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### Spiderweb honeycombs

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### ABSTRACT

Small and large deformation in-plane elastic response of a new class of hierarchical fractal-like honeycombs inspired by the topology of the "spiderweb" were investigated through analytical modeling, detailed numerical simulations, and mechanical testing. Small deformation elasticity results show that the isotropic in-plane elastic moduli (Young's modulus and Poisson's ratio) of the structures are controlled by dimension ratios in the hierarchical pattern of spiderweb, and the response can vary from bending to stretching dominated. In large deformations, spiderweb hierarchy postpones the onset of instability compared to stretching dominated triangular honeycomb (which is indeed a special case of the proposed spiderweb honeycomb), and exhibits hardening behavior due to geometrical nonlinearity. Furthermore, simple geometrical arguments were obtained for large deformation Poisson's ratio of first order spiderweb honeycombs, which show good agreement with numerical and experimental results. Spiderweb honeycombs exhibit auxetic behavior depending on the non-dimensional geometrical ratio of spiderweb.

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### 1. Introduction

The spider's web is a highly efficient network of natural fibers where the geometry plays a major role in unique properties such as significant strength, toughness and reversible extensibility. From the structural point of view, the current state of literature on the spiderweb includes evaluation of the elastic properties of spider silk (Blackledge et al., 2005a,b; Cranford et al., 2012; Eisoldt et al., 2011; Koski et al., 2013; Qin and Buehler, 2013; Termonia, 1994; Vollrath, 2010) and the out-of-plane mechanical properties of the structure under various static, dynamic, and impact loadings produced by wind, insects, or other natural sources (Ko and Jovicic, 2004; Ortega-Jimenez and Dudley, 2013; Pugno et al., 2013). In the current paper we incorporate the spiderweb structural organization into hexagonal honeycombs resulting in a centrosymmetrical fractal-like pattern.

Recently, it has been shown that engineered self-similarity can be exploited to control the mechanical properties of cellular structures (Ajdari et al., 2012; Banerjee, 2014; Fan et al., 2008; Haghpanah et al., 2014b; Oftadeh et al., 2014a,b; Sun et al., 2014; Sun and Pugno, 2013; Taylor et al., 2011; Vigliotti and Pasini, 2013; Wegst et al., 2015). Haghpanah et al. (2013) carried out a comprehensive study of hierarchical design which considered

multiple parameter enhancements of high order hierarchical honeycomb lattices and showed that remarkably favorable combinations of specific stiffness and specific strengths can be simultaneously achieved via hierarchical organization. However, unlike previously introduced geometries, the current topology has the advantage of controlling the response through a critical transition between two main structural responses in a cellular solid, namely the stretching and bending dominated behaviors. The transverse (i.e. in-plane) elastic modulus of a regular hexagonal honeycomb is governed mostly by the bending deformation of cell walls and is related to the structure's relative density through the closed-form expression:  $E/E_s = 1.5\rho^3$ , where E and  $E_s$ are respectively the Young's moduli of the structure and cell wall material, and  $\rho$  is the relative density of the structure (Gibson and Ashby, 1997). On the other hand, in an equilateral triangular honeycomb, the elastic deformation is dominated by the axial extension or compression of cell walls, so that the resulting elastic modulus is much higher than the regular hexagonal honeycomb and is given as:  $E/E_s = (1/3)\rho$  (Gibson and Ashby, 1997).

To this end, we introduce spiderweb hierarchy by adding smaller hexagons at the centers of cells in an underlying hexagonal network and connecting the adjacent vertices by straight beams. This procedure can be repeated at smaller scales to produce higher orders of spiderweb structure, yet the thickness of the cell walls is reduced simultaneously to conserve the overall relative density of the structure. The resulting structural organization has an

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isotropic in-plane linear elastic response due to the preservation of six-fold symmetry. Fig. 1 shows the evolution of a regular hexagonal honevcomb cell as the order of hierarchy is increased through the introduction of successive spiderweb topology. The structural organization of the spiderweb honeycomb at each order of hierarchy can be defined by the ratio of the newly added hexagonal edge length (*b* for first order and *c* for second order spiderwebs), to the original hexagonal edge length, a, as illustrated in Fig. 1 (i.e.,  $\gamma_1 = b/a$  and  $\gamma_2 = c/a$ ). For first order spiderweb honeycomb,  $0 \le b \le a$  and thus,  $0 \le \gamma_1 \le 1$ , where  $\gamma_1 = 0$  represents the equilateral triangular grid and  $\gamma_1 = 1$  denotes the regular hexagonal honeycomb structure where each cell wall consists of three separate cell walls with a thickness equal to one third of the overall wall thickness. For a second order spiderweb honeycomb,  $0 \le c \le b$  and thus,  $0 \le \gamma_2 \le \gamma_1$ . The relative density (equal to area fraction) can be given as:

$$\rho = 6/\sqrt{3} \cdot (t/a) \quad n = 0, 1$$
$$\rho = 2/\sqrt{3} \cdot (t/a) \cdot \left(3 + 2\sum_{i=1}^{n-1} \gamma_i\right) \quad n \ge 2$$

where *t* is the thickness of the cell walls and *n* is the order of hierarchy. Using this equation, one can easily obtain the wall thickness for a structure with specified geometry and relative density. For instance, for a second order hierarchical structure with  $\gamma_1 = 1/6$ ,  $\gamma_2 = (1/6)^2$ , and 5% relative density, assuming the original edge length to be equal to unity, Eq. (1) gives:

 $0.05 = 2/\sqrt{3} (t/1) (3 + 2 (1/6))$ , and then the thickness is obtained as t = 0.01299.

The rest of the paper is organized as follows: Analytical models based on energy methods were provided in Section 2 to determine the closed-form expressions of small deformation Young's modulus and Poisson's ratio of first order spiderweb honeycombs. The analytical results were then compared with finite element (FE) simulations. We provided numerical results for small deformation Young's modulus of second and higher orders of spiderweb hierarchy in Section 3. Furthermore, large deformation elastic response of first order spiderweb honeycomb was investigated in Section 4. Conclusions were drawn in Section 5.

### 2. First order spiderweb honeycombs under small deformations

### 2.1. Theoretical investigations

In this section, an analytical approach based on energy methods (Boresi and Schmidt, 2002) is used to determine closed-form expressions for small deformation in-plane elastic moduli (Young's modulus and Poisson's ratio) of the first order spiderweb honeycomb. The cell walls of the structure were assumed to have an isotropic linear elastic behavior with the Young's modulus,  $E_s$ . A six-fold symmetry seen within the structure makes it to exhibit macroscopic isotropy in the in-plane elastic behavior (Christensen, 1987). Therefore, for complete characterization of the in-plane elastic behavior of first order spiderweb honeycomb, we only need to determine two elastic constants. These constants can be obtained by employing any type of in-plane loading. Here without



(1)

Fig. 1. Schematics showing the evolution of the unit cell in a regular hexagonal honeycomb into first and second order spiderweb honeycombs.



**Fig. 2.** (A) First order spiderweb honeycomb under biaxial loading in *x* and *y* directions. A unit cell of the structure is shown by red. The area bounded by dotted lines is the effective area of the unit cell used in strain energy calculations. (B) Free body diagram of the unit cell. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

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loss of generality, we chose biaxial loading in the principal directions of the material, x and y in Fig. 2, to obtain the Young's modulus and Poisson's ratio.

To find the Young's modulus of a first order spiderweb honeycomb described earlier, we first imposed a far field biaxial state of stress ( $\sigma_{xx}$  and  $\sigma_{yy}$  in the x and y directions, respectively) as illustrated in Fig. 2(A). Next, we choose the triangular area shown by dashed lines in Fig. 2(A) as a structural unit cell of the structure. This implies that we can restore the entire structure solely by translating and/or reflecting this triangle. Considering small deformations, we can assume that the deformation is symmetrical due to inherent geometrical symmetries of the structure as well as the symmetry of the applied macroscopic stresses (i.e.  $\sigma_{xx}$  and  $\sigma_{yy}$ ). Therefore, the chosen unit cell can only be utilized for this part of the study and cannot be employed for large and unsymmetrical deformation analysis. A free body diagram (FBD) of the unit cell is shown in detail in Fig. 2(B). We assigned numbers (1 through 5) to the mid-points of the edges in the unit cell cut by the dashed lines,  $L_1$ - $L_3$ . The horizontal edge in the unit cell with length a/2 + (a - b) (see Fig. 2(B)) is an axis of symmetry for the unit cell. Thus, any pair of mirror points with respect to this edge experiences the same internal forces and moments, so same numbers were assigned to them. Consider the edges cut at their mid-points 1 and 3. Due to the 180° rotational symmetry of the structure and the components of macroscopic stresses, no bending moment is transmitted by these edges at their mid-points. For example, if the bar at point 1 bulges 'downward', rotating the structure in the x-y plane by 180° makes point 1 to bulge 'upward' breaking the symmetry of the structure and loading mentioned earlier. Next, considering the mid-points 1 and 2 and their corresponding edges, we can conclude that no vertical forces are transmitted by these edges through their mid-points, because that would again break the requirements of symmetry. For example, if the bar at point 1 transmits a 'downward' vertical force, reflecting the structure with respect to the *x* axis makes the vertical force to point 'upward'. Finally, symmetry of the structure also implies that points 4 and 5 transmit same forces and moments as shown in the figure. This is because point 5 can be mapped onto point 4 through a half plane rotation of the structure and loading around the intersection of lines  $L_1$  and  $L_2$  in Fig. 2(A), followed by a rigid body translation along the line  $L_2$  (in the down right direction by the magnitude  $a\sqrt{3} + b\sqrt{3}/2$ ).

The unknown forces and moments being transmitted through the points 1–5 are summarized in Fig. 2(B). They include four unknown horizontal forces,  $F_{x1}$ – $F_{x4}$ , two unknown vertical forces,  $F_{y3}$  and  $F_{y4}$ , and two unknown moments, M and M', thus representing eight unknown variables, which would be uniquely determined through eight appropriate equations.

Note that based on the FBD of the unit cell (see Fig. 2(B)), two out of three equations of equilibrium in the *x*-*y* plane, i.e.  $\Sigma F_y = 0$  and  $\Sigma M = 0$ , are automatically satisfied. Thus, we only need to take into account the *x* component of equilibrium equation; i.e.  $\Sigma F_x = 0$ . This gives us,  $F_{x1} = -(2F_{x2} + 2F_{x3} + 4F_{x4})$ . Then, neglecting the contribution of shearing energy, the strain energy stored in the unit cell can be written as a function of unknown forces and moments as,  $U = U(F_{x2}, F_{x3}, F_{x4}, F_{y3}, F_{y4}, M, M')$ , in which the equations of equilibrium are already satisfied. Next, considering the cut line  $L_1$ , the average force per unit length transmitted through this vertical line is  $\sigma_{xx}$ , or in other words we can write the following relation between  $\sigma_{xx}$  and the forces acting on  $L_1$ :

$$\sigma_{xx} = \frac{F_{x1} + 2F_{x2}}{a\sqrt{3}} \tag{2}$$

Similarly,  $\sigma_{vv}$  is related to  $F_{v3}$  and  $F_{v4}$  through the following relation:

$$\sigma_{yy} = \frac{-(F_{y3} + 2F_{y4})}{3a/2} \tag{3}$$

Line  $L_1$  is an axis of symmetry for the structure. So, the unit cell's three horizontal lines which are cut in half by  $L_1$  through their mid-points 1 and 2 must deform in a fashion in which their right and left halves are mirror images with respect to line  $L_1$ . This implies that by fixing the coordinate system at the center of the unit cell, point 2 must have zero displacement in the *x* direction and zero rotation with respect to the *z* axis. These constraints can be expressed mathematically using Castigliano's theorem (Boresi and Schmidt, 2002) as:

$$\frac{\partial U}{\partial F_{x2}} = 0 \quad \text{and} \quad \frac{\partial U}{\partial M'} = 0$$
(4)

Again, symmetries seen within the structure impose identical rotations of points 4 and 5. Since the direction of moment acting at point 4 is opposite to the direction of moment acting on point 5, the total amount of rotation of these two points in the direction of their moments must be equal to zero. Using Castigliano's theorem, this can be written as:

$$\frac{\partial U}{\partial M} = 0$$
 (5)

To be able to reconstruct the structure using deformed unit cells, points 3, 4, and 5, which are initially collinear must remain so during deformation. It can be shown that this constraint will be satisfied if the vector relation  $\vec{u_3} = (\vec{u_4} + \vec{u_5})/2$  is satisfied, where  $\vec{u_i}$  is the displacement vector of point *i* and i = 3, 4, 5. This equation actually includes two separate equations, i.e. one in the *x* and the other in the *y* directions as  $u_{x3} = (u_{x4} + u_{x5})/2$ , and  $u_{y3} = (u_{y4} + u_{y5})/2$ , where  $u_{xi}$  and  $u_{yi}$  are respectively the displacements of point *i* (*i* = 3, 4, 5) in the *x* and *y* directions. Using Castigliano's theorem, these relations can be expressed as:

$$\frac{\partial U}{\partial F_{x4}} = 2 \frac{\partial U}{\partial F_{x3}}$$
$$\frac{\partial U}{\partial F_{y4}} = 2 \frac{\partial U}{\partial F_{y3}}$$
(6)

Eqs. (2)-(6), make a system of seven equations with seven unknowns. We employed Matlab<sup>®</sup> (Mathworks Inc., Natick, MA) to solve this system of equations using symbolic variables.

Note that under a uniaxial state of stress (i.e.  $\sigma_{xx} \neq 0$  and  $\sigma_{yy} = 0$ ), the Young's modulus of the structure is defined as  $E = \sigma_{xx}^2/(2U_0)$ , where  $U_0$  is the strain energy density stored in the unit cell of the structure and is given as  $U_0 = U/(3\sqrt{3}a^2/4)$ . Then, closed-form relation for the Young's modulus (to be normalized by the Young's modulus of cell wall material) is obtained as follows:

$$\frac{E}{E_{\rm s}} = 4\sqrt{3}\delta^3 \frac{f_1(\gamma) + f_2(\gamma)\delta^2 + f_3(\gamma)\delta^4}{g_1(\gamma) + g_2(\gamma)\delta^2 + g_3(\gamma)\delta^4 + g_4(\gamma)\delta^6}$$
(7)

where  $\delta = t/a$ ,  $\gamma = \gamma_1$ , and the functions appearing in the equation are listed below:

$$\begin{split} f_1(\gamma) &= 6 - 16\gamma + 12\gamma^2 + 6\gamma^3 - 22\gamma^4 + 24\gamma^5 - 12\gamma^6 + 2\gamma^7 \\ f_2(\gamma) &= 12 - 23\gamma + 24\gamma^2 - 13\gamma^4 + 4\gamma^5 \\ f_3(\gamma) &= 6 - 7\gamma - \gamma^2 + 2\gamma^3 \\ g_1(\gamma) &= 18\gamma^3 - 60\gamma^4 + 72\gamma^5 - 36\gamma^6 + 6\gamma^7 \\ g_2(\gamma) &= 36 - 111\gamma + 88\gamma^2 + 78\gamma^3 - 169\gamma^4 + 132\gamma^5 - 60\gamma^6 + 10\gamma^7 \end{split}$$

$$\begin{split} g_{3}(\gamma) &= 48 - 100\gamma + 97\gamma^{2} + 6\gamma^{3} - 59\gamma^{4} + 20\gamma^{5} \\ g_{4}(\gamma) &= 12 - 5\gamma - 17\gamma^{2} + 10\gamma^{3} \end{split} \tag{8}$$

For the special case where  $\gamma = 0$  (and  $\delta \ll 1$ ), Eq. (7) reduces to  $E/E_s = (2/\sqrt{3})\delta$ , which is equivalent to the relative Young's modulus of an equilateral triangular honeycomb reported in the literature (Gibson and Ashby, 1997). Interestingly, for the other special case where  $\gamma = 1$  (and  $\delta \ll 1$ ), Eq. (7) reduces to  $E/E_s = 4\sqrt{3}\delta^3$ , which is three times the relative Young's modulus of a regular hexagonal honeycomb with thickness, *t* and edge length, *a*. This is because when  $\gamma = 1$ , the structure geometrically transforms to a regular hexagonal honeycomb with each cell wall consisting of three separate cell walls with thickness, *t*, and three-fold bending rigidity.

As mentioned earlier, to completely identify the linear elastic behavior of first order spiderweb honeycomb, we need to determine the Poisson's ratio of the structure, v. We again use energy methods and consider the same structure as in Fig. 2 under biaxial state of stress. It should be emphasized that we ignored the contribution of beam shear deformation in strain energy density of the structure; however the bending and stretching terms are fully considered.

Under equi-biaxial state of stress (i.e.  $\sigma_{xx} = \sigma_{yy} = \sigma$ ), the relation  $v = 1 - U_0 E/\sigma^2$  can be used to obtain the Poisson's ratio of the structure (Boresi and Schmidt, 2002), where again  $U_0$  is the strain energy density of the unit cell and *E* is the Young's modulus of the structure that was determined earlier (Eq. (7)). Therefore, closed-form relation for the Poisson's ratio of a first order spiderweb honeycomb can be derived as:

$$v = 1 - 4\delta^2 \frac{f_1(\gamma) + f_2(\gamma)\delta^2 + f_3(\gamma)\delta^4}{g_1(\gamma) + g_2(\gamma)\delta^2 + g_3(\gamma)\delta^4 + g_4(\gamma)\delta^6}$$
(9)

We now rewrite the above equation as a function of the Young's modulus of the structure as (see Eq. (7)):

$$v = 1 - \frac{E/E_s}{\delta\sqrt{3}} \tag{10}$$

Note that for two special cases where  $\gamma = 0$  and  $\gamma = 1$  (in both cases take  $\delta \ll 1$ ), Eq. (9) reduces to  $\nu = 1/3$  and  $\nu = 1$  which are the Poisson's ratio of equilateral triangular and hexagonal honeycombs, respectively.

### 2.2. Numerical investigations

In this section, the finite element (FE) method was used to verify the analytical formulations of elastic response of first order spiderweb honeycombs derived in the previous subsection of this paper. Commercially available FE software ABAQUS 6.11-2 (SIMULIA, Providence, RI) was used to carry out all the simulations in this study. 3D models of first order spiderweb honeycomb structure were meshed using 4-node shell elements (S4R). A mesh sensitivity analysis was also performed to ensure that the results are not dependent on the mesh size. Cell walls were assumed to have a rectangular cross section with unit length normal to the plane of loading (i.e. normal to the *x*-*y* plane (see Fig. 2)) and the thickness was adjusted to be consistent with the value of the relative density. Linear elastic properties of aluminum were assumed for the cell wall material with  $E_s = 70$  GPa, and  $v_s = 0.3$ .

Fig. 3(A) shows the schematic diagram of the FE model constructed in ABAQUS for simulating static uniaxial loading on first order spiderweb honeycomb. Vertical displacement of the top and bottom nodes of the structure was coupled to the corresponding rigid flat plates. A constant downward static displacement was then assigned to the top plate, while the bottom plate was fixed. To eliminate any boundary effects, periodic boundary conditions were imposed to the structure on the right and left side nodes (Harders et al., 2005). Also note that the horizontal displacement of an arbitrary node in the structure was constrained (i.e. set to zero) in order to prevent rigid body motion of the structure in that direction. The out-of-plane degrees of freedom of the model were also constrained to avoid the out-of-plane buckling of the structure.

Fig. 3(B) and (C) respectively show the normalized Young's modulus,  $\overline{E}$  and Poisson's ratio, v of first order spiderweb honeycombs for all possible values of  $\gamma_1$ . The Young's modulus is normalized by the Young's modulus of a regular hexagonal honeycomb with same relative density,  $E = 1.5E_s\rho^3$ . The results are presented for three different values of relative densities. 1%, 5%, and 10%. In the figures, solid lines represent the results obtained directly by using the theoretical closed-form expressions derived in the previous subsection of this paper, and markers denote the FE results. Excellent agreement between the analytical and numerical approaches was observed even though the contribution of shearing energy was neglected in the analytical method presented in the previous subsection. For the values of  $\gamma_1$  greater than  $\gamma_1 \simeq 0.25$ , the structure exhibits a bending dominated behavior (i.e.  $E/E_{\rm s} \propto \rho^3$ ), with the normalized Young's modulus independent of the relative density  $\overline{E} \propto \rho^0$ . In contrast, for the values of  $\gamma_1$  smaller than  $\gamma_1 \simeq 0.25$ , the honeycomb transforms into a stretching dominated structure with  $E/E_s \propto \rho$  or equally  $\overline{E} \propto \rho^{-2}$ . At  $\gamma_1 = 0$ , the normalized Young's modulus is obtained as 2222, 89, and 22 for the relative densities of 1%, 5%, and 10%, respectively. As the value of  $\gamma_1$  increases, since the structure transforms from stretching to bending dominated one, its normalized Young's modulus decreases. At  $\gamma_1 \cong 0.35$  the stiffness of the structure is about the stiffness of a regular hexagonal honeycomb with same relative density. After this point, the structure becomes more compliant compared to a regular honeycomb. At the  $\gamma_1 = 1$  limit, the structure transforms into a regular hexagonal honeycomb with each cell wall consisting of three separate cell walls. At this point the normalized Young's modulus can be obtained using Eq. (7) as,  $4\sqrt{3}(\rho\sqrt{3}/6)^3/(1.5\rho^3) = 1/9$ . Fig. 3(C) shows the Poisson's ratio of first order spiderweb honevcomb varving from 1/3 (equilateral triangular lattice) to 1 (regular hexagonal honeycomb). With decrease in the relative density, the Poisson's ratio for a constant value of  $\gamma_1$  ( $\gamma_1 \neq 0$ ) approaches unity.

# 3. Higher order spiderweb honeycombs under small deformations – Young's modulus

FE analysis was used to evaluate the small deformation elastic response in higher order spiderweb honeycombs. Finite size, 3D models of the structure were constructed in ABAQUS, and were subjected to uniaxial static compression along y direction. Material properties, FE models, boundary conditions, and loadings are similar to those explained in Section 2. The overall relative density of the structures was fixed at 5%.

Fig. 4 shows the FE results on the Young's modulus of second order spiderweb honeycombs normalized by that of a regular hexagonal honeycomb of equal relative density ( $E = 1.5E_s\rho^3$ ), versus  $\gamma_2$ . The results are plotted for four different values of  $\gamma_1 = 1/3$ , 1/2, 2/3, 5/6. Geometrically, the structural parameter  $\gamma_2$  is bound on the upper limit by the value of  $\gamma_1$ , i.e.  $\gamma_2 \leq \gamma_1$ . Similar to the first order spiderweb honeycombs, lower values of  $\gamma_1$  result in higher Young's modulus at a constant value of  $\gamma_2$ . At a constant value of  $\gamma_1$ , increasing the value of  $\gamma_2$  decreases the Young's modulus of the structure is increased and less portion of the strain energy is stored through the axial stretching of the beams.



**Fig. 3.** (A) Schematic of the finite element model of first order spiderweb honeycomb under in-plane compressive loading. (B) Normalized Young's modulus, and (C) Poisson's ratio, versus  $\gamma_1$  for honeycombs with three different relative densities (solid lines show the analytical estimates, markers show the finite element results).

To investigate the Young's modulus of higher order spiderweb honeycombs, we introduced a scalar geometrical ratio,  $\eta$ , defined as  $\gamma_i = \eta^i$  (e.g.  $\gamma_1 = 1/6$  and  $\gamma_2 = 1/36$  for a second order spiderweb honeycomb with  $\eta = 1/6$ ). This relation in fact describes a subclass of fractal-like spiderweb honeycomb with constant ratios between successive hexagonal sides. The normalized Young's modulus of higher order spiderweb honeycombs (up to fifth order) with different values of  $\eta$  is plotted in Fig. 5. The Young's modulus of the structures is normalized by the Young's modulus of a regular hexagonal honeycomb of equal relative density ( $E = 1.5E_s\rho^3$ ). For honeycombs with  $\eta < \sim 0.8$ , increasing the hierarchical order increases the Young's modulus of the structure. However, for  $\eta > \sim 0.8$  a negative correlation is found between the hierarchical order and Young's modulus. In fact, the mechanical response in spiderweb honeycombs is governed by the size of the smallest hexagonal feature (i.e.  $\eta^n$  for *n*th order of hierarchy). Therefore, based on the results obtained for first order hierarchy (i.e.  $\gamma_1 = 0.25$  as the boundary between stretching and bending dominated behaviors), as well as the results shown in Fig. 5 for higher order structures, we can define an empirical equation to estimate

the boundary between stretching and bending dominated behaviors of self-similar spiderweb honeycombs as  $\eta^n = 0.25$ . This empirical condition states that the transition from stretching to bending dominated behaviors occurs at increasing  $\eta$  value as the order of hierarchy increases. For instance,  $\eta = 0.25$  and  $\eta \sim 0.76$ , respectively for first and fifth orders of hierarchy. In general, in a fully stretching dominated regime (lower values of  $\eta$ ), a smaller hexagon will result in a more stretching dominated structure and increased Young's modulus. However, in the fully bending dominated regime (higher values of  $\eta$ ), the increase in Young's modulus due to higher stretching energy is somewhat offset by the fact that an addition to the order of hierarchy will only reduce the effective bending rigidity of the cell walls (beams) due to conservation of mass, resulting in a decreased Young's modulus.

### 4. First order spiderweb honeycombs under large deformations

In this section, we investigated the large deformation elastic response of first order spiderweb honeycombs under quasi-static compressive loading. Material properties, FE models, boundary



**Fig. 4.** Normalized Young's modulus versus  $\gamma_2$  for second order spiderweb honeycombs with different values of  $\gamma_1$ . The relative density of honeycombs was kept constant as 5%.



**Fig. 5.** Normalized Young's modulus versus  $\eta$  for higher order spiderweb honeycombs (up to fifth order). The relative density of honeycombs was kept constant as 5%.

conditions, and loadings are similar to those discussed in the previous sections, and similar to the previous section the overall relative density of the structure was kept constant at 5%. Similar to previous sections, a mesh sensitivity analysis was also performed to ensure that the results are not dependent on the mesh size. The deformation of first order spiderweb honeycomb with different values of  $\gamma_1$  was simulated up to at least 40% crushing strain.

It is well known (Gibson et al., 2010) that typical cellular solids have a distinctive sequence of practically linear elastic behavior, weakly stable or unstable collapse due to instability or plasticity, followed by a significant restiffening caused by densification. The

latter regime occurs at 70-80% strain for low density cellular structures. The compressive stress-strain response of spiderweb honeycomb obtained for different values of  $\gamma_1$  is plotted in Fig. 6(A). For  $\gamma_1 = 0$  (equilateral triangular honeycomb), the "stress plateau" regime begins at the very early stages of crushing (strain ~0.1%) as the result of elastic buckling (i.e. instability) of the cell walls. This very low buckling strain is due to the highly stretching-dominated behavior of the structure (Haghpanah et al., 2014a). As the value of  $\gamma_1$  increases, although the small deformation Young's modulus of the structure decreases dramatically, instability occurs at higher strains. For example, for  $\gamma_1 = 1/4$  the instability occurs at 8% crushing strain. This effect would cause the structures with  $\gamma_1$  equal or greater than 1/3 to not experience instability until 40% crushing strain. In fact, the large static deformation along with large lateral load components in the cell walls would entirely suppress instability in these structures (Haghpanah et al., 2014a). At  $\gamma_1 = 1/3$ , although the small deformation stiffness is much lower than that of the triangular honeycomb, the structure is much stronger in crushing strains greater than 7.5%

The load-displacement response of the spiderweb honeycombs promises potentially enhanced values of toughness and energy absorption at certain geometries, which we will investigate next. Fig. 6(B) shows plots of the strain energy density versus crushing strain. Note that the strain energy density shown in this figure is equivalent to the area of the region bounded by the graph of stress and x axis in Fig. 6(A). Smaller values of  $\gamma_1$  correspond to the higher elastic energy storage in the structure at small deformation range ( $\varepsilon$  < 2.5%). For  $\gamma_1 \leq 1/3$ , this behavior is reversed as deformation proceeds to larger strains. For example at 40% crushing strain, the elastic energy storage capacity of spiderweb honeycombs with  $\gamma_1 = 1/4$  and  $\gamma_1 = 1/3$  is equally about 40% greater than that of a triangular honeycomb ( $\gamma_1 = 0$ ). The noticeable difference between energy storage performance for structures  $\gamma_1 = 1/3$  and  $\gamma_1 = 1/4$  is due to cell wall buckling in the latter structure starting at  $\varepsilon$  = 8%. At 40% strain, the spiderweb structure with  $\gamma_1 = 1/3$  has not experienced instability, yet it has the greatest strain energy density among all the values of  $\gamma_1$  studied (equal to  $\gamma_1 = 1/4$ ).

As previously observed, occurrence of instability could significantly influence the deformation mechanisms and large deformation elastic response of first order spiderweb honeycombs. For further studying this effect, the effect of large deformation elasticity on Poisson's ratio is studied next. In Fig. 7(A), Poisson's ratio is plotted against the crushing strain. The solid lines denote the FE results and the dashed lines represent the Poisson's ratio estimations at 100% crushing strain obtained by a geometrical estimation which will be discussed shortly in this section. The markers show the experimental data for  $\gamma_1 = 0$ ,  $\gamma_1 = 1/5$ , and  $\gamma_1 = 1/2$ , which are in good agreement with numerical results. For the experimental investigations, the specimens were fabricated using PolyJet 3D printing (Objet24 3D printer, Stratasys Inc., Eden Prairie, MN) with VeroWhitePlus© (see Appendix A for material's stress-strain response). The specimen with  $\gamma_1 = 0$  has an overall size of 245 mm \* 243 mm with wall thickness of 0.5 mm and wall length of 35 mm. The specimens with  $\gamma_1 = 1/5$  and 1/2 have an overall size of 225 mm \* 243 mm with wall thickness of 0.4 mm and wall length of 28 mm. All specimens maintain a relative density of 5%. The specimens were then tested under uniaxial compression using an Instron 5582 testing machine at the rate of 5 mm/min (i.e. strain rate of 2%/min). Images of deformed configurations were taken to obtain the values of Poisson's ratio in the section of the structure far from the boundaries. For each specimen the measurement were repeated 4 times. The figure shows that the value of Poisson's ratio for each structure at very small strains is equal to the value predicted by theoretical analysis presented in Section 2.1. As the



**Fig. 6.** Nonlinear elastic response of first order spiderweb honeycombs under large deformations. (A) Normalized stress (with respect to the Young's modulus of cell wall material), and (B) strain energy density of the honeycombs versus crushing strain plotted for first order spiderweb honeycombs with different values of  $\gamma_1$ . The relative density of honeycombs was kept constant as 5%.



**Fig. 7.** (A) Poisson's ratio versus crushing strain for first order spiderweb honeycombs under large deformations. The solid lines show the finite element results and dashed lines denote the geometrical predictions at 100% crushing strain for honeycombs with different values of  $\gamma_1$ . The markers show the experimental results for  $\gamma_1 = 0$ ,  $\gamma_1 = 1/5$ , and  $\gamma_1 = 1/2$ . (B) Undeformed ( $\varepsilon = 0$ ) and deformed configurations of specimens at two different crushing strains,  $\varepsilon = 0.2$  and  $\varepsilon = 0.4$ . The specimen with  $\gamma_1 = 0$  has an overall size of 245 mm \* 243 mm with wall thickness of 0.5 mm and wall length of 35 mm. The specimens with  $\gamma_1 = 1/5$  and 1/2 have an overall size of 225 mm \* 243 mm with wall thickness of 0.4 mm and wall length of 28 mm. All specimens maintain a relative density of 5%.

crushing proceeds, Poisson's ratio decreases. The rate of reduction is higher for smaller values of  $\gamma_1$ . Fig. 7(B) shows the undeformed and deformed configurations of the experimental samples at two different stages,  $\varepsilon = 0.2$  and  $\varepsilon = 0.4$ , where two different behaviors are observed which will be discussed shortly. For values of  $\gamma_1$ greater than 1/3 the Poisson's ratio remains positive. On the other hand, an auxetic behavior (i.e. negative Poisson's ratio) is seen for the structures with  $\gamma_1$  less than 1/3.  $\gamma_1 = 1/4$  seems to be a threshold value of  $\gamma_1$  in which the deformation mechanism of the structure completely changes. To understand the difference between these two behaviors, we studied deformation mechanism of the structures at 40% strain. Schematics of undeformed configuration for the unit cell of spiderweb structures with eight different values of  $\gamma_1$  are shown in Fig. 8. FE results on deformed configurations at 40% strain are also depicted in this figure. Two different types of deformation mechanism were seen within the structures. In mechanism #1, the structure deformation is mostly governed by static deflection in the cell walls. No instability (i.e. elastic buckling) is observed in the structures deforming based on this deformation mechanism, which is dominant in the structures with  $\gamma_1 > 1/3$ . All nodal rotations are zero (or very close to zero approaching  $\gamma_1 = 1/3$ ) due to the reflection symmetry of the structure and

loading. The second mechanism takes place in the structures with  $\gamma_1 < 1/3$  (i.e. stretching dominated) and is mostly governed by elastic buckling of cell walls and rotation of the smaller hexagons which remain almost intact. The limit structure is the case of  $\gamma_1 = 0$ , where the periodic deformation is characterized by the equal rotation of all nodes in a row, while adjacent rows have opposite rotations.

To estimate the Poisson's ratio at 100% crushing strain for the structures whose deformation is governed by deformation mechanism # 1, we considered an undeformed unit cell of the structure with the geometrical ratio  $\gamma_1 = b/a$  as shown in Fig. 8. The simplified geometry of deformed unit cell at 100% strain is also shown in the bottom of Fig. 8. Using this, the transverse engineering strain ( $\varepsilon_x$ ) is obtained as ((3a + b) - (3a))/((3a)) = b/((3a)), while the axial engineering strain  $(\varepsilon_y)$  is given as  $(0 - a\sqrt{3})/(a\sqrt{3}) = -1$ . Thus, the Poisson's ratio can be estimated as  $-(b/(3a))/(-1) = b/((3a)) = \gamma_1/3$ . Regarding the deformation mechanism # 2, an undeformed unit cell of the structure with the geometrical ratio  $\gamma_1 = b/a$  is shown in Fig. 8. Simplified deformed configuration is also shown in this figure. Based on FE observations at 100% crushing strain, the midpoints in beams oriented initially at 60° (or 120°, based on rotation direction) become



**Fig. 8.** Nonlinear elastic response of first order spiderweb honeycombs under large deformations. Two different types of deformation mechanism are seen in the honeycombs under uniaxial compression. For each  $\gamma_1$ , the schematic of undeformed unit cell and finite element result of deformed unit cell at 40% crushing strain are shown (two top rows). For each deformed mechanism, the schematic of undeformed unit cell and predicted configuration of deformed unit cell at 100% crushing strain are also shown (two bottom rows).

in contact with midpoints of initially horizontal beams. Therefore, we considered that the smaller hexagon rotates 60° in the plane of loading at 100% crushing strain. It was also assumed that the deformation (u) in edges that are originally horizontal in the undeformed configuration will be a cubic function of the position (s) along the beam (i.e.  $\partial^4 u / \partial s^4 = 0$ ), since there is no distributed load acting on the edges. Using these assumptions it can be shown that a horizontal edge with length L will bend such that its final length in horizontal direction will be 0.8L (see Appendix B). So the transverse engineering strain ( $\varepsilon_x$ ) is obtained as (( $0.8(a - b) \times 2$ )  $+0.8(a/2) \times 2 + b) - 3a)/(3a) = -0.2(1 + \gamma_1)$ . The axial engineering strain  $(\varepsilon_v)$  is again evaluated as -1. Thus the Poisson's ratio is estimated as  $-(-0.2(1 + \gamma_1))/(-1) = -0.2(1 + \gamma_1)$ . Note that the value of  $\gamma_1$  in this deformed configuration is smaller than 1/3, so we can estimate the Poisson's ratio to be -0.2 for all the structures following this deformation mechanism. A very good agreement is observed between the FE results and the values estimated by geometrical predictions.

### 5. Conclusions

The effect of spiderweb hierarchical organization on the in-plane elastic response of honeycombs in small and large deformation regimes was studied. Analytical closed-form formulas for the Young's modulus and Poisson's ratio for the first order spiderweb honeycomb were obtained and verified numerically. It was shown that a relatively broad range of linear elastic response, varying from bending to stretching dominated, can be achieved by tailoring the structural organization of spiderweb honeycombs. While the geometrical parameters influence the linear elastic moduli in small deformations, they also significantly influence the mechanisms of deformation in large deformation regime. In structures with  $\gamma_1 > 1/3$ , large deformation is symmetrical and is formed by the static deflection in the cell walls. When  $\gamma_1 < 1/3$  (i.e. stretching dominated structures), deformation is nonlinear, asymmetric and is accompanied by elastic buckling of cell walls and rotation of

the nodes. The latter mechanism is not unique for a given macroscopic state of stress and is influenced by boundary conditions (Haghpanah et al., 2014a). Furthermore, a geometrical estimation for the large deformation Poisson's ratio of spiderweb honeycombs at 100% crushing strain was presented. Large deformation auxetic behavior was observed in first order spiderweb honeycombs with  $\gamma_1$  less than 1/3.

A unique feature in the spiderweb honeycomb is a combination of high stiffness and toughness. Toughness of the spiderweb honevcomb - a measure of structure's ability to absorb energy under quasi-static loading - is greater than that of a stretching dominated structure (e.g. triangular lattice). In a stretching dominated cellular solid under crushing, the capacity of the structure to absorb energy is limited by the early onset of buckling occurring at low crushing strains. A bending dominated structure, on the other hand, has a relatively low relative stiffness which makes it unsuitable for many in-plane applications. Spiderweb design can therefore provide required stiffness and toughness, as the geometrical parameters can be tuned to create a sweet spot between bending and stretching dominant responses. The elastic energy storage capacity of the spiderweb honeycombs with  $\gamma_1 = 1/4$  and  $\gamma_1 = 1/3$  was shown to be about 40% greater than triangular honeycomb ( $\gamma_1 = 0$ ) at 40% crushing strain.

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### **Appendix A. Material properties**

For the experiments we used VeroWhitePlus<sup>©</sup> material with the stress–strain response shown in Fig. A1.



Fig. A1. (Left) 3D printed dog-bone specimen for uniaxial tension test. (Right) Engineering stress vs. engineering strain for the tested material.

Note that five dog-bone samples (Fig. A1 (left)) were tested under uniaxial tensile loading to obtain the stress–strain response of the material (engineering stress vs. engineering strain). The tension tests were based on ASTM-D638-10 standard, which is the standard test method for tensile properties of plastics.

### **Appendix B. Geometrical estimation**

As shown in Fig. B1, consider a horizontal line with length L under a loading in which it bends such that it forms a cubic



Fig. B1. Undeformed and deformed line with original length L.

function (i.e.  $y = ax^3 + bx^2 + cx + d$  where *x* here denotes position along beam direction) with the slope of  $-60^\circ$  at both ends. Also assume that the length of the curved line remains *L* and the horizontal distance between two ends is *L'*. The following geometrical boundary conditions can be written based on the assumptions:

$$y(0) = 0, \quad y''(0) = 0, \quad y(L'/2) = 0, \quad y'(L'/2) = -\sqrt{3}$$
 (B.1)

where *y*' and *y*" are respectively the first and second derivatives of *y* with respect to *x*. Solving Eq. (B.1), the unknown constants appearing in *y* are obtained as, b = d = 0,  $a = -2\sqrt{3}/L^2$ , and  $c = \sqrt{3}/2$ . Using the assumption that the length of the curved line remains *L* we can finally obtain the unknown length *L*' by using the following relation:

$$2\int_{0}^{\frac{t'}{2}}\sqrt{1+y'^{2}}dx = L$$
 (B.2)

Solving this equation will result in L' = 0.8L.

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