Asymptotically Euclidean metrics without conjugate points are flat

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Conjugate points

Let (M, g) be a complete Riemannian manifold.

We say that $x, x' \in M$ are conjugate points if:

- $\exists$ a geodesic $\gamma(t)$ so that $\gamma(0) = x$, $\gamma(1) = x'$
- $\exists$ a Jacobi vector field Z along γ so that $Z(0) = 0$ and $Z(1) = 0$.

Recall: Jacobi field Z means

$$\nabla^2_{\partial_t} Z(t) + R(Z(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0$$

if R is curvature tensor.

An alternative definition for conjugate points:

Let $\varphi_t : SM \to SM$ is the geodesic flow on the unit tangent bundle SM, $\pi : SM \to M$ the projection and $V = \ker d\pi$ the vertical bundle.

Then x and x' are conjugate points if:

- $\exists v \in S_x M, v' \in S_{x'} M$, $\exists t_0 \geq 0$, $\varphi_{t_0}(x, v) = (x', v')$
- $d\varphi_{t_0}(V) \cap V \neq 0$.

Property: no conjugate points implies that $\forall x, x'$, there is a unique geodesic joining x, x' (if M simply connected).

Asymptotically Euclidean metrics

On $\mathbb{R}^n$, a (smooth) Riemannian metric g is said asymptotically Euclidean to order m if

$$g = g_0 + \frac{1}{|x|^m} \sum_{i,j=1}^{n} a_{ij}\Big(\frac{1}{|x|}, \frac{x}{|x|}\Big) dx_i dx_j$$

where $g_0 = \sum_{j=1}^n dx_i^2$ is Euclidean and $a_{ij} \in C^\infty([0,1] \times \mathbb{S}^{n-1})$.

This means homogeneous expansion in powers $|x|^{-m-j}$ at infinity.

Main result

Theorem (G-Mazzucchelli-Tzou '19)

An asymptotically Euclidean metric on $\mathbb{R}^n$ to order $m \geq 3$ with no conjugate points is isometric to the Euclidean metric g_0.

Remark: only expansion to finite order in powers of $|x|^{-1}$ is necessary in the proof (depending on n).

Remark: there are metrics $g = dr^2 + f(r)^2 d\theta^2$ on $\mathbb{R}^2$ with $f(r) = ar$ for $a > 1$ in the region $r > 2$ such that g has non-positive curvature, thus no conjugate points. These are asymptotically conic.

Previous related results

- Green-Gulliver ('85), Croke ('90): Compact perturbations of Euclidean metrics with no conjugate points are flat.
- Innami ('86): A metric on $\mathbb{R}^n$ with integrable Ricci curvature, $\int_{\mathbb{R}^n} \mathrm{Scal}_g dv_g = 0$ and no conjugate points must be flat.
- Michel ('81), Gromov ('83): a metric with no conjugate points, with same boundary distance on strictly convex ball in $\mathbb{R}^n$ is flat. In other words: Euclidean metric is boundary rigid
- Hopf('48), Burago-Ivanov ('94): a metric on $\mathbb{T}^n$ with no conjugate points is flat.

Radial coordinates

First, there exists radial coordinate for g near infinity:

$\exists$ a diffeo $\psi : [1,\infty)_r \times \mathbb{S}^{n-1} \to \mathbb{R}^n \setminus K$ (with K compact set) such that

$$\psi^* g = dr^2 + r^2 h(r;\theta, d\theta)$$

with $h(r)$ family of metrics on $\mathbb{S}^{n-1}$ with asymptotic expansion as $r \to \infty$

$$h(r) \simeq d\theta^2_{\mathbb{S}^{n-1}} + \sum_{j=0}^{\infty} \frac{1}{r^{m+j-2}} h_j(\theta, d\theta)$$

Denote $\rho := 1/\psi_*(r)$. Adding a sphere at $\rho = 0$ (ie. $r = \infty$) gives a radial compactification $\overline{\mathbb{R}}^n$ of $\mathbb{R}^n$ and

$$g \simeq \frac{d\rho^2}{\rho^4} + \sum_{j=0}^{\infty} \rho^{m+j-2} h_j(\theta, d\theta)$$

Scattering map

Using the asymptotic structure of g, one can show the

Lemma

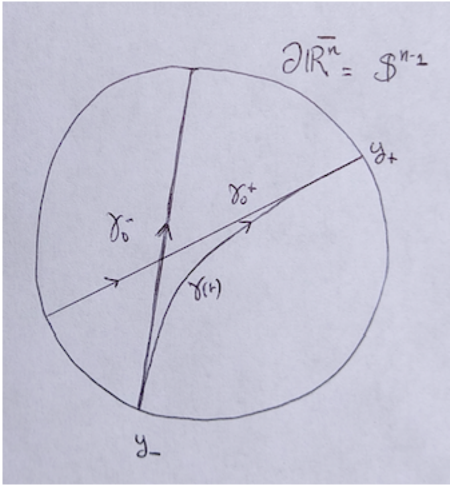
For each complete g-geodesic γ, there is $y_\pm \in \mathbb{S}^{n-1} = \partial\overline{\mathbb{R}}^n$ and $\eta_\pm \in T^*_{y_\pm}\mathbb{S}^{n-1}$ such that

$$\lim_{t\to\pm\infty}(\gamma(t),\dot\gamma(t)^\flat) = \Big(y_\pm, \mp\frac{d\rho}{\rho^2} + \eta_\pm\Big) = \Big(y_\pm, \pm dr + \eta_\pm\Big),$$

where $\dot\gamma(t)^\flat = g(\dot\gamma,\cdot)$. There exist Euclidean geodesics $\gamma_0^\pm$ such that

$$\lim_{t\to\pm\infty} d_{g_0}(\gamma(t),\gamma_0^\pm(t)) = 0, \qquad \lim_{t\to\pm\infty}\big(\dot\gamma(t) - \dot\gamma_0^\pm(t)\big) = 0. \tag{1}$$

Call $(y_\pm,\eta_\pm) \in T^*\mathbb{S}^{n-1}$ the past/future asymptotic vectors of γ.
Example: the Euclidean lines $\gamma_\eta(t) := (\eta, t) \in \mathbb{R}^{n-1}\times\mathbb{R}$ have past asymptotic vector $(0,\dots,0,-1;\eta) \in T^*\mathbb{S}^{n-1}$ and future asympt vector $(0,\dots,0,1;-\eta) \in T^*\mathbb{S}^{n-1}$.


$\partial \overline{\mathbb{R}^n} = \mathbb{S}^{n-1}$
y_+
γ_0^+
γ_0^-
$\gamma(t)$
y_-

For each $(y_-, \eta_-) \in T^*\mathbb{S}^{n-1}$ there is a unique Euclidean line γ_0^- so that

$$\lim_{t\to-\infty}(\gamma_0^-(t), (\dot\gamma_0^-(t))^\flat) = \Big(y_-, \frac{d\rho}{\rho^2} + \eta_-\Big).$$

Converse to the Lemma also holds: for each Euclidean line $\gamma_0^-(t)$, there is a unique g-geodesic $\gamma(t)$ asymptotic in the past, ie. such that

$$\lim_{t\to\pm\infty} |\gamma(t) - \gamma_0^-(t)| = 0, \quad \lim_{t\to-\infty} \big(\dot\gamma(t) - \dot\gamma_0^-(t)\big) = 0.$$

Conclusion: for each $(y_-, \eta_-) \in T^*\mathbb{S}^{n-1}$, there is a unique g-geodesic γ with past asymptotic vector (y_-, η_-).

The scattering map $S_g : T^*\mathbb{S}^{n-1} \to T^*\mathbb{S}^{n-1}$ is defined by

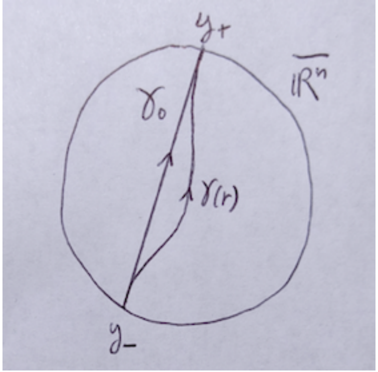
$$(y_-, \eta_-) \mapsto (y_+, \eta_+)$$

where (y_+, η_+) is the future asymptotic vector of the g-geodesic γ, whose past asymptotic vector is (y_-, η_-).

Fact: g has same scattering map as Euclidean metric g_0 if and only if for all g geodesic $\gamma(t)$, there is a Euclidean line $\gamma_0(t) = x + tv$ so that

$$\lim_{t\to-\infty} |\gamma(t) - \gamma_0(t)| = 0, \quad \lim_{t\to-\infty} |\dot{\gamma}(t) - v| = 0$$
$$\lim_{t\to+\infty} |\gamma(t) - \gamma_0(t + t_0)| = 0, \quad \lim_{t\to+\infty} |\dot{\gamma}(t) - v| = 0$$

for some $t_0 \in \mathbb{R}$.


y_+
$\overline{\mathbb{R}^n}$
γ_0
$\gamma(r)$
y_-

First reduction

We can show the

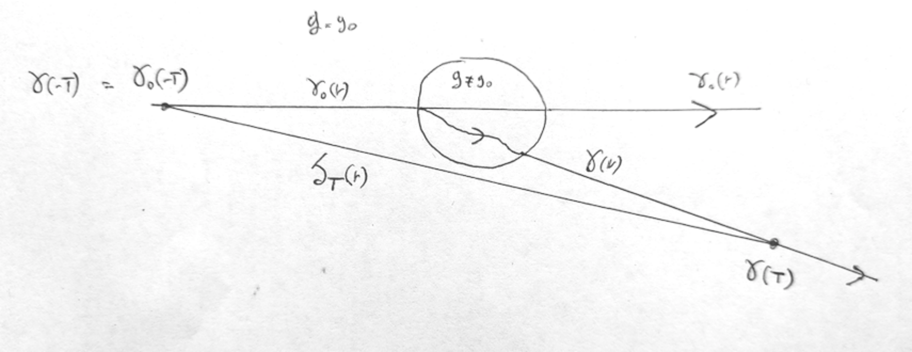
Theorem

An asymptotically Euclidean metric g on $\mathbb{R}^n$ with no conjugate points must have the same scattering map as the Euclidean metric g_0.

We use Croke's ideas, but applied to our case.

Main Difficulties: in order to get quantitative estimates, we need to analyse carefully the geodesic flow near infinity for the metric g.

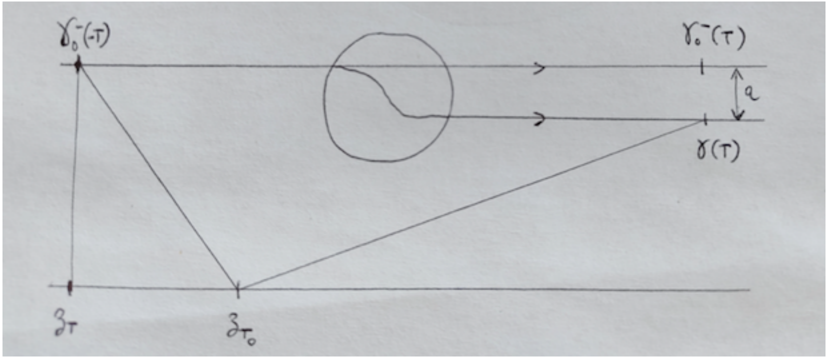
Idea of proof (based on Croke's ideas)



If $g = g_0$ outside compact set: for large $T \gg 1$, can show that if $\dot\gamma(t) \neq \dot\gamma_0(t)$ for t large,

$$\ell_g(\zeta_T) < \ell_g(\gamma([-T, T]))$$

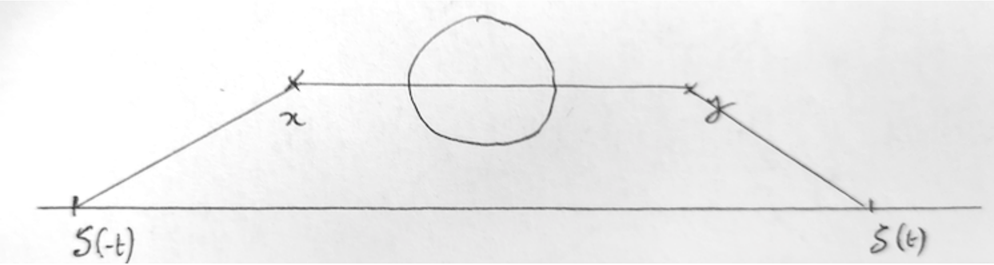
thus $\exists$ two g-geodesics with same endpoints, contradicting no conjugate points.
Same argument (but quantitative) if $g \neq g_0$ outside a compact.



Case $g = g_0$ outside a compact B: T_0 fixed, we can play with triangular inequalities (using z_{T_0}) and Euclidean geometry to obtain

$$a^2 \leq d_{g_0}(\gamma_0^-(-T), \gamma(T))^2 - d_g(\gamma_0^-(-T), \gamma(T))^2$$

Same holds modulo $O(T^{-m+1})$ if asymptotically Euclidean to order m.



If $g = g_0$ outside compact set B, for all x, y outside B:

$$\begin{aligned}|x-y| &= \lim_{T\to\infty} |\zeta(T)-\zeta(-T)| - |\zeta(-T)-x| - |\zeta(T)-y| \\ &= \lim_{T\to\infty} d_g(\zeta(T),\zeta(-T)) - d_g(\zeta(-T),x) - d_g(\zeta(T),y) \\ &\leq \lim_{T\to\infty} d_g(x,y) = d_g(x,y)\end{aligned}$$

Thus

$$a^2 \leq d_{g_0}(\gamma_0^-(-T),\gamma(T)) - d_g(\gamma_0^-(-T),\gamma(T))^2 \leq 0$$

General case: Make this quantative, up to $O(T^{-m+1})$ error, and let $T \to \infty$.

The second step is to prove a volume comparison

Theorem

If g is asymptotically Euclidean to order m without conjugate points, then for each x_0, when $R \to \infty$

$$\mathrm{Vol}_g(B_g(x_0, R)) - \mathrm{Vol}_g(B_{g_0}(x_0, R)) = O(R^{n-m+1}).$$

To prove this, we show that there is a conjugation $\Theta : S_g\mathbb{R}^n \to S_{g_0}\mathbb{R}^n$ conjugating the geodesic flow of g and g_0

$$\Theta^{-1} \circ \varphi_t^{g_0} = \varphi_t^{g} \circ \Theta^{-1}$$

such that

$$\|\Theta - \mathrm{Id}\|_{C^1(T(\mathbb{R}^n \setminus B(x_0,R))} = O(R^{-m+1})$$

and use some Santalo type identity for volumes.

A digression

Croke('92) proved that for universal covers of closed manifolds (M, g) without conjugate points,

$$\liminf_{R\to\infty} \frac{\mathrm{Vol}_g(B_g(x_0, R))}{\mathrm{Vol}_{g_0}(B_{g_0}(x_0, R))} \geq 1$$

(where g_0 is Euclidean metric) with equality if and only if g is flat.
This was also proven in dimension 2 by Bangert-Emmerich ('12) for metrics metrics witout conjugate points on $\mathbb{R}^2$, but the same problem is open for $n > 2$.

Rigidity when m is large

We take the cylinder

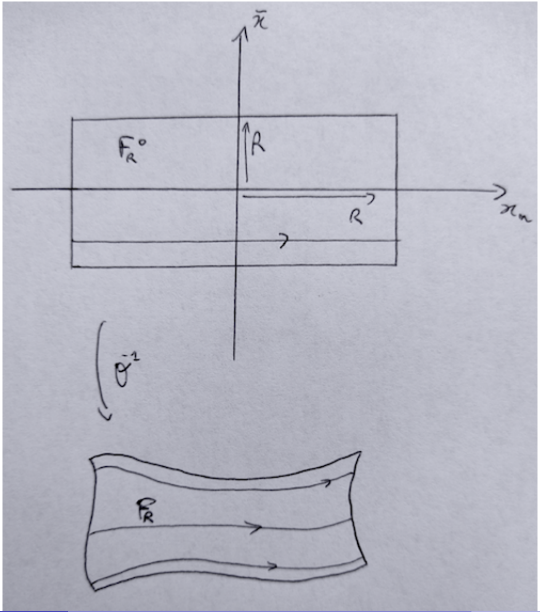
$$F_R^0 := \{x = (\bar{x}, x_n) \in \mathbb{R}^n \,|\, |x_n| \leq R, |\bar{x}| \leq R\}$$

foliated by Euclidean geodesics with tangent vectors ∂_{x_n} and its image

$$F_R := \pi(\Theta^{-1}(F_R^0 \times \{\partial_{x_n}\})).$$

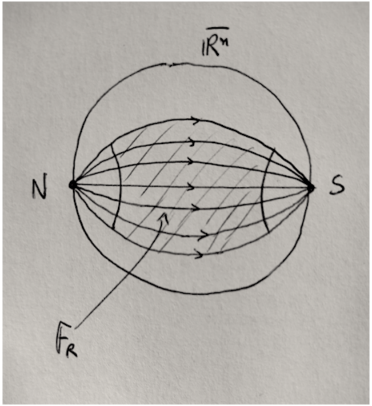
Just as before for $B_g(x_0, R)$, we can prove that

$$\mathrm{Vol}_g(F_R) - \mathrm{Vol}_{g_0}(F_R^0) = O(R^{n-m+1})$$



$\bar{x}$
F_R°
R
R
x_n
θ^{-1}
F_R

In terms of the compactification $\overline{\mathbb{R}}^n$, the geodesics foliating F_R correspond to the endpoints $(y, \eta) \in T^*\mathbb{S}^{n-1}$ given by $(N, \eta) \in T^*_N\mathbb{S}^{n-1}$ with $|\eta| \leq R$.



We can write

$$\mathrm{Vol}_g(F_R) = \int_{|\eta|<R} \int_{|t|\leq R} \det(A_\eta(t)) d\eta dt$$

where $A_\eta(t)$ is a matrix solution of the Jacobi equation

$$\partial_t^2 A_\eta + R_\eta A_\eta = 0$$

along the g-geodesic γ_η, where R_η is the curvature tensor viewed as an endomorphism. Moreover

$$A_\eta(t) = \mathrm{Id} + O(t^{-m}), \text{ as } t \to -\infty$$

$$A_\eta(t) = H_\eta + O(t^{-m}), \text{ as } t \to \infty$$

for some $H_\eta \in SO(n)$ with $H_\eta = \mathrm{Id} + O(1/|\eta|)$ as η large.

Playing with convexity properties of the map $X \mapsto \det(X)^{-1/2}$ on symmetric matrices and Hölder inequalities (with the case of equality) as in the proof of Blaschke conjecture by Berger, and using estimates on the Jacobi matrix A_η, we can prove

Proposition

If g is asymptotically Euclidean to order $m > n+1$ with no conjugate points, then the matrix A_η is equal to Id and the curvature tensor of g must then vanish.

Boundary determination

The last step is to prove the following

Theorem

Let g be asymptotically Euclidean to order $m \geq 3$ and assume that it has the same scattering ma as te Euclidean metric, $S_g = S_{g_0}$. Then g is asymptotically Euclidean to all order.

In radial coordinates $(\rho, y) \in (0, \varepsilon) \times \mathbb{S}^{n-1}$ near infinity, and writing the covectors $\xi = \xi_0 \frac{d\rho}{\rho^2} + \eta.dy$, the geodesics staying in $\rho \leq \varepsilon$ (close to infinity) look at first order like

$$(\rho(s), \xi_0(s), y(s), \eta(s)) = (\sin(s), \cos(s), e^{sH_0}(y, \eta))$$

for $s \in (0, \pi)$ and H_0 is the Hamilton vector field on the sphere (cf Melrose).

By expanding the flow equation at ∞ (ie $\rho = 0$) for the metric

$$g = g_0 + \rho^{m-2} h_m(\theta, d\theta) + O(\rho^{m-1}),$$

we obtain some attenuated ray transform of the tensor h_m on $\mathbb{S}^{n-1}$ with weights given by polynomials on $\mathbb{S}^{n-1}$.

In the end, this finally reduces to some analysis of X-ray transform on symmetric 3 tensors (applied to $\nabla^s h_m$) on the sphere, see Estezet, Goldschmidt, Eastwood. We also use some techniques of Joshi-Sa Barreto in inverse quantum scattering.