L^p Spaces

1 The Basics

Let $(X, \mathcal{M}, \mu)$ be a fixed measure space. If $f : X \to \mathbb{C}$ is a measurable function on X and $0 < p < \infty$, we define the L^p *norm of* f to be

$$\|f\|_{L^p} := \Big(\int_X |f|^p \, d\mu \Big)^{\frac{1}{p}}, \tag{1}$$

where we allow the possibility $\|f\|_p = \infty$. For each $p > 0$, we define the L^p space

$$L^p(X, \mathcal{M}, \mu) := \{ f : X \to \mathbb{C} : f \text{ is measurable and } \|f\|_p < \infty \}. \tag{2}$$

It is common to denote th space by $L^P(X, \mathcal{M}, \mu)$ by $L^p(\mu)$, $L^p(X)$, or even just L^p if the measure or base set is clear. As before, elements of L^p are equivalence classes of functions equal almost everywhere. Since any complex function is a linear combination of real functions, we will only consider the case when f is real-valued. However, all the theorems in this section extend to complex-valued functions. The L^p spaces are vector spaces since, for $f, g \in L^p$, we have

$$|f + g|^p \leq (2 \max\{|f|, |g|\})^p \leq 2^p(|f|^p + |g|^p)$$

implying $f + g \in L^p$. For each $p > 0$, $\|\cdot\|_p$ is a norm on L^p, which we leave as an exercise.

In practice, it is most common to consider only the cases $p \geq 1$ so we will do so in this note. The reason why is that the L^p spaces are *Banach spaces* for $p \geq 1$, which means they are really nice. A Banach space is a complete normed linear space. The nicest of the L^p spaces is L^2 since in this case it can be equipped with an inner product making it a Hilbert space; that is, a complete normed inner product space. In fact, L^2 is the only L^p space that is a Hilbert space, so in this sense it is special. The reason the L^p spaces for $p < 1$ are not Banach spaces is due to the fact that the triangle inequality fails when $p < 1$.[1] The cornerstone of the theory of L^p spaces is the *Hölder inequality.* It relies on a very useful inequality for positive real numbers:

Lemma 1 (Young's Inequality)**.** *If $a, b \geq 0$ and $0 < \lambda < 1$, then*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda) b$$

with equality if and only if $a = b$.

Often times, you will see Young's inequality presented as

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b \geq 0$. Another useful variant is *Cauchy's inequality with* ϵ, which says that for any a, b and any $\epsilon > 0$ we have

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

Using Young's Inequality, one can prove the Hölder inequality:

[1] See the beginning of page 182 in Folland.

Theorem 1 (Hölder's Inequality)**.** *Suppose $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. If f and g are measurable functions on X, then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q. \tag{3}$$

In particular, if $f \in L^p$ and $g \in L^q$, then $fg \in L^1$, and in this case equality holds in (3) *if and only if $\alpha|f|^p = \beta|g|^q$ a.e. for some constant $\alpha, \beta \neq 0$.*

We call the exponents p, q *Hölder conjugates*. An important consequence of the Hölder inequality is *Minkowski's Inequality*. It can be viewed as the triangle inequality for L^p spaces.

Theorem 2 (Minkowski's Inequality)**.** *If $1 \leq p < \infty$ and $f, g \in L^p$, then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Using the dominated convergence theorem, it is not hard to show that the integrable simple functions are dense in L^p for $p \geq 1$. A very useful fact is that $C_0(\mathbb{R})$ and $C_0^\infty(\mathbb{R})$ are both dense in L^p too.

Up to this point, we have not considered the case $p = \infty$ since this case is a little more delicate. We now define the space L^∞, which can be seen as the limiting value $p = \infty$ (see Problem 3 below). If f is a measurable function on X, define

$$\|f\|_\infty := \inf\{a \geq 0 : \mu(\{x : |f(x)| > a\}) = 0\}, \tag{4}$$

with the convention that $\inf \emptyset = \infty$. The infimum is attained since

$$\{x : |f(x)| > a\} = \cup_1^\infty \{x : |f(x)| > a + n^{-1}\},$$

and if the sets on the right-hand side are null, so is the one on the left. The value $\|f\|_\infty$ is called the *essential supremum* of $|f|$. We can now define the space

$$L^\infty(X, \mathcal{M}, \mu) := \{f : X \to \mathbb{C} : f \text{ is measurable and } \|f\|_\infty < \infty\}, \tag{5}$$

with the usual convention that functions are defined a.e. Examining the definitions, we see that $f \in L^\infty$ if and only if there is a bounded measurable function g such that $f = g$ a.e. In fact, we can take $g = f\chi_E$ where $E := \{x : |f(x)| \leq \|f\|_\infty\}$. Two remarks are in order:

1. For X and $\mathcal{M}$ fixed, $L^\infty(\mu)$ depends on μ only in that μ determines the null sets.
2. It is common to view 1 and ∞ as holder conjugates also, since $1 + \infty^{-1} = 1$. The following theorem justifies this convention.

Theorem 3. *(a) If f and g are measurable functions on X, then $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.*

(b) $\|\cdot\|_\infty$ is a norm on L^∞.

(c) $\|f_n - f\|_\infty \to 0$ iff there exists $E \in \mathcal{M}$ such that $\mu(E^c) = 0$ and $f_n \to f$ uniformly on E.

(d) L^∞ is a Banach space.

(e) The simple functions are dense in L^∞.

As an exercise, it is worthwhile to prove Theorem 3.

2 Inclusions for L^p Spaces

To understand the intuition behind what L^p spaces measure, it is instructive to study the inclusion of various L^p spaces in each other. In general, we have $L^p \neq L^q$ for all $p \neq q$. To see why, it is helpfu l to consider the functions $f_a(x)$ on $(0, \infty)$, where $a > 0$. By elementary calculus, $f_a\chi_{(0,1)} \in L^p$ iff $p < a^{-1}$ and $f_a\chi_{(1,\infty)}$ iff $p > a^{-1}$. This suggests two reasons why a function f may fail to be L^p:

1. $|f|^p$ blows up too rapidly near some point, or

2. $|f|^p$ fails to decay rapidly enough at infinity.

In the first case, the behavior of $|f|^p$ becomes worse as p increases, and in the second case it gets better. In other words, if $p < q$, then functions in L^p can be locally more singular than functions in L^q, whereas functions in L^q can be globally more spread out than functions in L^p. We state three useful results which follow directly from Hölder's inequality.

Proposition 1. *If $1 \leq p < q < r \leq \infty$, then $L^q \subset L^p + L^r$; that is, each $f \in L^q$ is the sum of a function in L^p and a function in L^r.*

Proposition 2. *If $1 \leq p < q < r \leq \infty$, then $L^p \cap L^r \subset L^q$ and $\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$, where $\lambda \in (0,1)$ is defined by*

$$\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}.$$

Proposition 3. *If $\mu(X) < \infty$ and $1 \leq p < q \leq \infty$, then $L^p(\mu) \supset L^q(\mu)$ and $\|f\|_p \leq \|f\|_q \mu(X)^{\frac{1}{p}-\frac{1}{q}}$.*

Propositions 1 and 2 are covered in the homework. As an additional exercise, it is helpful to prove Proposition 3.

2.1 Some Remarks

Before we proceed, some remarks about the significance of L^p spaces will be useful. The three most important L^p spaces are L^1, L^2, and L^∞; however, L^1 and L^∞ can still be rather pathological. As mentioned before, L^2 is special because it is a Hilbert space and L^∞ is important because of its relation to the topology of uniform convergence. Due to the pathological nature of L^1 and L^∞, it is often more fruitful to deal with intermediate L^p spaces. One manifestation of this is the duality theory of L^p spaces, and another is that many operators of interest in Fourier analysis and differential equations are bounded (i.e. continuous) on L^p for $p \in (1,\infty)$ but not on L^1 or L^∞.

3 Useful Inequalities

In this section, we state some useful inequalities that are helpful on qual problems and in research. The following inequality is a great exercise:

Lemma 2 (Chebyshev's Inequality)**.** *If $f \in L^p$ for $p \in [1,\infty)$, then for any $\alpha > 0$ we have*

$$\mu(\{x : |f(x)| > \alpha\}) \leq \left(\frac{\|f\|_p}{\alpha}\right)^p.$$

We have already learned of Minkowski's inequality, which states that the L^p norm of a sum is less than the sum of the L^p norms. We can generalize this to the case that sums are replaced by integrals.

Theorem 4 (Minkowksi's Inequality for Integrals)**.** *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces, and let be an $(\mathcal{M} \otimes \mathcal{N})$-measurable function on $X \times Y$.*

(a) If $f \geq 0$ and $1 \leq p < \infty$, then

$$\left(\int \left(\int f(x,y)\, d\nu(y)\right)^p d\mu(x)\right)^{\frac{1}{p}} \leq \int \left(\int f(x,y)^p\, d\mu(x)\right)^{\frac{1}{p}} d\nu(y).$$

(b) If $1 \leq p < \infty$, $f(\cdot, y) \in L^p(\mu)$ for a.e. y, and the function $y \mapsto \|f(\cdot,y)\|_p$ is in $L^1(\nu)$, then $f(x,\cdot) \in L^1(\nu)$ for a.e. x and the function $x \mapsto \int f(x,y)\, \nu(y)$ is in $L^p(\mu)$, and

$$\left\|\int f(\cdot,y)\, d\nu(y)\right\|_{L^p(\mu)} \leq \int \|f(\cdot,y)\|_{L^p(\mu)}\, d\nu(y).$$

Theorem 4(b) gives a nice expression for the p-norm of an integral.

3.1 Additional Material

To close this note, we mention that some useful material for anyone interested in analysis is contained in sections 6.4 and 6.5 in Folland. Section 6.4 deals with distribution functions and weak L^p, while section 6.5 proves some useful interpolation inequalities for L^p spaces. Both are important in, for example, the regularity theory for differential equations and Fourier analysis. However these sections are a little bit outside the scope of the course. At the very least, it is worthwhile to be aware that the results in those sections exist.

4 Problems

1. All of the suggested exercises in the notes.
2. When does equality hold in Minkowski's inequality? (The answer is different for $p = 1$ and for $1 < p < \infty$. What about $p = \infty$?)
3. If $f \in L^p \cap L^\infty$ for some $p < \infty$, so that $f \in L^q$ for all $q > p$, then $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.
4. Suppose $1 \leq p < \infty$. If $\|f_n - f\|_p \to 0$, then $f_n \to f$ in measure, and hence some subsequence converges to f a.e. On the other hand, if $f_n \to f$ in measure and $|f_n| \leq g \in L^p$ for all n, then $\|f_n - f\|_p \to 0$.

Reference: *Real Analysis: Modern Techniques and Their Applications*, 2nd ed., Gerald B. Folland.