Inverse problems for real principal type operators

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Outline

1. Motivation
2. Results for real principal type operators
3. Methods of proof

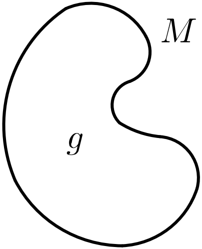
PDEs in a geometric setting

Consider the Laplace equation

$$\Delta u = 0 \text{ in } \Omega \subset \mathbb{R}^n.$$

More generally, let (M, g) be a compact Riemannian manifold with boundary. The Laplace-Beltrami operator Δ_g is

$$\Delta_g u = \sum_{j,k=1}^{n} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_j} \left(\sqrt{\det g}\, g^{jk} \frac{\partial u}{\partial x_k} \right)$$

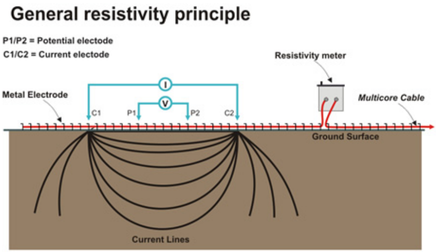


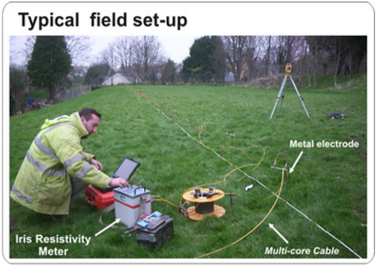
where $g = (g_{jk}(x))$, $g^{-1} = (g^{jk}(x))$.

Models an inhomogeneous medium.

1. Calderón problem

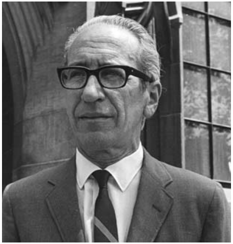
Electrical Resistivity Imaging in geophysics (1920's) [image: TerraDat]





A.P. Calderón (1980):

- mathematical formulation
- solution of the linearized problem



1. Calderón problem (elliptic PDE)

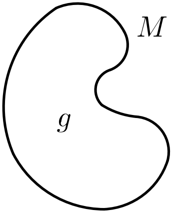
Laplace-Beltrami equation

$$\begin{cases} \Delta_g u = 0 & \text{in } M, \\ u = f & \text{on } \partial M \end{cases}$$

where (M, g) is a compact Riemannian manifold with boundary ($g \leftrightsquigarrow$ electrical conductivity).

Boundary measurements given by the Dirichlet-to-Neumann (DN) map

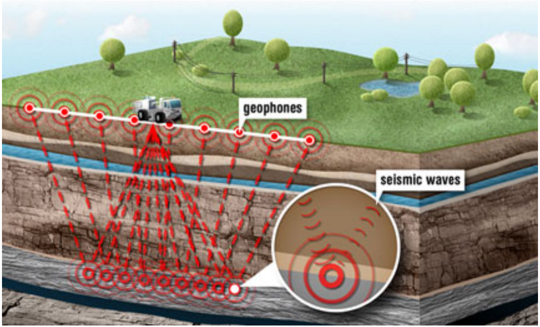
$$\Lambda_g : C^\infty(\partial M) \to C^\infty(\partial M), \ f \mapsto \partial_\nu u|_{\partial M}.$$



Inverse problem: given Λ_g, recover g.

1. Gel'fand problem

Exploration seismology:



I.M. Gel'fand (1950s):

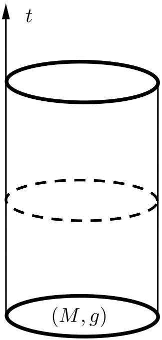
- 1D inverse scattering (Gel'fand, Levitan, Marchenko)
- proposed the question for $n \geq 2$

2. Gel'fand problem (hyperbolic PDE)

Wave equation ($g \leftrightsquigarrow$ sound speed)

$$\begin{cases} (\partial_t^2 - \Delta_g)u = 0 & \text{in } M \times (0,T), \\ u = f & \text{on } \partial M \times (0,T) \\ u|_{\{t<0\}} = 0 & \end{cases}$$

where (M, g) is compact with boundary.



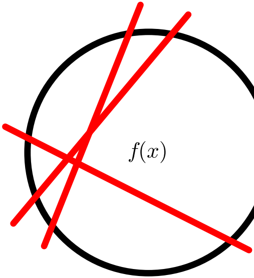
Boundary measurements given by the hyperbolic DN map

$$\Lambda_g^{\mathrm{Hyp}} : f \mapsto \partial_\nu u|_{\partial M \times (0,T)}.$$

Inverse problem: given Λ_g^{Hyp}, recover g.

3. X-ray transform problems

X-ray computed tomography:

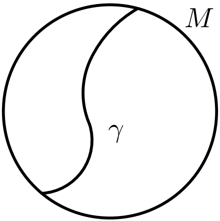


Recover a function $f(x)$ from its integrals over straight lines [Radon 1917]. Seismic imaging: lines are replaced by geodesics.

3. X-ray transform / scattering relation

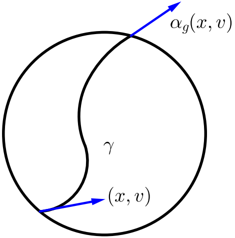
Try to recover a function f in (M, g) from its *geodesic X-ray transform* If, where

$$If(\gamma) = \int_\gamma f \, dt, \quad \gamma \text{ maximal geodesic.}$$



Related scattering rigidity problem: recover (M, g) from its scattering relation α_g, relating initial and final data of maximal geodesics:

$$\alpha_g : (x, v) \mapsto \alpha_g(x, v)$$



Can formulate both questions in terms of (transport) PDEs. These are highly nonlinear questions related to linear PDEs!

Connections

Unexpected connections in special geometries:

- Calderón problem reduces to geodesic X-ray transform [Dos Santos-Kenig-S-Uhlmann 2009]
- Calderón problem reduces to Gel'fand problem [Dos Santos-Kurylev-Lassas-S 2016]
- scattering rigidity problem reduces to Calderón problem [Pestov-Uhlmann 2005]

What are the general structural conditions and mechanisms behind this?

Goal

Propose to study inverse problems for general differential operators. Hope to understand:

- structural conditions for treating classes of operators
- fundamental mechanisms for solving inverse problems
- the extent to which it is possible to push existing methods.

Approach in the spirit of general theory for linear PDEs [Hörmander 1983–1985]. Earlier results for constant coefficients [Isakov 1991, . . .].

We will consider variable coefficients (=microlocal analysis).

Outline

1. Motivation
2. Results for real principal type operators
3. Methods of proof

Real principal type operators

Let M be compact with boundary. A differential operator P on M, of order $m \geq 1$, is real principal type if

- it has real principal symbol $\sigma_{\mathrm{pr}}(P) = p_m$,[1] and
- the null bicharacteristic flow is nontrapping.

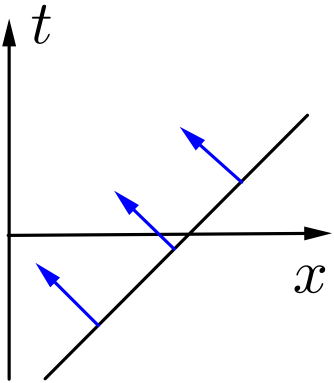
Null bicharacteristic curves $\gamma(t) = (x(t), \xi(t))$ are integral curves of Hamilton field H_{p_m} in $p_m^{-1}(0)$. They solve the ODE

$$\begin{cases} \dot{x}(t) &=& \nabla_\xi p_m(x(t), \xi(t)), \\ \dot{\xi}(t) &=& -\nabla_x p_m(x(t), \xi(t)). \end{cases}$$

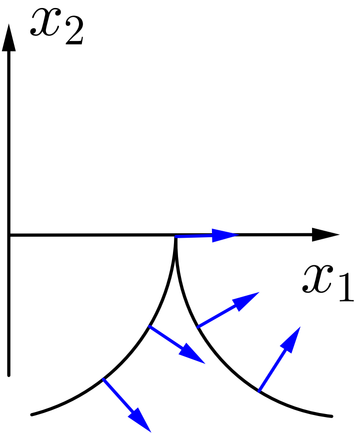
Nontrapping means that any such $\gamma(t)$ reaches ∂M in finite time in both directions.

[1] If $P = \sum_{|\alpha| \leqslant m} p_\alpha(x) D^\alpha$, then $\sigma_{\mathrm{pr}}(P) = \sum_{|\alpha|=m} p_\alpha(x)\xi^\alpha$.

Null bicharacteristics



Wave operator $\partial_t^2 - \Delta$



Tricomi operator $x_2 D_{x_1}^2 + D_{x_2}^2$

Real principal type operators

Examples:

- real vector fields with no trapped integral curves
- wave operator in $M \times (0, T)$, Lorentzian wave operators with nontrapping condition, strictly hyperbolic operators
- Tricomi type operators, e.g. $x_2 D_{x_1}^2 + D_{x_2}^2$
- Schrödinger operator $i\partial_t + \Delta$, plate equation $\partial_t^2 + \Delta^2$ with suitable (anisotropic) weighting for ∂_t

Real principal type operators can be microlocally conjugated to normal form D_{x_1}. Singularities of solutions propagate along null bicharacteristics, solvability theory for $Pu = f$ [Duistermaat-Hörmander 1972].

Boundary measurements

It is not clear how to define an analogue of DN map for a general operator. However, we consider the Cauchy data set

$$C_P = \{(u|_{\partial M}, \ldots, \nabla^{m-1} u|_{\partial M})\,;\, Pu = 0 \text{ in } M,\ u \in H^m(M)\}.$$

This is equivalent to knowing the DN map e.g. in the Calderón and Gel'fand problems.

Inverse problem: given C_P, determine information about P.

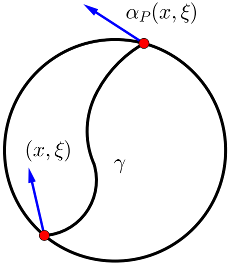
From now on, all operators will be real principal type in M.

Determining (sub)principal information

Theorem 1

If $C_{P_1} = C_{P_2}$ and if $P_1 = P_2$ to infinite order on ∂M, then

$$\alpha_{P_1} = \alpha_{P_2}$$



where α_P is the bicharacteristic scattering relation, mapping an initial point of a maximal null bicharacteristic to its final point.

Moreover, if P_1 and P_2 have the same principal symbol, then

$$\exp\left[i\int \sigma_{\mathrm{sub}}(P_1)(\gamma(t))\,dt\right] = \exp\left[i\int \sigma_{\mathrm{sub}}(P_2)(\gamma(t))\,dt\right]$$

for any maximal null bicharacteristic γ in T^*M.

Lower order coefficients

The conclusion $\exp[i \int \cdots] = \exp[i \int \cdots]$ is equivalent with

$$\int \sigma_{\mathrm{sub}}(P_1)(\gamma(t))\,dt = \int \sigma_{\mathrm{sub}}(P_2)(\gamma(t))\,dt \mod 2\pi\mathbb{Z}.$$

This is related to the Aharonov-Bohm effect in determining subprincipal terms on domains with nontrivial topology. For lower order coefficients, this effect does not appear:

Theorem 2 (Bicharacteristic ray transforms)

If $C_{P+Q_1} = C_{P+Q_2}$ where Q_j are operators of order $\leq m-2$, then

$$\int \sigma_{\mathrm{pr}}(Q_1)(\gamma(t))\,dt = \int \sigma_{\mathrm{pr}}(Q_2)(\gamma(t))\,dt$$

for any maximal null bicharacteristic γ in T^*M.

Real principal type operators

The results are general: they extend results for wave equations [Rakesh-Symes 1988, . . . , Stefanov-Yang 2018], and are valid for

- operators of any order, with real principal symbol and nontrapping condition (no wellposedness assumptions)
- any maximal bicharacteristic, even with cusps (Tricomi) and tangential reflections

However, the results are conditional: in order to recover coefficients of P, one still needs to analyze the scattering relation α_P or bicharacteristic ray transforms.

Boundary determination

Determine Taylor serier of coefficients of P at null points $(x, \xi) \in T^*(\partial M)$, based on zeros of characteristic polynomial

$$t \mapsto p_m(x, \xi + t\nu).$$

Two methods:

1. Elliptic region. If there is a simple non-real zero, use exponentially decaying solutions (analogue of boundary determination for Laplace equation).
2. Hyperbolic region. If there are two distinct real zeros, use solutions concentrating near two null bicharacteristics (analogue of boundary determination for wave equation).

Boundary determination

Theorem 3 (Determining Taylor series of a potential)

If $V_1, V_2 \in C^\infty(M)$ and $C_{P+V_1} = C_{P+V_2}$, then

$$\nabla^k V_1(x_0) = \nabla^k V_2(x_0), \qquad k \geq 0,$$

at any $x_0 \in \partial M$ so that for some $\xi \in T_x^*(\partial M)$, the map $t \mapsto p_m(x_0, \xi + t\nu)$ either has a simple non-real root, or two distinct real roots[1].

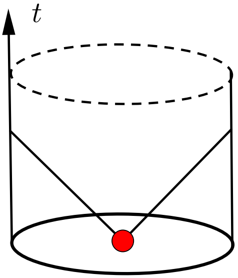
In particular, if M and V_j are real-analytic and there is one such x_0, then $V_1 = V_2$ everywhere in M.

[1] with corresponding bicharacteristics intersecting nicely at x_0

Boundary determination

Observations:

- boundary determination in general not possible for $m = 1$
- even for wave equation, can do boundary determination in the elliptic region as for elliptic operators (local argument)
- boundary determination in the hyperbolic region is global in character



$$P = \partial_t^2 - \Delta + V$$

Nonlinear equations

If $q \in C^\infty(M)$, consider the semilinear equation

$$Pu + q(x)u^k = 0 \text{ in } M.$$

Let C_q^{small} be the Cauchy data set for small solutions.

Theorem 4 (Semilinear equations)

Let $q_1, q_2 \in C^\infty(M)$ and $k \geq 3$. If $C_{q_1}^{\mathrm{small}} = C_{q_2}^{\mathrm{small}}$, then $q_1 = q_2$ in B where

$$B = \{x \in M\,;\ \text{there are two null bicharacteristics that intersect only once at } x \text{ transversally}\}.$$

Nonlinearity helps (proof fails if $k = 1$)! Wave equations: [Kurylev-Lassas-Uhlmann 2018, Lassas-Uhlmann-Wang 2018, Hintz-Uhlmann 2018]

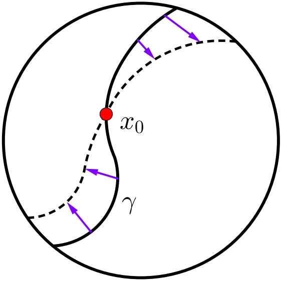
Nonlinear equations

Can recover the coefficient $q(x)$ in the set

$$B = \{x_0 \in M\,;\ \text{there are two null bicharacteristics that intersect only once at } x_0 \text{ transversally}\}.$$

If there is a nice[1] bicharacteristic $\gamma(t)$ through x_0 having a variation field only vanishing at $t = 0$, can recover $q(x_0)$.

Works e.g. if some $\gamma(t)$ through x_0 has "no conjugate points". May fail if there is a "maximally conjugate" point.



[1] nontangential, no cusp at x_0, $x(t)$ does not self-intersect

Outline

Methods

1. Use Cauchy data of special solutions concentrating along a null bicharacteristic (propagation of singularities).

2. For boundary determination, also use exponentially decaying solutions concentrating at a boundary point.

3. Use integral identities and a mix-and-match construction to pass from Cauchy data set C_P to scattering relation / bicharacteristic ray transforms / pointwise information.

Quasimode construction

Theorem 5

Let P have real principal symbol in an open mfld X, and let $\gamma : [0, T] \to T^*X$ be an injective null bicharacteristic segment. There is $u = u_h \in C_c^\infty(X)$ with

$$WF_{\mathrm{scl}}(u) = \gamma([0, T]), \qquad WF_{\mathrm{scl}}(Pu) = \gamma(0) \cup \gamma(T),$$

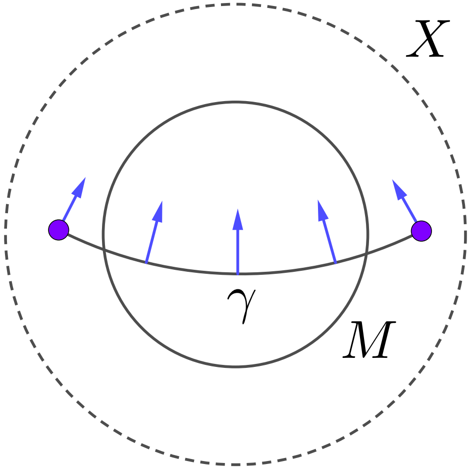
having semiclassical defect measure (with $c_\gamma(t) > 0$)

$$\lim_{h \to 0} (Op_h(a)u_h, u_h)_{L^2(X)} = \int_0^T a(\gamma(t))c_\gamma(t)\, dt$$

whenever $a \in S^0$ vanishes near endpoints of γ.

Semiclassical counterpart of [Duistermaat-Hörmander 1972].

Quasimode construction



$$WF_{\mathrm{scl}}(Pu) = \gamma(0) \cup \gamma(T) \implies Pu = O(h^\infty) \text{ in } M$$

Methods for constructing quasimodes

1. Locally, enough to use geometrical optics:

$$u_h(x) = e^{i\varphi(x)/h}a(x)$$

where φ is real and $p_m(x, d\varphi(x)) = 0$ (eikonal equation).

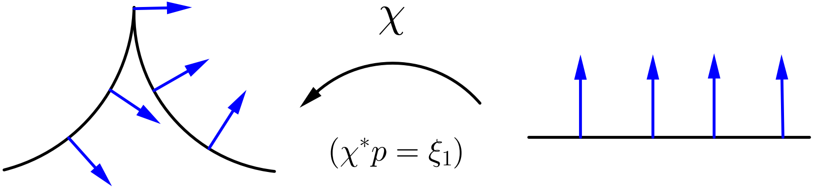
2. No cusps ($\dot{x}(t) \neq 0$): can use a Gaussian beam construction

$$u_h(x) = e^{i\Phi(x)/h}a(x)$$

where Φ is complex and solves eikonal equation to infinite order on the curve $x(t)$. Main point: $\nabla^2\Phi|_{x(t)}$ solves a matrix Riccati equation and $\mathrm{Im}(\nabla^2\Phi) > 0$.

Quasimodes [Duistermaat-Hörmander 1972]

3. If $\gamma(t)$ is injective but may have cusps, can straighten $\gamma(t)$ in phase space by a canonical transformation χ.



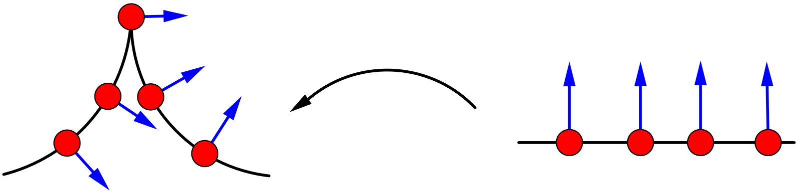
Multiply P by an elliptic ΨDO so that P becomes of order 1. Construct Fourier integral operators A, B such that

$$BPA \approx D_{x_1} \text{ microlocally near } \gamma.$$

Quasimode U for $D_{x_1} \implies u = AU$ is a quasimode for P.

Quasimodes (direct construction)

4. Think of quasimodes as superpositions of wave packets $\approx e^{i\frac{\xi(t)\cdot(x-x(t))}{h}} e^{-\frac{|x-x(t)|^2}{2h}}$ at $x(t)$ oscillating in direction $\xi(t)$.



Look for u_h with $Pu_h = O(h^\infty)$ directly in the form

$$u_h(x) = \int_0^T e^{i\Phi(x,t)/h} a(x,t)\, dt.$$

Cf. Gaussian beam construction along $(x(t), t)$ in $X \times \mathbb{R}$.

Future directions

1. Inversion of scattering relation α_P? If $P = X_g$,[1] studied in [Pestov-Uhlmann 2005, Stefanov-Uhlmann-Vasy 2017].

2. Inversion of bicharacteristic ray transform? Cf. geodesic ray transform [Uhlmann-Vasy 2016, Paternain-S-Uhlmann 2015] ($P = X_g$), and light ray transform [Lassas et al 2019] ($P = \Box_g$).

3. Results for mild trapping? For $P = X_g$ and hyperbolic trapping, studied in [Guillarmou / Guillarmou-Monard 2017].

4. Can one associate a symbol directly to C_P?

5. The results are in the spirit of using singularities of the integral kernel of DN map. Can one extract information from the C^∞ part of the kernel?

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[1] geodesic vector field on unit sphere bundle