Mathematics

Transmission Eigenvalues and Non-scattering in Euclidean and Hyperbolic Geometry

Fioralba Cakoni

Research supported by grants from AFOSR and NSF

RUTGERS

Outline

Part I

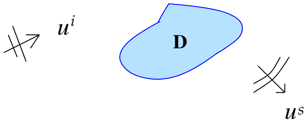
Non-scattering and transmission eigenvalues in scattering by an inhomogeneous medium in Euclidean geometry

Part II

Non-scattering and transmission eigenvalues in scattering for automorphic forms on fundamental domains generated by discrete groups acting on the hyperbolic upper half complex plane

This part is a joint work with Sagun Chanillo

Scattering by an Inhomogeneous Media



Let the incident field $u^i := v$ satisfy

$$\Delta v + k^2 v = 0 \qquad \text{in } \mathbb{R}^3.$$

The total field $u = u^s + v$ satisfies

$$\Delta u + k^2(1 + m(x))u = 0 \qquad \text{in } \mathbb{R}^3.$$

and the scattered field u^s is outgoing, i.e. satisfies

$$\lim_{r \to \infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0$$

The contrast in the medium $m \in L^\infty(\mathbb{R}^3)$ has support $\overline{D}$ and

$m = \Re(m) + \frac{i}{k}\Im(m)$, $1 + \Re(m) > 0$ and $\Im(m) \geq 0$.

$k > 0$ is the wave number proportional to the frequency.

Scattering by an Inhomogeneous Media

The scattered field $u^s \in H^2_{loc}(\mathbb{R}^3)$ satisfies

$$\Delta u^s + k^2(1 + m)u^s = -k^2 m v \qquad \text{in } \mathbb{R}^3$$

and for $\Im(k) \geq 0$ it has the asymptotic behavior

$$u^s(x,k) = \frac{e^{ikr}}{r} u^\infty(\hat{x};k) + O\left(\frac{1}{r^2}\right) \qquad \text{as } r := |x| \to \infty$$

The function $u^\infty(\hat{x};k)$, $\hat{x} := x/|x|$ is the far field pattern.

Question: Is there an incident field v that does not scatter?

Rellich's Lemma: $u^\infty(\hat{x};k) = 0$ implies $u^s(x;k) = 0$.

Scattering Operator

In particular, we consider free space waves as incident field v, i.e entire solutions to Helmholtz equation in $\mathbb{R}^3$ satisfying

$$\|v\|_{B^*} := \sup_{R>0} \frac{1}{\sqrt{R}} \|v\|_{L^2(B_R)} < \infty$$

Such solutions have the asymptotic behavior:

$$v(x;k) = \frac{e^{ikr}}{r} \Theta(\hat{x};k) + \frac{e^{-ikr}}{r} \Theta(-\hat{x};k) + O\left(\frac{1}{r^2}\right) \qquad \text{as } r := |x| \to \infty$$

The scattering operator for $k \in \mathbb{C}$ with $\Im(k) \geq 0$ is

$$\mathcal{S}_k : \Theta(\hat{x};k) \mapsto u^\infty(\hat{x};k)$$

The scattering operator has a meromorphic continuation to $k \in \mathbb{C}$.

Non-scattering question is related to the kernel of scattering operator.

Example of Spherical Geometry

Consider scattering of $v = j_\ell(k|x|)Y_\ell(\hat{x})$ by the ball $B_1(0)$ and $m(r)$ real

$$u^s(x) := \frac{C(k; m, \ell)}{W(k; m, \ell)} h_\ell^{(1)}(k|x|)Y_\ell(\hat{x}), \quad u^\infty(x) := \frac{C(k; m, \ell)}{W(k; m, \ell)} \frac{1}{k} Y_\ell(\hat{x})$$

$$C(k; m, \ell) = \text{Det} \begin{pmatrix} y_\ell(1; k, m) & j_\ell(k) \\ y'_\ell(1; k, m) & kj'_\ell(k) \end{pmatrix}$$

$$W(k; m, \ell) = \text{Det} \begin{pmatrix} y_\ell(1; k, m) & h_\ell^{(1)}(k) \\ y'_\ell(1; k, m) & kh_\ell^{(1)'}(k) \end{pmatrix}$$

with $y_\ell(r; k, m)$ the solution (regular at $r = 0$) of

$$y'' + \frac{2}{r} + \left(k^2(1 + m(r)) - \frac{\ell(\ell+1)}{r^2}\right) y = 0.$$

If k is such that $C(k; m, \ell) = 0$ then $v = j_\ell(k|x|)Y_\ell(\hat{x})$ does not scatter.

Such k are non-scattering wave numbers.

The Eigenvalue Transmission Problem

These non-scattering wave numbers are eigenvalues of the transmission eigenvalue problem:

$$
\begin{aligned}
\Delta v + k^2 v &= 0, & |r| < 1 \\
\Delta u + k^2 (1 + m(r)) u &= 0, & |r| < 1 \\
u &= v & |r| = 1 \\
\frac{\partial u}{\partial r} &= \frac{\partial v}{\partial r} & |r| = 1
\end{aligned}
$$

These eigenvalues are called the transmission eigenvalues

For spherically symmetric inhomogeneities the set of transmission eigenvalues coincide with the set of non-scattering frequencies.

Results for Spherical Geometry

Provided $m(1) \neq 0$ one can prove:

- There exist an infinite set of real and complex transmission eigenvalues.
- All transmission eigenvalues lie in a strip $|\Im(k)| < C$.
- Weyl's law: Asymptotic number of all transmission eigenvalues in $|k| < r$ is of order $r^3 + O(r^{2+\epsilon})$, whereas of transmission eigenvalues with spherically symmetric eigenfunctions, i.e. for $\ell = 0$ is of order r.
- Inverse spectral problem: All transmission eigenvalues (counting multiplicity) uniquely determine $m(r)$. Transmission eigenvalues with spherically symmetric eigenfunctions, i.e. for $\ell = 0$, (counting multiplicity) uniquely determine $m(r) < 0$.

Cakoni-Colton-Gintides (2010), Aktosun-Gintides-Papanicolaou (2011), Colton-Leung (2012), (2013)

Colton-Leung-Meng (2015), Sylvester (2013), Petkov-Vodev (2016).

Transmission Eigenvalues in General

Question: Is there an incident field v that does not scatter?

Recall $\Delta u^s + k^2(1+m)u^s = -k^2 m v$ in $\mathbb{R}^3$. If yes, $u^s \in H_0^2(D)$, hence $v|_D$ and u solve:

The transmission eigenvalue problem

$$\begin{aligned} \Delta v + k^2 v &= 0 && \text{in} \quad D \\ \Delta u + k^2(1+m)u &= 0 && \text{in} \quad D \\ u &= v && \text{on} \quad \partial D \\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} && \text{on} \quad \partial D \end{aligned}$$

- Values of $k \in \mathbb{C}$ for which the transmission eigenvalue problem has a non trivial solution are called transmission eigenvalues
- A transmission eigenvalue is a non-scattering wave number if the part v of the eigenfunction solves the Helmholtz equation in all of $\mathbb{R}^3$.

Transmission Eigenvalues in General

More generally for $A_j \in (L^\infty(D))^{d\times d}$, $n_j \in L^\infty(D)$

$$
\begin{aligned}
\nabla \cdot A_1 \nabla v + k^2 n_1 v &= 0 && \text{in } D \\
\nabla \cdot A_2 \nabla u + k^2 n_2 u &= 0 && \text{in } D \\
u &= v && \text{on } \partial D \\
\nu \cdot A_2 \nabla u &= \nu \cdot A_1 \nabla v && \text{on } \partial D
\end{aligned}
$$

The transmission eigenvalue problem was first introduced in

KIRSCH (1986), COLTON-MONK (1988)

From first results on the existence of real transmission eigenvalues

PÄIVÄRINTA-SYLVESTER (2009), CAKONI-GINTIDES-HADDAR (2010)

the subject has taken a multitude of directions:

- F. CAKONI, D. COLTON, H. HADDAR, Inverse Scattering Theory and Transmission Eigenvalues, CBMS-SIAM (2016)
- D. COLTON, R. KRESS, Inverse Acoustic and Electromagnetic Scattering Theory, Springer, 4th Edition (2019).

The State-of-the-Art of the TEP

Discreteness

1. $m \in L^\infty(D)$ and $\Re(m)$ does not change sign in a neighborhood of ∂D or coefficients are smooth only near the boundary, and
2. A_1, A_2, n_1, n_2 are C^1 only near ∂D and real-valued
   i) A_1, A_2 satisfy the complementing boundary condition, i.e., for all $\xi \cdot \nu = 0$ where ν is the normal vector to ∂D

   $$\langle A_2\nu,\nu\rangle\langle A_2\xi,\xi\rangle - \langle A_2\nu,\xi\rangle^2 \neq \langle A_1\nu,\nu\rangle\langle A_1\xi,\xi\rangle - \langle A_1\nu,\xi\rangle^2 \text{ on } \partial D$$

   ii) $\langle A_2\nu,\nu\rangle n_2 \neq \langle A_1\nu,\nu\rangle n_1$ on ∂D.

BONNET-BEN DHIA-CHESNEL-HADDAR (2011), SYLVESTER (2012), CAKONI-HADDAR-MENG (2015), LAKSHTANOV-VAINBERG (2015), KIRSCH (2016), NGUYEN-NGUYEN (2016), CAKONI-NGUYEN (2020)

Open Problems

Is the spectrum of the transmission eigenvalue problem still discrete if m changes sign up to the boundary?

The State-of-the-Art of the TEP

Existence

1. Existence of real TE and monotonicity properties: If
   - $m \in L^\infty(D)$, m real and of one sign uniformly in D
   - $A_1 - A_2$ of one sign uniformly in D and $n_1 = n_2$ or both $A_1 - A_2$, $n_1 - n_2$ of one sign uniformly in D

   PÄIVÄRINTA-SYLVESTER (2009), CAKONI-GINTIDES-HADDAR (2010), CAKONI-KIRSCH (2010)

2. Existence and completeness of generalized eigenfunctions for $m \in C^\infty(\overline{D})$ real and $m \neq 0$ on ∂D.

   BLÅSTEN-PÄIVÄRINTA (2013), ROBBIANO (2013), HADDAR-MENG (2018) (for Maxwell's)

Open Problems

- Do complex transmission eigenvalues exist for general media?
- Do real transmission eigenvalues exists for media with one sign contrast only near the boundary?
- Spectral analysis for $m = \Re(m) + i/k\Im(m)$ is open.

The State-of-the-Art of the TEP

Location of Transmission Eigenvalues in the Complex Plane

1. If $m \in C^\infty$ and $m \neq 0$ on ∂D, all TEs satisfy $|\Im(k)| < C$.
2. If A_1, A_2, n_1, n_2 are scalar in C^∞, all TEs lie in a strip if

$$(A_1 - A_2)(A_1 n_1 - A_2 n_2) < 0, \text{ on } \partial D$$

and the imaginary part of TEs grow at most logarithmically if

$$(A_1 - A_2)(A_1 n_1 - A_2 n_2) > 0, \text{ and } A_1 n_2 \neq A_2 n_1 \text{ on } \partial D$$

HITRIK-KRUPCHYK-OLA-PÄIVÄRINTA (2011), VODEV (2018), CAKONI-NGUYEN (2020) (for Maxwell equations)

Weyl's Asymptotics

$$\#\{\text{TE in the ball } B_r(0)\} \sim \frac{1}{6\pi^2}\left(1 + \int_D (1+m)^{3/2}\right) r^3 \quad \text{as } r \to \infty$$

PHAM-STEFANOV (2011), LAKSHTANOV-VAINBERG (2013), ROBBIANO (2016), PETKOV-VODEV (2017), VODEV (2018)

The State-of-the-Art of the TEP

Transmission Eigenvalues and Non-scattering Frequencies

Can the part v of the eigenfunctions (corresponding to the background) extend outside D as solution to the background equation?

1. If D is a ball all transmission eigenvalues are non-scattering frequencies
2. If D has a corner there exits NO non-scattering frequencies
   (exceptional open case appear in $\mathbb{R}^3$ and for the case with A, n)

Inability to extend v outside D is related to unique determination of the support of inhomogeneity D with one incident wave.

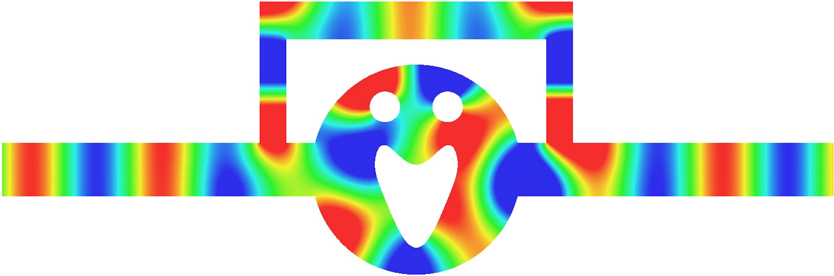
BLÅSTEN-PÄIVÄRINTA-SYLVESTER (2014), HU-SALO-VESALAINEN (2016), BLÅSTEN-LIU (2016), PÄIVÄRINTA-SALO-VESALAINEN (2017), ELSCHNER-HU (2018), BLÅSTEN-LIU-XIAO (2017), CAKONI-XIAO (2019)

Open Problems

What happens between D being a ball and D being non-smooth?

Non-reflected, Non-transmitted Modes in Waveguides

Thanks to Luca Chesnel, CMAP [Click]



A-S BONNET-BEN DHIA, L. CHESNEL AND V. PAGNEUX, Trapped modes and reflectionless modes as eigenfunctions of the same spectral problem (2018).

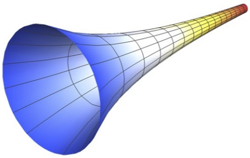
Scattering Theory in Hyperbolic Geometry

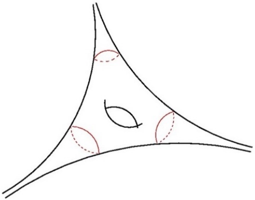
- The concept of transmission eigenvalues can also be considered in connection with scattering theory for automorphic solutions of the wave equation in the hyperbolic plane with isometries corresponding to a specific group.
- Faddeev-Pavlov (1972) made a connection between: harmonic analysis of automorphic functions with respect to group $SL_2(\mathbb{R})$, scattering theory for non-Euclidean wave equations, and Selberg's pioneering work (1956) on spectral theory for compact and finite-area Riemann surfaces.
- Lax-Phillips (1976) redid this work and further developed it in particular for non-compact hyperbolic domains of finite area, which led to more recent development of scattering theory for hyperbolic surfaces of infinite area.

Scattering Theory in Hyperbolic Geometry

Limited to automorphic forms with respect to Fuchsian groups of the first kind that has only cusps at infinity, scattering theory has profound connection to fundamental results from analytic number theory.

H. Iwaniec, Spectral Methods of Automorphic Forms, AMS (2014).





Hyperbolic Plane

$$\mathbb{H} = \left\{ z = x + iy : \quad x \in \mathbb{R},\, y \in \mathbb{R}^+ \right\}.$$

- $\mathbb{H}$ is a Riemannian manifold with the complete metric

$$ds^2 = y^{-2}(dx^2 + dy^2)$$

- This differential form on $\mathbb{H}$ is invariant with respect to Möbius transformations $SL_2(\mathbb{R})$ acting on the whole compactified complex plane $\hat{\mathbb{C}}$, which are fractional linear functions

$$g(z) = \frac{az + b}{cz + d} \qquad a, b, c, d \in \mathbb{R},\ ad - bc = 1$$

- $\mathbb{H}$ is obtained as the orbit of a point (modulo of rotation)

$$\mathbb{H} = Gz = \{gz : \ g \in SL_2(\mathbb{R})\}$$

Hyperbolic Surface

Given large isometry group of $\mathbb{H}$, a natural way to obtain a hyperbolic surface is by a quotient $\Gamma \setminus \mathbb{H}$ (the sets of orbits), for some subgroup Γ of $PSL_2(\mathbb{R}) := SL_2(\mathbb{R}) \setminus (\pm I)$.

A fundamental domain $F := \Gamma \setminus \mathbb{H}$ is a region in $\mathbb{H}$, whose distinct points are not equivalent (different modulo Γ) and any orbit Γz for some $z \in \mathbb{H}$, contains points in the closure of F in the $\hat{\mathbb{C}}$ topology.

$f : \mathbb{H} \to \mathbb{C}$ is called automorphic with respect to Γ if

$$f(\gamma z) = f(z) \ \text{ for all } \gamma \in \Gamma,$$

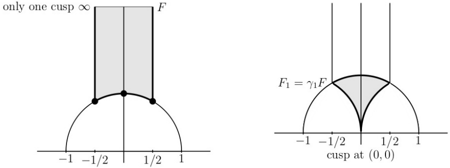
i.e. f lives on $F := \Gamma \setminus \mathbb{H}$.

Fuchsian Groups of First Kind

- A Fuchsian group Γ is a discrete subgroup of $PSL_2(\mathbb{R})$, or equivalently (due to Poincaré) that acts discontinuously on $\mathbb{H}$ (the orbit $\Gamma z := \{\gamma z : \gamma \in \Gamma\}$ of any $z \in \mathbb{H}$ has no limit point in $\mathbb{H}$)
- Fuchsian group Γ is of the first kind if every point of $\partial\mathbb{H} = \hat{\mathbb{R}}$ is a limit (in the $\hat{\mathbb{C}}$-topology) of the orbit Γz for some $z \in \mathbb{H}$. In particular they have a fundamental domain of finite volume.
- We further consider Fuchsian groups of type I that are non co-compact, i.e the closure in $\hat{\mathbb{C}}$ of the fundamental domain is not compact.

For such groups a fundamental domain F must have at least a vertex on $\hat{\mathbb{R}}$ that is a cusp, where the two sides of F are meeting at this vertex orthogonally to $\hat{\mathbb{R}}$. F can be chosen as a polygon all of whose cuspidal vertices are inequivalent.

Examples of Fuchsian Group of First Kind



Two equivalent fundamental domains

Example (Modular Group)

$SL_2(\mathbb{Z})$ is the subgroup of 2×2 matrices with integer entries.

Example $\gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$, which acts according to $z \to -\frac{1}{z}$.

Fundamental domain has one (non-equivalent) cusp.

Applying γ_1 to F we get an equivalent fundamental domain to F by periodicity, $F_1 = \gamma_1 F$.

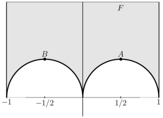
F is Ford fundamental domain, the image of F under Γ tesselate $\mathbb{H}$

Examples of Fuchsian Group of First Kind

Example (Principal Congruence Group)

$$\Gamma(N) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{c} a,b,c,d \in \mathbb{Z}, \quad ad - bc = 1 \\ a \equiv d \equiv 1\,(mod\,N) \\ b \equiv c \equiv 0\,(mod\,N) \end{array} \right\}$$

The number of inequivalent cusps is $h = N^2 \prod_{p/N} (1 - p^{-2})$



The fundamental domain for $\Gamma(2)$. Three non-equivalent cusps at $z = 0$ and $z = \infty$ and $z = 1$ ($z = \pm 1$ are equivalent).

Examples of Fuchsian Group of First Kind

Example (Hecke Congruence Group)

The Hecke congruence subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$ of level N are defined as

$$\Gamma_0(N) = \left\{ \gamma \in SL_2(\mathbb{Z}),\ \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (mod\ N) \right\}$$

and

$$\Gamma_1(N) = \left\{ \gamma \in SL_2(\mathbb{Z}),\ \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (mod\ N) \right\},$$

respectively.

If N is prime than $\Gamma_0(N)$ has only two inequivalent cusps at 0 and ∞.

One has the following inclusions as subgroups

$$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL_2(\mathbb{Z})$$

Waves in Hyperbolic Plane

The Laplace-Beltrami operator in this case is

$$\Delta_{\mathbb{H}} u := y^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

Governing equation of wave propagation on the hyperbolic plane $\mathbb{H}$ is

$$\Delta_{\mathbb{H}} u + s(1-s)u = 0 \qquad \text{or} \qquad y^2 \Delta u + s(1-s)u = 0.$$

Solutions y^s and y^{1-s}, $s \in \mathbb{C}$ are invariant under $z \mapsto z + 1$.

If $\Im(s) > 0$, $\Im(1-s) < 0$ then y^s is incoming (away from the cusp) and y^{1-s} is outgoing (toward the cusp): y^s satisfies

$$\left.\frac{\partial u}{\partial \nu}\right|_{\mathbb{H}} - su = y\frac{\partial u}{\partial y} - su = 0.$$

Scattering by Fundamental Domain

Given a fundamental domain $\Gamma \setminus \mathbb{H}$ and $\Re(s) > 1$, and incoming wave sent at a cusp **a**, the scattering problem for $u := y^s + u_{scat}$ reads

$$\begin{aligned} y^2 \Delta u + s(1-s)u = 0, & \qquad z = (x, y) \in F := \Gamma \setminus \mathbb{H} \\ u(\gamma z) = u(z), & \qquad z \in \partial F, \ \gamma \in \Gamma \\ \frac{\partial u}{\partial \nu}(\gamma z) = \frac{\partial u}{\partial \nu}(z) & \qquad z \in \partial F, \ \gamma \in \Gamma. \end{aligned}$$

Scattering happens because the outgoing free wave package y^s needs to be periodified on F.

Scattering by Fundamental Domain

For $\Re(s) > 1$, the above problem has a solution given by Eisenstein series, such that, as $y \to \infty$ within the cusp **a** uniformly in $z \in \mathbb{H}$

$$u = \delta_{\mathbf{ab}} y^s + \varphi_{\mathbf{ab}}(s) y^{1-s} + O((1 + y^{-\Re(s)})e^{-2\pi y})$$

$\delta_{\mathbf{ab}}$ is Kronecker delta, vanishing when $\mathbf{a}, \mathbf{b}$ are inequivalent cusps.

In a similar manner as for the Euclidean scattering operator $\mathcal{S}$, here the "incoming-outgoing" scattering matrix is

$$\Phi(s) := (\varphi_{\mathbf{ab}}(s)), \quad \text{where } \mathbf{a} \text{ and } \mathbf{b} \text{ run over all cusps,}$$

The scattering matrix has a meromorphic continuation to $s \in \mathbb{C}$ (Maass-Selberg)

Explicit Scattering Matrix

Theorem (Scattering Matrix)

- For $F := SL_2(\mathbb{Z}) \setminus \mathbb{H}$

$$\varphi_{\infty\infty} = \pi^{1/2} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}$$

  where $\zeta(s)$ is the Riemann zeta function.

- For the cusp $\mathbf{a} := \infty$ and $\Gamma := \Gamma(N)$ we have that

$$\varphi_{\infty\infty}(s) = \pi^{1/2} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)} \left[\frac{\varphi(N)}{N^{4s}} \prod_{p/N} \left(1 - \frac{1}{p^{2s}}\right)^{-1} \right]$$

  where $\varphi(N)$ is Euler's totient function.

  Note that here the scattering matrix is $(\Phi(s))_{h \times h}$ and we compute only one entry.

Explicit Scattering Matrix

Theorem (Scattering Matrix)

The scattering matrix for $\Gamma_0(p)$, p prime, is

$$\Phi(s) = \begin{pmatrix} \varphi_{\infty\infty} & \varphi_{\infty 0} \\ \varphi_{0\infty} & \varphi_{00} \end{pmatrix} = \psi(s) N_p(s)$$

where

$$\psi(s) = \pi^{1/2} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}$$

and

$$N_p(s) = \left(p^{2s} - 1\right)^{-1} \begin{pmatrix} p - 1 & p^s - p^{1-s} \\ p^s - p^{1-s} & p - 1 \end{pmatrix}.$$

Transmission Eigenvalues

Example (Definition of Transmission Eigenvalues)

A transmission parameter for the cusp **a** is $s \in \mathbb{C}$ such that

$$\varphi_{\mathbf{aa}}(s) = 0.$$

Such s is said to be a transmission parameter for the transmission eigenvalue $s(1-s)$.

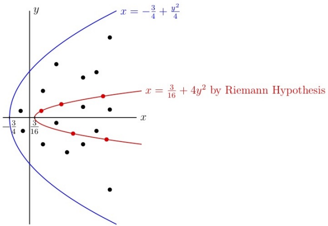
In this definition we ask for invisibility to backscattering data at the cusp **a**.

Remark: One could also ask that all the the entries of the scattering matrix $\Phi(s) := (\varphi_{\mathbf{ab}}(s))$ are zero. While it is proven that poles of the main diagonal terms are also poles for off-diagonal terms (see Iwaniec's book), this is open for the zeros.

Location Transmission Eigenvalues

$$\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} = 0 \qquad \lambda := s(1-s)$$

$s = 1$ and $s = 1/2$ are the trivial zeros (only 2 real transmission eigenvalues $\lambda = 1$ and $\lambda = 1/4$).
The rest of the transmission eigenvalues come from non-trivial zeros, i.e. $\zeta(2s-1) = 0$ such that $\Im(2s-1) \neq 0$.



For the above arithmetic gorups, the Riemann hypothesis is equivalent to the statement that all transmission eigenvalues lie on the parabola

$$x = 3/16 + 4y^2$$

except for the trivial eigenvalues $\lambda = 0$ and $\lambda = 1/4$

Weyl's Law for Transmission Eigenvalues

Let $N_\lambda(r)$ be the number of TE $\lambda := s(1-s)$ such that $|\lambda| < r$

Using Riemann-Von Mangoldt on the zeros of the Riemann zeta function, we can show

If Γ is any of the arithmetic groups above

$$N_\lambda(r) \sim \frac{\sqrt{r}}{2\pi} \log \frac{r}{\pi^2 e^2} + O(\log r), \qquad r \to \infty$$

For general discrete groups, Poisson-Jenssen formula for entire functions applied to $\varphi_{\mathbf{aa}}(s)$ which has at most order 2, gives

$$N_\lambda(r) \leq Cr^{1+\epsilon},\ \forall \epsilon > 0 \quad \text{and} \quad \sum_\lambda \frac{1}{|\lambda|^{1+\eta}} < +\infty,\ \forall \eta > 0$$

Counting of Transmission Eigenvalues

Define the density for transmission eigenvalue $\lambda = s(1-s)$

$$\rho = \limsup_{r\to\infty} \frac{\log |\{\lambda : \quad |\lambda| < r\}|}{\log r}.$$

The above discrepancy on the counting function translates to

- $\rho = \frac{1}{2} + \epsilon, \quad \forall\epsilon > 0$, for Γ any of the arithmetic groups above

  Here $\varphi_{\mathbf{aa}}(s)$ is meromorphic of order 1
- $\rho \leq 1 + \epsilon$ for general discrete groups Γ.

  Here $\varphi_{\mathbf{aa}}(s)$ is meromorphic of order at most 2

Thus ρ is tied in with the existence of cusp forms which relies on the growth order of $\varphi_{\mathbf{aa}}(s)$.

R. S. PHILLIPS, P. SARNAK, On cusp forms for cofinite subgroups of $PSL_2(\mathbb{R})$, *Invent. Math.*, (1980).

Literature and Future Directions

F. CAKONI, S. CHANILLO , Transmission eigenvalues and the Riemann zeta function in scattering theory for automorphic forms on Fuchsian groups of type I, *Acta. Math. Sinica* (2019).

M LEVITIN, A STROHMAIE, Computations of eigenvalues and resonances on perturbed hyperbolic surfaces with cusps, *Int. Math. Research Notices* (2019).

Next one could consider non-scattering and transmission eigenvalues for hyperbolic surfaces $\Gamma/\mathbb{H}$ with Γ Fuchsian groups of type II. Such surfaces have both cusps and funnels and the corresponding scattering matrix is a pseudo-differential operator.

D. BORTHWICK, Spectral Theory of Infinite Area Hyperbolic Surfaces, Birkhäuser (2000).

BLÅSTEN-VESALAINEN, Non-Scattering Energies and Transmission Eigenvalues in $\mathbb{H}^n$, arXiv.