The Radon-Nikodym Theorem

1 Signed Measures

Let $(X, \mathcal{M})$ be a measure space. A *signed measure* on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that

(i) $\nu(\emptyset) = 0$;

(ii) ν assumes at most one of the values $\pm\infty$;

(iii) If $\{Ej\}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu(\cup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$, where the latter sum converges absolutely if $\nu(\cup_1^\infty E_j) < \infty$.

Thus, every measure has a signed measure. Measures are sometimes referred to as *positive measures* when there is possibility for confusion. Two important examples of signed measures are as follows: If μ_1, μ_2 are signed measures and at least one of them is finite, then $\nu = \mu_1 - \mu_2$ is a signed measured. Also, if μ is a measure on $\mathcal{M}$ and $f : X \to [-\infty, \infty]$ is a measurable function such that at least one of $\int f^+$ or $\int f^-$ is finite (in which case, we say that f is *extended μ-integrable*), then $\nu(E) := \int_E f \, d\mu$ is a signed measure. In fact, these are the *only* examples since every signed measure can be represented in one of these two forms, as we will soon see. Arguing precisely as for positive measures, one can shows that signed measures are continuous from above and below.[1]

If ν is a signed measure on $(X, \mathcal{M})$, a set $E \in \mathcal{M}$ is called *positive for* ν if $\nu(F) \geq 0$ for all $F \in \mathcal{M}$ such that $F \subset E$. *Negative sets* and *null sets* are defined similarly, except we require $\nu(F) \leq 0$ or $\nu(F) = 0$, respectively. For example, if

$$\nu(E) := \int_E f \, d\mu,$$

then a set E is positive when $f \geq 0$, negative when $f \leq 0$, and null when $f = 0$ μ-a.e. on E. Using the definitions, it is not hard to show that any measurable subset of a positive set is positive, and the union of any countable family of positive sets is positive. An analogous result holds for negative and null sets.

The two big theorems for signed measures are the decomposition theorems. The first is the *Hahn Decomposition Theorem* and says that, given a signed measure, X can be decomposed as the disjoint union of a positive and negative set:

Theorem 1 (Hahn Decomposition Theorem)**.** *If ν is a signed measure on $(X, \mathcal{M})$, then there exist a positive set P and a negative set N for ν such that $X = P \cup N$ and $P \cap N = \emptyset$. If P' and N' is another pair, then $P \triangle P'$ and $N \triangle N'$ are ν-null.*

We call the decomposition in Theorem 1 the *Hahn decomposition for* ν. It is not usually unique since we can alter P and N on ν-null sets, but it leads to a canonical representation of ν as the difference of two positive measures. This is called the *Jordan decomposition for* ν.

To state the theorem we need a couple of definitions. We say that two signed measures μ and ν on $(X, \mathcal{M})$ are *mutually singular*, or that ν *is singular with respect to* μ (or vice versa) if there exist $E, F \in \mathcal{M}$ such that $E \cap F = \emptyset$, $X = E \cup F$, E is μ-null, and F is ν-null. Informally, if μ and ν are mutually singular their supports do not intersect. It is common to write $\mu \perp \nu$.

Theorem 2 (Jordan Decomposition Theorem)**.** *If ν is a signed measure, there exist unique positive measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.*

[1] I encourage you to work this out.

We call the measures ν^+ and ν^- the *positive variation* and *negative variation* of ν, respectively, and $\nu = \nu^+ - \nu^-$ is called the *Jordan decomposition of* ν. The *total variation of* ν is the measure $|\nu|$ given by

$$|\nu| = \nu^+ + \nu^-.$$

One can show that $E \in \mathcal{M}$ is ν-null iff $|\nu|(E) = 0$, and $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^- \perp \mu$ and $\nu^- \perp \mu$.[2] Note that ν is bounded above by ν^+ and below by $-\nu^-$. In particular, if the range of ν is contained in $\mathbb{R}$, then ν is bounded. We also note that ν is of the form $\nu(E) = \int_E f\,d\mu$, where $\mu = |\nu|$, $f = \chi_P - \chi_N$, and $X = P \cup N$ is a Hahn decomposition for ν. Integration with respect to a signed measure ν is defined in the obvious way. We set

$$L^1(\nu) := L^1(\nu^+) \cap L^1(\nu^-)$$

and

$$\int f\,d\nu = \int f\,d\nu^+ - \int f\,d\nu^- \text{ for } f \in L^1(\nu).$$

A signed measure is called *finite* (resp, *σ-finite*) if $|\nu|$ is finite (resp. σ-finite).

2 The Lebesgue-Radon-Nikodym Theorem

We can now discuss the Lebesgue-Radon-Nikodym Theorem, which is often simply referred to as the Radon-Nikodym Theorem. Put simply, the Radon-Nikodym Theorem can be viewed as differentiation theorem for measures.

Suppose that ν is a signed measure and μ is a positive measure on $(X, \mathcal{M})$. We say that ν is *absolutely continuous* with respect to μ and write $\nu \ll \mu$ if $\nu(E) = 0$ for every $E \in \mathcal{M}$ for which $\mu(E) = 0$. It is a good exercise to show that $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$. Absolutely continuity is, in some sense, the antithesis of mutual singularity. To see this, note that if $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$. One can also extend the definition of absolute continuity to the case when μ is a signed measure, but we will not need to do so for our purposes.

So far, it may not be clear what the absolute continuity condition has to do with continuity. The next two results make the connection explicit.

Theorem 3. *Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ iff for every $\epsilon > 0$ there is a $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$.*

If μ is a measure and f is an extended μ-integrable function, the signed measure ν defined by $\nu(E) = \int_E f\,d\mu$ is absolutely continuous with respect to μ and finite iff $f \in L^1(\mu)$. Since any complex function f is a linear combination of real functions, the following corollary holds for all complex $f \in L^1(\mu)$:

Corollary 1. *If $f \in L^1(\mu)$, for every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$\Big|\int_E f\,d\mu\Big| < \epsilon \text{ whenever } \mu(E) < \delta.$$

The corollary above is often referred to as *absolute continuity of the integral.* For short, we will write

$$d\nu = f\,d\mu$$

to denote the relationship $\nu(E) = \int_E f d\mu$.

Theorem 4 (Lebesgue-Radon-Nikodym Theorem)**.** *Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(X, \mathcal{M})$. There exist unique σ-finite signed measures λ, ρ on $(X, \mathcal{M})$ such that*

$$\lambda \perp \mu, \ \rho \ll \mu, \text{ and } \nu = \lambda + \rho.$$

Moreover, there is an extended μ-integrable function $f : X \to \mathbb{R}$ such that $d\rho = f d\mu$ and any two such functions are equal μ-a.e.

[2] See Homeowrk 8.

We call the decomposition $\nu = \lambda + \rho$ the *Lebesgue decomposition of* ν with respect to μ. In the case $\nu \ll \mu$, Theorem 4 says that $d\nu = f\,d\mu$ for some f. This result is often called the *Radon-Nikodym Theorem* and f is called the *Radon-Nikodym derivative of* ν with respect to μ. We denote the Radon-Nikodym derivative $f = \frac{d\nu}{d\mu}$ by

$$d\nu = \frac{d\nu}{d\mu}\,d\mu.$$

One can check that

$$\frac{d(\nu_1 + \nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}.$$

The following chain rule is very useful:

Proposition 1 (Chain Rule)**.** *Suppose that ν is a σ-finite signed measure and μ, λ are σ-finite measure on $(X, \mathcal{M})$ such that $\nu \ll \mu$ and $\mu \ll \lambda$.*

(a) If $g \in L^1(\nu)$, then $g\frac{d\nu}{d\mu} \in L^1(\mu)$ and

$$\int g\,d\nu = \int g\frac{d\nu}{d\mu}\,d\mu.$$

(b) We have $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda - a.e.$$

Corollary 2. *If $\mu \ll \lambda$ and $\lambda \ll \mu$, then*

$$\frac{d\lambda}{d\mu}\frac{d\mu}{d\lambda} = 1 \text{ a.e. with respect to } \mu \text{ or } \lambda.$$

An important non-example is as follows. Let μ be the Lebesgue measure and ν the point mass at zero on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Then $\nu \perp \mu$. However, the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ does not exist. This non-existent derivative is more commonly known as the Dirac δ-function.

3 Problems

1. Prove Proposition 3.1 on page 86 of the textbook: Let ν be a signed measure on $(X, \mathcal{M})$. Then

$$\nu\Big(\bigcup_{n\in\mathbb{N}} E_n\Big) = \lim_{n\to\infty} \nu(E_n) \text{ and } \nu\Big(\bigcap_{n\in\mathbb{N}} F_n\Big) = \lim_{n\to\infty} \nu(F_n),$$

for any increasing sequence $(E_n)_{n\in\mathbb{N}}$ of measurable sets and any decreasing sequence $(F_n)_{n\in\mathbb{N}}$ of measurable sets with $\nu(F_1) < \infty$.

Solution. First, suppose $\{E_n\} \subset \mathcal{M}$ is an increasing sequence of sets. Set $E_0 = \emptyset$. Then, by disjoint additivity of ν and the well-definedness (since ν takes on only one of the values $\pm\infty$) and absolute convergence (if finite) of the sum $\sum_1^\infty \nu(E_n \setminus E_{n-1})$

$$\begin{aligned}\nu\Big(\bigcup_1^\infty E_n\Big) &= \nu\Big(\bigcup_1^\infty E_n \setminus E_{n-1}\Big) \\ &= \sum_1^\infty \nu(E_n \setminus E_{n-1}) \\ &= \lim_{n\to\infty} \sum_1^n \nu(E_j \setminus E_{j-1}) \\ &= \lim_{n\to\infty} \nu(E_n)\end{aligned}$$

since $E_n = \bigcup_1^n E_j \setminus E_{j-1}$. Suppose now that $\{E_n\}$ is a decreasing sequence of sets and that $\nu(E_1)$ is finite. For each n, set $F_n = E_1 \setminus E_n$ and note that $\{F_n\} \subset \mathcal{M}$ is an increasing sequence of sets. Moreover, $\nu(E_1) = \nu(F_n) + \nu(E_n)$ for each n, where the sum is well defined since ν takes on only one of the values $\pm\infty$. Since $\bigcup_1^\infty F_n = E_1 \setminus (\bigcap_1^\infty E_n)$,

$$\begin{aligned}\nu(E_1) &= \nu(\bigcap_1^\infty E_n) + \nu(\bigcup_1^\infty F_n) \\ &= \nu(\bigcap_1^\infty E_n) + \lim_{n\to\infty} \nu(F_n) \\ &= \nu(\bigcap_1^\infty E_n) + \lim_{n\to\infty} [\nu(E_1) - \nu(E_n)].\end{aligned}$$

Once again, each of the sums are well-defined by the properties of the signed measure ν. Since $\nu(E_1)$ is finite, we may subtract it from each side of the equality above to get the desired result. □

2. Let ν be given by

$$\nu(E) = \int_E f\,d\mu,\ E \in \mathcal{M},$$

for a positive measure μ on $(X, \mathcal{M})$ and an extended μ-integrable function f. Determine $\nu^\pm$ and $|\nu|$ in terms of f and μ.

Solution. To determine ν^+, ν^-, and $|\nu|$, we need a Hahn decomposition for ν. Set

$$P := \{x \in X : f(x) \geq 0\} \text{ and } N := \{x \in X : f(x) < 0\}.$$

Then $X = P \cup N$ and $P \cap N = \emptyset$. Furthermore,

$$\nu(P) = \int_P f d\mu \geq 0 \text{ and } \nu(N) = \int_N f d\mu \leq 0,$$

so $X = P \cup N$ is a Hahn decomposition for ν. For any $E \in \mathcal{M}$, we have

$$\nu^+(E) = \nu(E \cap P) = \int_{E\cap P} f d\mu = \int_E f^+ d\mu.$$

Similarly,

$$\nu^-(E) = -\nu(E \cap N) = -\int_{E\cap N} f d\mu = \int_E f^- d\mu.$$

Then

$$\begin{aligned}|\nu|(E) &= \nu^+(E) + \nu^-(E) \\ &= \int_E f^+ d\mu + \int_E f^- d\mu \\ &= \int_E (f^+ + f^-) d\mu \\ &= \int_E |f| d\mu.\end{aligned}$$

□

3. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $\mathcal{N}$ a sub-σ-algebra of $\mathcal{M}$, and $\nu = \mu|_{\mathcal{N}}$. If $f \in L^1(\mu)$, there exists $g \in L^1(\nu)$(thus g is $\mathcal{N}$-measurable) such that $\int_E f d\mu = \int_E g d\nu$ for all $E \in \mathcal{N}$; if g' is another such function then $g = g'$ ν-a.e.. (In probability theory, g is called the conditional expectation of f on $\mathcal{N}$.)

Solution. Define $\lambda : \mathcal{N} \to (-\infty, \infty)$ by $\lambda(E) = \int_E f d\mu$ for each $E \in \mathcal{N}$. Then λ is a signed measure on $\mathcal{N}$. Indeed, $\lambda(\emptyset) = 0$ (since the integral of f over any μ-null set is zero), $|\lambda(E)| < \infty$ for each $E \in \mathcal{N}$, and for any pairwise disjoint collection $\{E_j\} \subset \mathcal{N}$ we find $\lambda(\bigcup_1^\infty E_j) = \sum_1^\infty \lambda(E_j)$ by properties of the integral. Moreover $f \in L^1(\mu)$ so that the sum converges absolutely. Since $\nu = \mu|_{\mathcal{N}}$, we see that $\lambda \ll \nu$. Hence, by the Radon-Nikodym theorem there is an extended ν integrable function g such that $d\lambda = g d\nu$. That is, for each $E \in \mathcal{N}$, $\lambda(E) = \int_E f d\mu = \int_E g d\nu$. Since λ is finite and $X \in \mathcal{N}$, it must be that $\int_E g^\pm d\nu < \infty$. In particular, $g \in L^1(\nu)$. Uniqueness is also guaranteed by the Radon-Nikodym theorem. ☐

4. Let $X = [0,1]$, $\mathcal{M} = \mathcal{B}_{[0,1]}$, let m be the Lebesgue measure, and let μ be the counting measure on $\mathcal{M}$. Show that:

   (a) $m \ll \mu$ but $dm \neq f\, d\mu$ for any f.

   (b) The measure μ has no Lebesgue decomposition with respect to m.

   Solution. (a) If E is μ-null, then $E = \emptyset$ by definition of the counting measure. Hence, $m \ll \mu$. Suppose $dm = f d\mu$ for some $f \in L^1(\mu)$. Fix any $x \in [0,1]$. Then

$$0 = m(\{x\}) = \int_x f\, d\mu = f(x).$$

   Hence, $f \equiv 0$ which is impossible since $m([0,1]) = 1$. It follows that $dm \neq f\, d\mu$ for any $f \in L^1(\mu)$.

   (b) Suppose μ has the Lebesgue decomposition

$$\mu = \lambda + \rho$$

   where $\lambda \perp m$ and $\rho \ll m$. Then

$$\infty = \mu([0,1]) = \lambda([0,1]) + \rho([0,1]) = \rho([0,1]),$$

   where the last equality follows from the fact that $\lambda \perp m$. Since $\rho \ll m$, we can write

$$\rho([0,1]) = \int_{[0,1]} f\, dm$$

   where $f \in L^1(m)$. Hence, the right-hand side is finite which contradicts the equalities derived above.

☐

Reference: *Real Analysis: Modern Techniques and Their Applications*, 2nd ed., Gerald B. Folland.