Function Spaces (2): What Does It Mean?

Recall from Function Spaces (1) that we can write a Fourier series in vector notation as

$$|f\rangle = \sum_{n=0}^{\infty} a_n |c_n\rangle + b_n |s_n\rangle . \tag{1}$$

Question: That's cute @InertialObservr, but what happened to all the x's?

Answer: This point is initially side-stepped by many textbooks, but let's have a wack at it.

Indeed, there are no longer any x's in equation (1). It is no longer a relation between functions but rather is *purely* a vector relation.

Where was I sneaky?

The 'sneakiness' was in [1] when I said $f(x) \to |f\rangle$. Note that this is **not** an equals sign. **The full relation between functions and vectors is**[2]:

$$\boxed{\langle x|f\rangle = f(x)} \tag{2}$$

Question: Okay, but then what exactly is $|f\rangle$, and for that matter what is $|x\rangle$?

Answer:

(a) We think of $|f\rangle$ as an abstract object living in a vector space that has the properties of functions that are **basis independent** [3]

(b) We haven't really seen objects like $|x\rangle$ before. The set of all $\{|x\rangle\}$ are called the **position space basis vectors**—the dimension of this space is uncountably infinite.

If I'm being honest, while going into the abstract details is *neat*, it's not necessary for the physicist. These are tools for us to use, and we will learn how to use them correctly.

Question: Wait, if $f(x) = \langle x|f\rangle$ then how is it the case that in [1] you said that

$$\langle g|f\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)f(x)\ \mathrm{d}x \tag{3}$$

Answer: Ugh, yes okay you're right. While it **is true** I need to tell you why. If you go to the next section on Completeness ,what I'm about to say will make more sense.

The $\{|x\rangle\}$ form a **complete basis** for our function space. Which means that they satisfy the following relation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |x\rangle\langle x|\ \mathrm{d}x = 1 \tag{4}$$

Comments & Exercises

[1] Function Spaces (1)

[2] Furthermore, we define $\langle f|x\rangle$ as the complex conjugate: $\langle f|x\rangle = f^*(x)$

[3] The easiest way to think about this is actually through Quantum Mechanics. In QM, $|f\rangle$ is the **state** of a particle. That is, all the information about the particle is contained in $|f\rangle$.

There are many ways of describing the state such as Where is it? What's it's momentum? etc. These questions all have answers which are 'zipped' up into $|f\rangle$, and we can extract those answers by projecting onto a certain basis. That is, if we wish to answer 'Where is it?', we take $\langle x|f\rangle$; If we wish to answer 'What's its momentum?' we take $\langle p|f\rangle$.

Exercise:
Starting with the LHS of equation (3), show that by insertion of unity of the form (4) returns the the RHS of (3). (To see this you will need to recall equation (2) and look at [2])