Lagrangian Mechanics: Equivalence to Newton's Laws

We now prove that Lagrangian Mechanics is equivalent to Newton's Second law. We find the equations of motion for particle β, using equation (2) with the $\mathcal{L}$ defined in (1).

Proof: Taking the derivative of the LHS of (1) with respect to $\mathbf{r}_i^\beta$, we get that

$$\frac{\partial \mathcal{L}}{\partial \mathbf{r}_i^\beta} = -\frac{\partial V}{\partial \mathbf{r}_i^\beta}. \tag{a}$$

Now, the RHS of (2) becomes

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_i^\beta} = \frac{d}{dt}\left(\frac{1}{2}m_\beta \sum_{i=1}^{3} \frac{\partial}{\partial \dot{\mathbf{r}}_i^\beta}(\dot{\mathbf{r}}_i^\beta)^2\right) = \frac{d}{dt} m_\beta \dot{\mathbf{r}}_i^\beta = \frac{d\mathbf{p}_i^\beta}{dt} \tag{b}$$

where we have used the facts that $|\mathbf{r}|^2 = \mathbf{r}\cdot\mathbf{r} = \sum_i \mathbf{r}_i^2$, and that the sum collapses to $\alpha = \beta$.

Following the Euler-Lagrange equations we set (a) equal to (b). Then we obtain, for each component i, that

$$\frac{d\mathbf{p}_i^\beta}{dt} = -\frac{\partial V}{\partial \mathbf{r}_i^\beta}$$

for each particle β. Writing this more compactly in vector notation, we get

$$\boxed{\frac{d\mathbf{p}^\beta}{dt} = -\nabla^\beta V}$$

which is Newton's second law for particle β.