The inverse problem for the X-ray transform

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based on work with Plamen Stefanov and Gunther Uhlmann
as well as work with/of Hanming Zhou, Maarten de Hoop, Joey Zou and Evangelie Zachos

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In this talk we analyze variants of X-ray transforms: for a family of curves $\mathcal{C}$, such as geodesics, on a domain or manifold with boundary X one knows the integrals If of a function f (or perhaps tensor, evaluated on the tangent vector) along elements of $\mathcal{C}$:

$$If(\gamma) = \int f(\gamma(s))\, ds,$$

can one find the function f? This can be interpreted in various ways:

Question

- *is the transform $f \mapsto If$ injective?*
- *can one estimate f in terms of If (stability)?*
- *is there a reconstruction algorithm?*

One can further tweak this in various ways, such as spatial localization: if O is an open set in X, can one achieve the analogous objectives for $f|_O$, using only O-local curves, i.e. those staying in O.

To be more concrete, we consider families of curves (over)parameterized by the sphere bundle SX (or $(TX \setminus o)/\mathbb{R}^+$, so norms are irrelevant):

write $\gamma = \gamma_{x,v}$, and have $I : C^\infty(X) \to C^\infty(SX)$,

$$(If)(x, v) = (If)(\gamma_{x,v}).$$

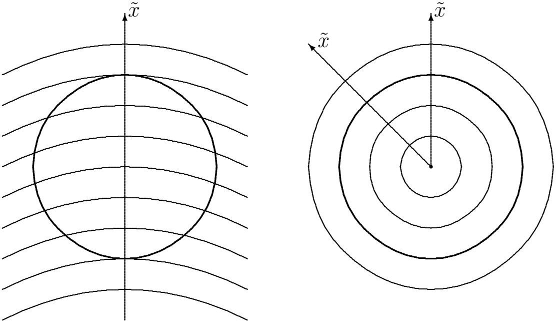
The overparameterization refers to the fact that a fixed geodesic, for instance, is $\gamma_{x,v}$ for numerous x, v, namely the various point-tangent-vector pairs along it.

Dimension counting: space of geodesics has dimension $2n - 2$.

- I is formally determined if $n = 2$.
- I is formally overdetermined if $n \geq 3$.

So: one expects some flexibility if $n \geq 3$, little flexibility if $n = 2$.

The localization (as in O-local curves) proved crucial for even some of the *global* results. This proceeds via level sets of a function $\tilde{x}$, localizing to the super-level sets $\{\tilde{x} > -c\}$. The level set $\tilde{x} = -c$ acts as an *artificial boundary*.



Analytically, as we shall see, the artificial boundary corresponds to a spatial infinity via Melrose's scattering pseudodifferential algebra.

In this talk we discuss modifications to this framework which in certain situation gives global results without the need for localization.

Even if one does use localization, the *size* of localization required becomes purely geometric, given by conjugate points.

Theorem (Rough version, V., 2020)

Suppose that there is a function $\tilde{x}$ with strictly convex level sets and non-degenerate differential on X, and suppose that geodesics do not have points conjugate to their points of tangency to the level sets of $\tilde{x}$.
Then the suitably defined modified normal operator of the geodesic X-ray transform is a left-invertible elliptic pseudodifferential operator on distributions supported in X.

This is a linear problem that has nonlinear applications. For instance, corresponding to the X-ray transform of symmetric 2-tensors along geodesics as the linearization (but actually using the Stefanov-Uhlmann pseudolinearization):

Question

Let d be the distance function on a domain X in a Riemannian manifold $\tilde{X}$. Does $d|_{\partial X \times \partial X}$ determine g on X?

There is an obvious obstacle: diffeomorphisms (changes of coordinates) preserving the boundary; the question is if this is all.

The conformal problem is the same question if one knows that g is in the conformal class of g_0, i.e. $g = c^{-2} g_0$. No diffeomorphism obstacles then. This has linearization given by a scalar X-ray transform.

One expects that there are also some geometric obstacles.

Recall that the distance function is given by length minimizing geodesics. So changing the metric in some region away from the boundary to make the region appear large will cause distance minimizing geodesics avoid it.

Michel's conjecture (early 1980s) is that for *simple* manifolds, i.e. manifolds with strictly convex boundary on which the Riemannian exponential map is a diffeomorphism around each point, the manifold is determined by the boundary distance function, up to the diffeomorphism invariance.

Very little was known except in the real analytic case (Vargo 2009) and 2-dimensions (Croke, Otal 1990, Pestov and Uhlmann 2005) which have a different flavor.

On simple manifolds the boundary distance function determines the *lens relation*, which in addition to the unique geodesic length between the boundary points, also contains the *direction* in which these geodesics leave the boundary.

The natural generalization of the travel-time problem to non-simple manifolds is if the lens relation determines the metric, up to diffeomorphisms.

There is a prototypical result due to Herglotz (1905), Wiechert and Zoeppritz (1907): for the ball, in the fixed conformal class problem, with *spherically symmetric* sound speed (function of r only), the sound speed is determined if $(r/c(r))' > 0$; one can use 1-dimensional techniques (Abel-type transform).

This condition means *exactly* that the spheres of radius r are strictly convex for the metric given by the sound speed!

Theorem (Stefanov-Uhlmann-V., 2017)

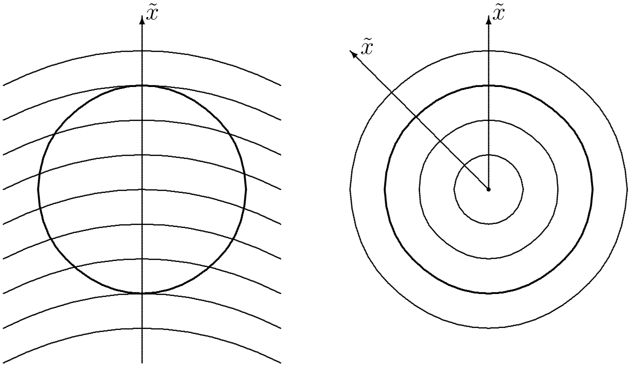
Suppose $n \geq 3$, X compact with boundary, and let $g, \tilde{g}$ be Riemannian metrics.

Assume that ∂X is strictly convex with respect to both g and $\tilde{g}$.

Assume also that one has a function $\tilde{x}$ whose level sets are nondegenerate, strictly concave for g from the superlevel sets, and $\{\tilde{x} \geq 0\} \cap X \subset \partial X$.

If the lens relations of g and $\tilde{g}$ are the same then there exists a diffeomorphism ψ fixing ∂X such that $g = \psi^* \tilde{g}$.

Such $\tilde{x}$ exists e.g. if X is simply connected with non-positive sectional curvature, or more generally if it has no focal points.



The RHS has a degenerate level set at the origin, but is covered by a slightly modified version of the theorem... and then the assumption holds if X has non-negative curvature! (Paternain, Salo, Uhlmann, Zhou, 2016).

The result is proved by *localization*: one only needs stricts convexity of ∂X and constructs $\tilde{x}$ locally.

Other nonlinear results concern for instance the recovery of elastic parameters in anisotropic elasticity, concretely transversely isotropic elasticity. In this case the related X-ray transform is along integral curves of the Hamilton vector field corresponding to different types of wave speeds (SH, qS and qP). One way this is an interesting setting is that the curves are naturally parameterized by phase space, S^*X, and the map $S^*X \to SX$ is often not injective (wave triplication).

With de Hoop and Uhlmann, we obtained results on the recovery of *certain* elastic parameters; these are basically analogues of the results discussed below.

However, there are new results by Joey Zou which go significantly beyond this framework by developing a new parabolic (rather than elliptic) version of this technique, in analogy with the work of Boutet de Monvel.

Back a little to a linear but geometric inverse problem: consider the large ends of cones: $dr^2 + r^2 h$, h a metric on the cross section Y, and more generally metrics asymptotic to these as $r \to \infty$. An example, if h is the unit sphere, is the Euclidean metric. This metric induces Melrose's scattering algebra, already referred to above.

In 2019 Guillarmou, Lassas and Tzou showed that if there are no conjugate points (or the curvature is negative) then the X-ray transform is injective on appropriate polynomially weighted spaces.

A remarkable paper of Guillarmou, Mazzucchelli and Tzou also showed that asymptotically Euclidean metrics without conjugate points are flat, so the absence of conjugate points is a somewhat strong hypothesis.

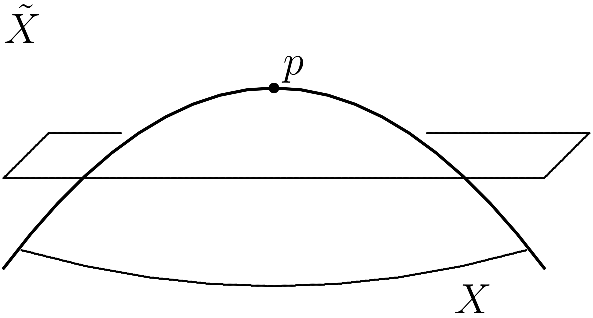
Methods close to those discussed in this talk, but using a yet different pseudodifferential algebra, called 1-cusp, and its semiclassical version, show:

Theorem (Zachos-V, 2020)

Suppose M is a manifold of dimension ≥ 3, g is an asymptotically conic metric on M for which the cone's cross section (link) is close to a standard unit sphere. Then on a collar neighborhood of infinity the geodesic X-ray transform is injective on appropriate Gaussian weighted function spaces.

For the rest of this talk we consider the scalar transform.

- If $\tilde{X} = \mathbb{R}^n$ with the Euclidean metric, then the Euclidean X-ray transform is well-understood (Radon transform, Fourier transform).
- In fact, if $n \geq 3$, one can also invert it by slicing $\mathbb{R}^n$ by 2-planes, and inverting the transform just using the geodesics in the 2-planes.



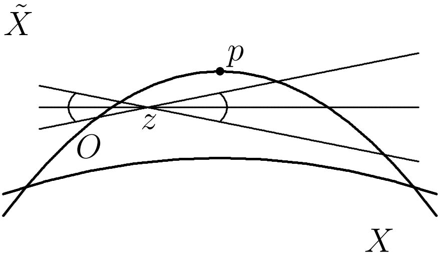
Notice that for X a strictly convex domain, $n \geq 3$, if $p \in \partial X$, then one can just use planes parallel to the tangent plane at p: this means that only local geodesics are used to reconstruct f near p.

Question

Can one do some weaker version of this in general for the O-local problem?

It turns out that answering this question for certain ‘small’ sets O actually yields an answer for the global question as well in many cases by a ‘layer stripping’ argument.

Thus, one would like O small, e.g. a small neighborhood of a point $p \in \partial X$. In general, there are no O-local geodesics. However, if X is strictly convex (in the Hessian sense), there are plenty which are ‘almost tangent’ to the boundary.



One needs more: in order to conclude even the local injectivity of I, one needs that there are enough O-local geodesics through *every point* of O.

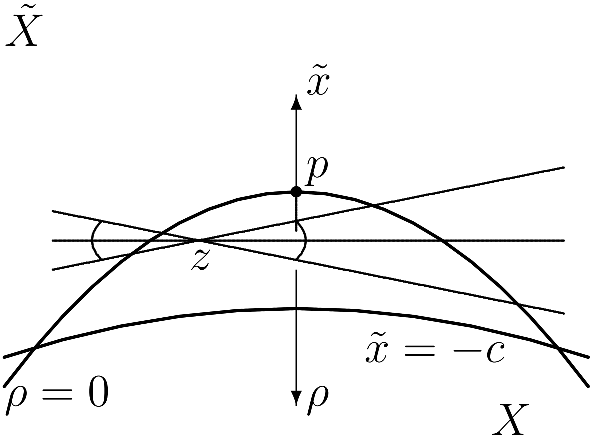
We can achieve this by letting O be the super-level set of a function $\tilde{x}$ on $\tilde{X}$ with level sets strictly concave from the (non-degenerate) super-level sets, normalized so that

- $\{\tilde{x} \geq 0\} \cap \overline{X} \subset \partial X$,
- $\{\tilde{x} \geq 0\} \cap \partial X \neq \emptyset$.

For a (small) constant c, let

$$O_c = \{\tilde{x} > -c\} \cap \overline{X}.$$

We call $\tilde{x} = -c$ the *artificial boundary*; it is much more important in the analysis than the actual boundary which simply constrains supports.



Example (ρ a boundary defining function):

$$\tilde{x}(z) = -\rho(z) - \epsilon|z - p|^2,$$

with $\epsilon > 0$ small; $O_c = \{z : \ \rho(z) \geq 0, \ \rho(z) + \epsilon|z - p|^2 < c\}$.

The local result is that for $c > 0$ small, If determines f:

Theorem (Uhlmann-V., 2012)

Suppose $n \geq 3$. Let $s \geq 0$. For $O = O_c$ as above, for sufficiently small $c > 0$, $I|_{^O SX}$ is injective on $H^s(O)$.

Further, with $H^s_F(O) = e^{F/(\tilde{x}+c)} H^s(O)$, and $H^s(^O SX)$ the restriction of elements of $H^s(S\tilde{X})$ to $^O SX$,

$$\|f\|_{H_F^{s-1}(O)} \leq C \|If|_{^O SX}\|_{H^s(^O SX)}.$$

This theorem has been extended by Hanming Zhou to more general families of curves.

Notice that the estimate is on a space giving very weak control at the 'artificial boundary' (exponential), but the latter is rather arbitrary.

Here one can also work with $\rho \geq \rho_0$ in place of $\overline{X}$ and replace O_c by

$$\{\rho \geq \rho_0\} \cap \{\tilde{x} > -c\},$$

and one can also perturb g; the results hold for small c, ρ_0 with a *uniform* constant C; this allows for the 'layer stripping' procedure below.

Moreover, the method is constructive, so one obtains a method for computing f from If, of which the least explicit part is summing a convergent Neumann series!

One can globalize the result via a layer stripping for $\overline{X}$ compact.

Suppose that one has a function $\tilde{x}$, and $T > 0$ such that for $t \in [0, T)$, $d\tilde{x} \neq 0$ on $\Sigma_t = \{\tilde{x} = -t\}$, and Σ_t is strictly concave from the super-level sets, $\tilde{x} \leq 0$ on X.

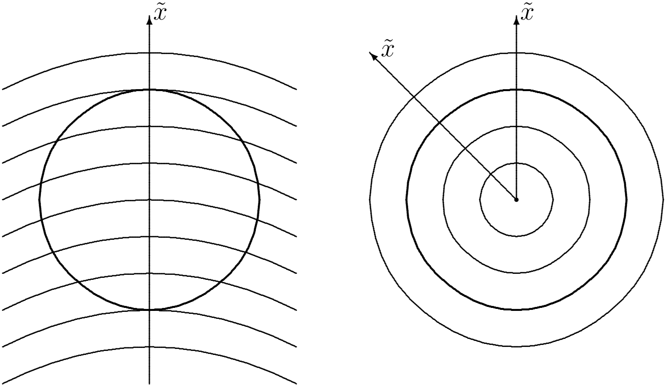
Theorem

If $\overline{X} \cap \{\tilde{x} \leq -T\}$ has measure 0, then I is injective on $L^2(X)$.

If $\overline{X} \cap \{\tilde{x} \leq -T\}$ has empty interior, then I is injective on $H^s(X)$, $s > n/2$.

In both cases, one has a stability result and reconstruction in sets $\{\tilde{x} \geq -T'\}$, $T' < T$.

The idea is to first recover $f|_{\tilde{x} \geq -t_1}$ for $t_1 > 0$ small by the theorem.



Then one can go on to recover f in $\{-t_1 \geq \tilde{x} \geq -t_2\}$ if $t_2 > t_1$ is close to t_1 by again applying the theorem, using that one knows the integrals of f along geodesic segments in $\{-t_1 \geq \tilde{x} \geq -t_2\}$ with endpoints on $\tilde{x} = -t_1$, as one knows the integrals of f along the rest of these geodesic segments with endpoints on ∂X, and proceed inductively.

Prior *global* uniqueness and stability results (no local results even for metrics conformal to the Euclidean one!):

- Mukhometov in 2 dimensions for simple manifolds (strictly convex boundary, the exponential map is a diffeomorphism),
- Mukhometov and Romanov in the higher dimensional analogue of this setting,
- Frigyik, Stefanov and Uhlmann in the real analytic simple metric setting, $n \geq 3$.
- Reconstruction is only known in a few cases, such as symmetric spaces (Helgason), and real analytic curves (Frigyik, Stefanov, Uhlmann).
- In the spherically symmetric case, the convexity condition is that of Herglotz, Wiechert and Zoeppritz: $\frac{d}{dr}(r/c(r)) > 0$.
- Our condition allows conjugate points, and is implied by the absence of focal points.
- More recently, Guillarmou, partly with collaborators, also proved X-ray injectivity and related results in negative curvature, e.g. proving deformation rigidity in general.

In order to motivate the proof of the local theorem, consider a localizer (multiplication operator) Q on $S\tilde{X}$, and the 'normal operator' I^*QI. (Technically, a replacement for I^* is used.)

Under a non-conjugacy condition for the geodesics on supp Q, which is always satisfied locally, $I^*QI \in \Psi^{-1}(\tilde{X})$. Here $\Psi^m(\tilde{X})$ stands for *pseudodifferential operators* of order m on $\tilde{X}$.

Recall that the principal symbol captures the 'leading behavior' as $|\zeta| \to \infty$, which is what matters for capturing an operator modulo relatively compact operators on a manifold without boundary. Moreover, a non-vanishing (classical) principal symbol states that the operator is elliptic, indeed Fredholm between appropriate spaces, and has an approximate inverse.

For $\zeta \in T^*_z\tilde{X} \setminus o$, writing the coordinate on the kernel (in $S_z\tilde{X}$) of ζ as $v^\perp$, the principal symbol of I^*QI is a multiple of

$$|\zeta|^{-1} \int q(z, v^\perp)\sigma(z, v^\perp)\, dv^\perp$$

for a positive density σ, $\Rightarrow$ (Stefanov and Uhlmann) I^*QI is elliptic provided $q \geq 0$ and for all ζ there is a vector v annihilated by it such that $q(z, v) \neq 0$.

This suffices for a semi-Fredholm theory: one sees that the nullspace of I in $\dot{H}^s$ (supported distributions) is finite dimensional and one has a stability estimate for f in a complementary subspace of this finite dimensional space:

$$\|f\|_s \leq C\|I^*QIf\|_{s+1},$$

with the latter norm taken on an enlarged version of the domain X.

In order to remove the possibility of this finite, but possibly huge, dimensional nullspace, it is useful to have a small parameter.

There are at least two options; both of these use a convex foliation, denoted by x, writing tangent vectors as $\lambda\,\partial_x + \omega \cdot \partial_y$, with y coordinates along the level sets, and λ sufficiently small, i.e. working with almost x-level set tangent geodesics.

A key point is that if $n \geq 3$ then for any $\zeta \in S^*X$ there is a v annihilated by it that is tangent to the level set of x.

- Work in small (thin) region via adding an artificial boundary; in this case the thinness is the small parameter that appears analytically due to the rapid vanishing at the artificial boundary of the errors of the parametrix construction. In this case Q depends on the distance to the artificial boundary, $x = 0$; it can be taken as $\chi(\lambda/x)$, χ compactly supported. This conservatively assures that geodesics do not cross the artificial boundary, and is the essence of the approach of Uhlmann-V.
- Work globally, but with Q depending on a semiclassical parameter h. In this case there need not (but can!) be an artificial boundary, and one takes Q to be $\chi(\lambda/h^{1/2})$. This is the new approach.

In the combined approach the cutoff is $\chi(\lambda/(xh^{1/2}))$.

In both cases, one also uses exponential weights, i.e. works with $e^{-\Phi} I^* Q I e^{\Phi}$:

- $e^{\Phi} = e^{F/x}$ for the artificial boundary, i.e. allowing this much growth, so the estimates are weaker when $x > 0$ is small;
- $e^{\Phi} = e^{-x/h}$ for the semiclassical version, again allowing this much growth, so the estimates are weaker as x decreases (matching case above).

With the strict convexity of the level sets of x, this means that the conjugated operator puts little weight (vanishing portion as the parameters tend to 0) to the not-completely-local portion of the Schwartz kernel.

In the semiclassical setting, one needs to use semiclassical foliation pseudodifferential operators $A \in \Psi^m_{\hbar,\mathcal{F}}$, i.e. operators that (in)formally have the form $A = a(x, y, hD_x, h^{1/2}D_y, h)$, built of h times all vector fields plus $h^{1/2}$ times foliation tangent vector fields. This corresponds to the quantization

$$(Au)(x,y,h) = (2\pi)^{-n}h^{-n/2-1/2}\int e^{i((x-x')\xi/h+(y-y')\cdot\eta/h^{1/2})} a(x,y,\xi,\eta,h)u(x',y')\,d\xi\,d\eta\,dx'\,dy',$$

where a is a standard symbol smoothly (or simply with an expansion at $h=0$) depending on h:

$$|D_z^\alpha D_\zeta^\beta a(z,\zeta,h)| \leq C_{\alpha\beta}\langle\zeta\rangle^{m-|\beta|},$$

$z = (x,y)$, $\zeta = (\xi,\eta)$. This still allows for a symbol calculus, gaining $h^{1/2}$ in each step of the asymptotic expansions.

Now the principal symbol is both in the sense of $|\zeta| \to \infty$ (standard), and $h \to 0$ (semiclassical), i.e. the space is $S^m / h^{1/2} S^{m-1}$.

If the principal symbol is elliptic, then there is a parametrix G, and its error $E = GA - I$ is $O(h^\infty)$, so $I + E$ is invertible for small h.

Proposition

For appropriate $\chi \geq 0$,

$$A_h = e^{x/h} I^* Q I e^{-x/h}$$

is in $\Psi^{-1}_{\hbar,\mathcal{F}}$, with the full (standard+semiclassical) principal symbol elliptic.

Although this is in a localized setting (to a neighborhood of X in $\tilde{X}$), a standard modification of the parametrix argument, as in the earlier work with Uhlmann, still gives a left inverse on supported distributions in X.

The artificial boundary analogue is:

Proposition

For appropriate $\chi \geq 0$, *and for* $F > 0$,

$$A_F = e^{-F/x} I^* Q I e^{F/x}$$

is in $\Psi_{\mathrm{sc}}^{-1}(\{x \geq 0\})$, *with the full (standard+boundary) principal symbol elliptic.*

Here $\Psi_{\mathrm{sc}}(M)$ is Melrose's algebra of scattering ps.d.o.'s – locally this is just a standard Hörmander algebra, and goes back to Parenti and Shubin in the $\mathbb{R}^n$ case.

The key point is that the geometrically finite artificial boundary analytically is at infinity thus tools of asymptotic analysis become available!

In our compactified world we have (in)formally the form $A = a(x, y, x^2 D_x, xD_y) \in \Psi_{\mathrm{sc}}^{m,l}$:

$$(Au)(x,y) = (2\pi)^{-n} \int e^{i((x-x')\tilde{\xi}/x^2 + (y-y')\cdot\tilde{\eta}/x)} a(x,y,\tilde{\xi},\tilde{\eta})u(x',y')\,d\tilde{\xi}\,d\tilde{\eta}\,\frac{dx'\,dy'}{(x')^{n+1}},$$

where a is a standard symbol conormal to $x = 0$:

$$|(xD_x)^j D_y^\alpha D_{\tilde{\zeta}}^\beta a(x,y,\tilde{\zeta})| \leq C_{j\alpha\beta}\langle\tilde{\zeta}\rangle^{m-|\beta|}x^{-l},$$

$\tilde{\zeta} = (\tilde{\xi}, \tilde{\eta})$. This also allows for a symbol calculus as $x \to 0$, gaining x in each step of the asymptotic expansions.
Cf. the semiclassical version, with the analogy being $x \sim h^{1/2}$.

$$(Au)(x,y,h) = (2\pi)^{-n}h^{-n/2-1/2} \int e^{i((x-x')\xi/h + (y-y')\cdot\eta/h^{1/2})} a(x,y,\xi,\eta,h)u(x',y')\,d\xi\,d\eta\,dx'\,dy'!$$

The more familiar version for 'classical' microlocal analysts is:

- When $M = \overline{\mathbb{R}^n}$, the radial compactification of $\mathbb{R}^n$ as a ball $\mathbb{B}^n$, $\Psi_{\mathrm{sc}}(M)$ is just a standard ps.d.o. algebra with $\Psi_{\mathrm{sc}}^{m,l}(M)$ corresponding to symbols $a \in S^{m,l}$:

$$|(D_z^\alpha D_\zeta^\beta a)(z,\zeta)| \leq C_{\alpha\beta}\langle z\rangle^{l-|\alpha|}\langle\zeta\rangle^{m-|\beta|}.$$

- In general, one can localize on M, and reduce to this case.
- Thus, if the identification is with the region where $z_n > C|z_j|$ for $j < n$, the identification is $x = z_n^{-1}$ and $y_j = z_j/z_n$.
- The standard principal symbol is a modulo $S^{m-1,l}$; this does *not* capture compactness.
- A_F is elliptic, essentially for the same reasons as in the Stefanov-Uhlmann argument.

- To capture full ellipticity, one also needs to consider a modulo $S^{m,l-1}$, i.e. altogether $S^{m,l}/S^{m-1,l-1}$. This is the *boundary principal symbol*.
- If the boundary principal symbol is also invertible, then $\mathrm{Op}(a)$ has a parametrix G with a Schwartz error (so in particular compact error).
- In our setting we can choose χ to achieve this: χ approximates a Gaussian, $s \mapsto e^{-s^2/(2\nu)}$, with ν appropriately chosen.
- One can actually make the error $GA_F - \mathrm{Id}$ small in, say, L^2, by making $c > 0$ small. Then the error can be iterated away by a Neumann series.

The treatment of the non-linear, fixed conformal class is completely similar, with a microlocally weighted transform.

Thank you!