ALGEBRA REVIEW

From test 3 (6 problems)

22. If b and c are elements in a group G, and if $b^5 = c^3 = e$, where e is the unit element of G, then the inverse of $b^2cb^4c^2$ must be

(A) b^3c^2bc (B) $b^4c^2b^2c$ (C) $c^2b^4cb^2$ (D) $cb^2c^2b^4$ (E) cbc^2b^3

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32. Let R denote the field of real numbers, Q the field of rational numbers, and Z the ring of integers. Which of the following subsets F_i of R, $1 \le i \le 4$, are subfields of R?

$$F_1 = \{a/b: \; a, b \in Z \text{ and } b \text{ is odd}\}$$
$$F_2 = \{a + b\sqrt{2}: \; a, b \in Z\}$$
$$F_3 = \{a + b\sqrt{2}: \; a, b \in Q\}$$
$$F_4 = \{a + b\sqrt[4]{2}: \; a, b \in Q\}$$

(A) No F_i is a subfield of R.
(B) F_3 only
(C) F_2 and F_3 only
(D) F_1, F_2, and F_3 only
(E) F_1, F_2, F_3, and F_4

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48. Let G_n denote the cyclic group of order n. Which of the following direct products is NOT cyclic?

(A) $G_{17} \times G_{11}$
(B) $G_{17} \times G_{11} \times G_5$
(C) $G_{17} \times G_{33}$
(D) $G_{22} \times G_{33}$
(E) $G_{49} \times G_{121}$

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51. An automorphism ϕ of a field F is a one-to-one mapping of F onto itself such that $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in F$. If F is the field of rational numbers, then the number of distinct automorphisms of F is

(A) 0 (B) 1 (C) 2 (D) 4 (E) infinite

55. Let p and q be distinct primes. There is a proper subgroup J of the additive group of integers which contains exactly three elements of the set $\{p, p+q, pq, p^q, q^p\}$. Which three elements are in J?

(A) pq, p^q, q^p
(B) $p+q, pq, p^q$
(C) $p, p+q, pq$
(D) p, p^q, q^p
(E) p, pq, p^q

59. Two subgroups H and K of a group G have orders 12 and 30, respectively. Which of the following could NOT be the order of the subgroup of G generated by H and K?

(A) 30 (B) 60 (C) 120 (D) 360 (E) Countable infinity

From test 2 (6 problems)

17. Let $*$ be the binary operation on the rational numbers given by $a * b = a + b + 2ab$. Which of the following are true?

I. $*$ is commutative.
II. There is a rational number that is a $*$-identity.
III. Every rational number has a $*$-inverse.

(A) I only (B) II only (C) I and II only (D) I and III only (E) I, II, and III

18 A group G in which $(ab)^2 = a^2b^2$ for all a, b in G is necessarily

(A) finite
(B) cyclic
(C) of order two
(D) abelian
(E) none of the above

25. Let x and y be positive integers such that $3x + 7y$ is divisible by 11. Which of the following must also be divisible by 11 ?

(A) $4x + 6y$ (B) $x + y + 5$ (C) $9x + 4y$ (D) $4x - 9y$ (E) $x + y - 1$

49. If the finite group G contains a subgroup of order seven but no element (other than the identity) is its own inverse, then the order of G could be

(A) 27 (B) 28 (C) 35 (D) 37 (E) 42

54 The map $x \to axa^2$ of a group G into itself is a homomorphism if and only if

(A) G is abelian (B) $G = \{e\}$ (C) $a = e$ (D) $a^2 = a$ (E) $a^3 = e$

66. Which of the following subsets are subrings of the ring of real numbers?

I. $\{a + b\sqrt{2} \mid a$ and b are rational$\}$

II. $\left\{\frac{n}{3^m} \mid n\right.$ is an integer and m is a non-negative integer$\left.\right\}$

III $\{a + b\sqrt{5} \mid a$ and b are real numbers and $a^2 + b^2 \le 1\}$

(A) I only (B) I and II only (C) I and III only (D) II and III only (E) I, II, and III

<u>From test 1 (7 problems)</u>

21. Let P_1 be the set of all primes, $\{2, 3, 5, 7, \ldots\}$, and for each integer n, let P_n be the set of all prime multiples of n, $\{2n, 3n, 5n, 7n, \ldots\}$. Which of the following intersections is nonempty?

(A) $P_1 \cap P_{23}$ (B) $P_7 \cap P_{21}$ (C) $P_{12} \cap P_{20}$ (D) $P_{20} \cap P_{24}$ (E) $P_5 \cap P_{25}$

32. Suppose that two binary operations, denoted by $\oplus$ and $\odot$, are defined on a nonempty set S, and that the following conditions are satisfied for all x, y, and z in S:

(1) $x \oplus y$ and $x \odot y$ are in S.

(2) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ and $x \odot (y \odot z) = (x \odot y) \odot z$.

(3) $x \oplus y = y \oplus x$

Also, for each x in S and for each positive integer n, the elements nx and x^n are defined recursively as follows:

$1x = x^1 = x$ and

if kx and x^k have been defined, then $(k+1)x = kx \oplus x$ and $x^{k+1} = x^k \odot x$.

Which of the following must be true?

I. $(x \odot y)^n = x^n \odot y^n$ for all x and y in S and for each positive integer n.

II. $n(x \oplus y) = nx \oplus ny$ for all x and y in S and for each positive integer n.

III. $x^m \odot x^n = x^{m+n}$ for each x in S and for all positive integers m and n.

(A) I only (B) II only (C) III only (D) II and III only (E) I, II, and III

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33. The Euclidean algorithm is used to find the greatest common divisor (gcd) of two positive integers a and b

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input(a)
input(b)
while b > 0
  begin
    r := a mod b
    a := b
    b := r
  end
gcd := a
output(gcd)
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When the algorithm is used to find the greatest common divisor of a = 273 and b = 110, which of the following is the sequence of computed values for r ?

(A) 2, 26, 1, 0

(B) 2, 53, 1, 0

(C) 53, 2, 1, 0

(D) 53, 4, 1, 0

(E) 53, 5, 1, 0

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40. For which of the following rings is it possible for the product of two nonzero elements to be zero?

(A) The ring of complex numbers

(B) The ring of integers modulo 11

(C) The ring of continuous real-valued functions on $[0, 1]$

(D) The ring $\{a + b\sqrt{2} : a \text{ and } b \text{ are rational numbers}\}$

(E) The ring of polynomials in x with real coefficients

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46. Let G be the group of complex numbers $\{1, i, -1, -i\}$ under multiplication. Which of the following statements are true about the homomorphisms of G into itself?

I. $z \mapsto \overline{z}$ defines one such homomorphism, where $\overline{z}$ denotes the complex conjugate of z.

II. $z \mapsto z^2$ defines one such homomorphism.

III. For every such homomorphism, there is an integer k such that the homomorphism has the form $z \mapsto z^k$.

(A) None (B) II only (C) I and II only (D) II and III only (E) I, II, and III

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49. Up to isomorphism, how many additive abelian groups G of order 16 have the property that $x + x + x + x = 0$ for each x in G?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 5

60. The group of symmetries of the regular pentagram shown above is isomorphic to the

(A) symmetric group S_5

(B) alternating group A_5

(C) cyclic group of order 5

(D) cyclic group of order 10

(E) dihedral group of order 10