$ are measurable and $\nu(E) = 0$, we have

$$0 = \nu(E \cap P) = \nu^+(E \cap P).$$

Similarly,

$$\nu^-(E \cap N) = 0.$$

Combining each case shows $|\nu|(E) = 0$, since

$$|\nu|(E) = \nu^+(E \cap P) + \nu^-(E \cap N).$$

If $|\nu|(E) = 0$, then

$$\nu^+(E \cap P) = \nu^-(E \cap N) = 0$$

so $\nu(E) = 0$.

Suppose now that $\nu \perp \mu$. Then there exist sets $E, F \in \mathcal{M}$ such that $E \cap F = \emptyset$, $E \cup F = X$, E is null for μ, and F is null for ν. From above, F is null for $|\nu|$ also so $|\nu| \perp \mu$. Similarly, if $|\nu| \perp \mu$ and E and F are as above, except F is null for $|\nu|$, then F is null for ν and $\nu \perp \mu$. An analogous argument holds for $\nu^\pm$.

2. Suppose $\nu \ll \mu$ and let $X = P \cup N$ be a Hahn decomposition for μ. Let $E \in \mathcal{M}$ be μ-null. Then $\mu(E \cap P) = \mu(E \cap N) = 0$ since both $E \cap P \in \mathcal{M}$ and $E \cap N \in \mathcal{M}$ so

$$\nu(E \cap P) = \nu(E \cap N) = 0$$

implying $\nu^+(E) = \nu^-(E) = 0$. This proves that $|\nu| \ll \mu$, $\nu^+ \ll \mu$, and $\nu^- \ll \mu$. If $\nu^+ \ll \mu$ and $\nu^- \ll \mu$, then it is clear $\nu \ll \mu$ since $\nu = \nu^+ - \nu^-$. Hence, the proof is complete.

3. (a) For each j, there are $E_j, F_j \in \mathbb{N}$ such that for each j we have $X = E_j \cup F_j$, $E_j \cap F_j = \emptyset$, E_j is μ-null, and F_j is ν_j-null. Set $F := \cap_1^\infty F_j$ and $E := \cup_1^\infty (E_j \backslash F)$. Then $X = E \cup F$ and $E \cap F = \emptyset$ by construction. The set E is μ-null since the countable union of μ-null sets is μ-null. Furthermore,

$$\sum_1^\infty \nu_j(F) = \lim_{n\to\infty} \sum_1^n \nu_j(F) = 0$$

since $\nu_j(F) = 0$ for all j. It follows that $\sum_1^\infty \nu_j \perp \mu$.

(b) Suppose $\nu_j \ll \mu$ for each j and let $E \in \mathcal{M}$ be μ-null. Then $\nu_j(E) = 0$ for each j. Hence,

$$\sum_1^\infty \nu_j(E) = \lim_{n\to\infty} \sum_1^n \nu_j(E) = 0.$$

Since E is arbitrary, $\sum_1^\infty \nu_j \ll \mu$ also.

4. The idea is to use convolution.[1] Set

$$M := \Big(\cup_{n\in\mathbb{N}} (x_n + E) \Big)^c.$$

We show that M is a null set. Consider the convolution

$$\chi_M * \chi_E(x) = \int_{\mathbb{R}^d} \chi_M(y)\chi_E(y-x)\, dx.$$

Without loss of generality, we may assume $m(E) < \infty$ since we can always restrict our attention to a subset of E with finite positive measure. Since χ_M and χ_E are bounded measurable functions and $\chi_E \in L^1(\mathbb{R}^d)$, their convolution is a continuous function on $\mathbb{R}^d$ (this was proved in an earlier homework). Evaluating at x_n gives

$$\chi_M * \chi_E(x_n) = 0$$

since $y - x_n \in E$ iff $y \in M^c$ iff $\chi_M(y) = 0$. Continuity of the convolution and density of $\{x_n\}_{n\in\mathbb{N}}$ implies $\chi_M * \chi_E \equiv 0$. Thus,

$$0 = \int_{\mathbb{R}^d} \chi_M(y)\chi_E(y-x)\, dy.$$

Integrating over $\mathbb{R}^d$ in x and applying the Tonelli theorem gives:

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_M(y)\chi_E(y-x)\, dydx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_M(y)\chi_E(y-x)\, dxdy \\ &= m(M)m(E). \end{aligned}$$

Since $m(E) > 0$, this implies $m(M) = 0$.

5. First, note that $\nu \le \mu$ implies $\nu \ll \mu$ since $\mu(A) = 0$ forces $\nu(A) = 0$. By the Radon-Nikodym Theorem, $d\nu = f d\mu$ where $f = \frac{d\nu}{d\mu}$. Set

$$E := \{x \in X : f = 1\}.$$

Since

$$\nu(A) = \int_A f\mu$$

for any measurable A, ν restricts to μ on E. In particular, $\mu(E) - \nu(E) = 0$. Since $\nu \ll \mu - \nu$, we have $\nu(E) = 0$ also. Thus, $\mu(E) = 0$ and the proof is complete.

6. Let $E \in \mathcal{M}$ be λ-null. Then $\lambda(E) = \mu(E) + \nu(E) = 0$ implying $\mu(E) = 0$ since both μ and ν are positive. Thus, $\mu \ll \lambda$ also. On the other hand, $\lambda \ll \mu$ since $\nu \ll \mu$ and $\lambda = \mu + \nu$. By Proposition 3.9, we find $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}\ \lambda - \text{a.e.}$$

Fix $E \in \mathcal{M}$ and suppose $\nu(E) > 0$. In particular, this implies $\mu(E) > 0$ since $\nu \ll \mu$. Observe that, since $\lambda \ll \mu$ and $\mu \ll \nu$, it suffices to show $0 \le f < 1$ λ-a.e.. Suppose $f \ge 1$ for some $E \in \mathcal{M}$ with $\lambda(E) > 0$. Applying Proposition 3.9 with $g = \chi_E$, we find

$$\nu(E) = \int_E d\nu = \int_E \frac{d\nu}{d\lambda} d\lambda \ge \int_E d\lambda = \lambda(E).$$

But $\lambda(E) = \mu(E) + \nu(E)$ so we must have $\mu(E) = 0$ which implies $\nu(E) = 0$ since $\nu \ll \mu$. If $\frac{d\nu}{d\lambda} < 0$ on for $\lambda(E) > 0$, then a similar computation shows

$$\nu(E) = \int_E \frac{d\nu}{d\lambda} d\lambda < 0$$

[1]This is a useful qual trick and can be used on several of the earlier problem sets from the first quarter.

whenever $\nu(E) \neq 0$. It follows that $0 \leq f < 1$ λ-a.e., since the assumptions $f < 0$ on a set of λ-positive measure and $f \geq 1$ on a set of λ-positive measure both lead to a contradiction.

To prove the remaining assertion, we will show that $(1-f)\frac{d\nu}{d\mu} = f$ μ-a.e. Indeed, using the identities

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda}\frac{d\lambda}{d\mu} \text{ and } \frac{d\lambda}{d\mu} = 1 + \frac{d\nu}{d\mu},$$

we find

$$(1-f)\frac{d\nu}{d\mu} = \frac{\nu}{d\lambda}(\frac{d\lambda}{d\mu} - \frac{d\nu}{d\mu}) = \frac{d\nu}{d\lambda} \quad \mu-\text{a.e.},$$

as was to be shown.

7. Let E be a $\mu_1 \times \mu_2$-null set. We need to show that E is $\nu_1 \times \nu_2$ null. By the Fubini Theorem, we have:

$$\begin{aligned} 0 &= \mu_1 \times \mu_2(E) \\ &= \int\int \chi_E(x,y)\, d\mu_1(x) d\mu_2(y) \\ &= \int\int \chi_E(x,y)\, d\mu_2(y) d\mu_1(x) \\ &= \int \mu_2(E_x)\, d\mu_1(x) \end{aligned}$$

implying $\mu_2(E_x) = 0$ for μ_1-a.e. $x \in X$. Here, $E_x := \{y \in Y : (x,y) \in E\}$. Since $\nu_i \ll \mu_i$ for $i = 1,2$, we thus have

$$\begin{aligned} \nu_1 \times \nu_2(E) &= \int\int \chi_E(x,y)\, d\nu_1(x) d\nu_2(y) \\ &= \int\int \chi_E(x,y)\, d\nu_2(y) d\nu_1(x) \\ &= \int \nu_2(E_x)\, d\nu_1(x) \\ &= 0. \end{aligned}$$

It follows that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ so the Radon-Nikodym Theorem applies.

We now show that

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x,y) = \frac{d\nu_1}{d\mu_1}(x)\frac{d\nu_2}{d\mu_2}(y)$$

for $\mu_1 \times \mu_2$-a.e. (x,y). By the first part, we may apply the Radon-Nikodym Theorem to obtain a $\mu_1 \times \mu_2$-measurable function f such that $d(\nu_1 \times \nu_2) = f\, d(\mu_1 \times \mu_2)$. In addition, since $\nu_i \ll \mu_i$ for $i = 1,2$ we may apply the Radon-Nikodym Theorem for each i to obtain μ_i-integrable functions f_i such that $d\nu_i = f_i\, d\mu_i$. Since the Radon-Nikodym derivative is unique up to a $\mu_1 \times \mu_2$-null set, it suffices to show that

$$d(\nu_1 \times \nu_2)(x,y) = f_1(x) f_2(y)\, d\mu_1(x) d\mu_2(y).$$

Furthermore, by Caratheodory's Theorem we can do the computation on a measurable rectangle; i.e. a set $E = A \times B \subset X \times Y$ where A is μ_1-measurable and B is μ_2-measurable. We have:

$$\begin{aligned} \nu_1 \times \nu_2(A \times B) &= \int\int_{A \times B} d\nu_1 d\nu_2 \\ &= \int_A d\nu_1 \int_B d\nu_2 \\ &= \nu_1(A)\nu_2(B) \\ &= \int_A f_1(x)\, d\mu_1 \int_B f_2(y)\, d\mu_2 \\ &= \int\int_{A \times B} f_1(x) f_2(y)\, d\mu_1 d\mu_2. \end{aligned}$$

This completes the proof.