## Homework 1 (Due Friday, January 12, 2024, 11:59 pm)

**Problem 1.** Let $A \subset \mathbb{R}$ be Lebesgue measurable with $m(A) > 0$. Prove that there exist $k, n \in \mathbb{N}$ and $x, y \in A$ with $|x - y| = \frac{k}{2^n}$.

**Problem 2.** Let $(X, \mathcal{A})$ be a measurable space, and let $\mu$ and $\nu$ be measures on $(X, \mathcal{A})$. The measure $\nu$ is called absolutely continuous with respect to $\mu$ if every $\mu$-null set is also a $\nu$-null set, i.e.

$$A \in \mathcal{A} \quad \text{and} \quad \mu(A) = 0 \quad \Longrightarrow \nu(A) = 0.$$

(i) Show that if $\nu$ is a finite measure, then $\nu$ is absolutely continuous with respect to $\mu$ if and only if $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall A \in \mathcal{A}$, $\mu(A) < \delta \Longrightarrow \nu(A) < \varepsilon$.

(ii) Show that this equivalence is not true without the assumption that $\nu$ is finite.

**Problem 3.** Let $f_n \in L^1(\mathbb{R})$, $n = 1, 2, \ldots$, and assume that the sequence

$$\int |f_n(x)| dx$$

is bounded. Assume that $f_n \to f$ a.e. as $n \to \infty$. Show that $f \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} |f_n(x)| dx - \int_{\mathbb{R}} |f_n(x) - f(x)| dx \to \int_{\mathbb{R}} |f(x)| dx$$

as $n \to \infty$.

**Problem 4.** Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous. Prove that

$$\lim_{n\to\infty} \int_0^1 x^n f(x) dx = 0, \quad \lim_{n\to\infty} n \int_0^1 x^n f(x) dx = f(1).$$

**Problem 5.** Let $f \in L^1_{loc}(\mathbb{R})$ be such that

$$\int_{|x|\le r} |f(x)| dx \le (r + 1)^a,$$

for all $r \ge 0$ and for some $a \in \mathbb{R}$. Let $g$ be a Lebesgue measurable function such that

$$|g(x)| \le e^{-|x|}.$$

Show that

$$x \mapsto f(x)g(tx) \in L^1(\mathbb{R})$$

for any $t \in \mathbb{R} \setminus 0$.