Exact Differential Equations

Up to this point we have learned to solve differential equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y = f(x) \tag{1}$$

which we called a **first order linear** differential equation (DE). But what if nonlinear powers of y occur? As it stands, we are out of machinery to solve such a problem.

Never worry though! In this section we will learn how to solve DEs of the form

$$N(y,x)\frac{\mathrm{d}y}{\mathrm{d}x} + M(y,x) = 0 \tag{2}$$

As always, we state our goal loud and clear:

Goal

Solve the differential equation

$$N(y,x)\frac{\mathrm{d}y}{\mathrm{d}x} + M(y,x) = 0$$

Exact Differential Equations (2): The Setup

Our line of attack involves by first recalling the **multivariable chain rule**. For a multivariable function $\Phi(y, x)$ the **total derivative** with respect to x is given by:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} = \frac{\partial \Phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial \Phi}{\partial x}\underbrace{\frac{\mathrm{d}x}{\mathrm{d}x}}_{1}$$

$$\boxed{\frac{\mathrm{d}\Phi}{\mathrm{d}x} = \frac{\partial \Phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial \Phi}{\partial x}} \tag{3}$$

Note how this final equation resembles equation (2)–the one we with to solve. Now hear comes the kicker. **Suppose** that we can find a function $\Phi(y, x)$ such that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} = N(y, x)\frac{\mathrm{d}y}{\mathrm{d}x} + M(y, x) = 0. \tag{4}$$

What would follow from that and why would it be useful? It might not be clear to you at this moment, but it's indeed the case that if we **can** find a function Φ that satisfies equation (4) then we will have solved our differential equation!

Exact Differential Equations (3): The Setup (cont.)

But why is this? Why does this solve our problem? Well, if we integrate equation (4) with respect to x then we have that [2]

$$\int \frac{\mathrm{d}\Phi}{\mathrm{d}x}\mathrm{d}x = \int 0 \ \mathrm{d}x$$

$$\implies \boxed{\Phi(y,x) = 0} \tag{5}$$

Which is an equation involving only y and x (not its derivatives). We can then presumably solve this equation for y as a function of x algebraically, and [cheff's kiss] we will have found the function $y(x)$ that satisfies or original DE (2) [1].

Assuming you're convinced that this solves our problem we now focus on exactly **how** to go about finding the function $\Phi(y,x)$.

Warning: this derivation can get a bit a bit hairy, but it's extremely worthwhile & I will only use tools that we have developed up to this point.

Exact Differential Equations (4): Solving for $\Phi(y, x)$

Recall our goal is to find a function $\Phi(y, x)$ such that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} = N(y, x)\frac{\mathrm{d}y}{\mathrm{d}x} + M(y, x) \tag{6}$$

The only reasonable thing to do at this point is to set this equation equal to our expression we obtained using the multivariable chain rule, which was:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} = \frac{\partial\Phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial\Phi}{\partial x} \tag{7}$$

If these two equations are to be equal, we therefore need that

$$\boxed{\frac{\partial\Phi}{\partial y} = N(y, x) \qquad \& \qquad \frac{\partial\Phi}{\partial x} = M(y, x)} \tag{8}$$

Furthermore, since the order in which we take partial derivatives doesn't matter we will also need

$$\frac{\partial^2\Phi}{\partial x\partial y} = \frac{\partial^2\Phi}{\partial y\partial x}$$

$$\implies \boxed{\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}} \tag{9}$$

Note: If Eq (9) is true, then the differential equation is said to be **exact**. Eq 9 **always** the first thing you should check, since if it's not true then the function Φ we're looking for doesn't exist...***almost***

Exact Differential Equations (5): Solving for $\Phi(y, x)$ cont.

What do I mean by almost? There are two cases, **1**) either equation (9) is satisfied or **2**) it's not. For now, lets consider case **1**) first.

Case 1) Suppose that our N and M satisfy

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Then, we can simply integrate N and M so that

$$\Phi(y, x) = f(x) + \int N(y, x)\ \partial y \tag{10}$$

$$\Phi(y, x) = g(y) + \int M(y, x)\ \partial x \tag{11}$$

A couple remarks:

(a) The notation ∂x, ∂y means the same thing as it does in partial differentiation (i.e. that we treat the other variables as constant–don't overthink this).

(b) The functions $f(x), g(y)$ are just the "$+C$" of partial derivatives. That is, when we take normal derivatives the result is correct *up to a constant*; when we take partial derivatives the answer is correct up to *a function of the variable we held constant.*

Exact Differential Equations (6): Solving for $\Phi(y, x)$ cont.

After obtaining equations (10) and (11) through integration you should be able to just look at both expressions and solve for $f(x)$ and $g(y)$ by eye such that both equations are equal.

There is of course a more rigorous way to do this last bit, but it will become clear what exactly I mean by this (if it's not already) in the future slides where we work through examples. After solving for $f(x)$ and $g(y)$ our general solution is given by

$$\boxed{\Phi(y, x) = 0} \tag{12}$$

This concludes how to solve an **exact** differential equation. In the next lesson we'll learn what to do if equation (9) is not true of our equation (i.e. if it's not exact). Spoiler: There are ways to **make it exact**.

If all of this seemed abstract and cumbersome, don't worry it's must easier in practice. Let's do an example to get our hands dirty.

Example

We wish to solve the differential equation

$$(2yx - \sin(y))\frac{dy}{dx} + y^2 - 2x = 0 \tag{13}$$

Solution: First we identify N and M.

$$N(y, x) = 2yx - \sin(y) \tag{14}$$
$$M(y, x) = y^2 - 2x \tag{15}$$

Now we check for exactness

$$\frac{\partial N}{\partial x} = 2y \tag{16}$$
$$\frac{\partial M}{\partial y} = 2y \quad \checkmark \tag{17}$$

Nice, our equation is exact. Now we proceed by writing down the relations between Φ ,N and M.

Example cont.

From equation (8) we have that

$$\frac{\partial \Phi}{\partial y} = 2yx - \sin(y) \tag{18}$$

$$\frac{\partial \Phi}{\partial x} = y^2 - 2x \tag{19}$$

Now, we integrate each of them so that

$$\Phi(y,x) = \int 2yx - \sin(y)\partial y = xy^2 + \cos(y) + f(x) \tag{20}$$

$$\Phi(y,x) = \int y^2 - 2x\partial x = xy^2 - x^2 + g(y) \tag{21}$$

By inspection we therefore see that $f(x) = -x^2$ and $g(y) = \cos(y)$. Hence our solution is given by

$$\boxed{\Phi(y,x) = xy^2 + \cos(y) - x^2 + C} \tag{22}$$

Exercise: Show that this expression for $\Phi(y,x)$ is an implicit solution to (13). ***Hint***: Take d/dx to both sides and use implicit differentiation.

[1] It's worth noting that you will not *always* be able to solve for y as a function of x explicitly, but that's okay and it's not our fault–it's just the nature of the solution and we can just leave it like that. Such a solution is called an **implicit solution** to the differential equation. An example of what this might look like is $y = e^{yx}$. Try as you might, you will never be able to solve for y but the equation is still a perfectly good equation.

[2] Without this footnote, technically equation (5) should read $\Phi(y, x) = C$. However, it's convention to absorb the $+C$ into the definition of Φ so there's no loss in generality. This will become clear after working through the example.