The non-Abelian X-ray transform

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Outline

1) Overview and injectivity results.

2) Neutron tomography and a statistical algorithm for inversion.

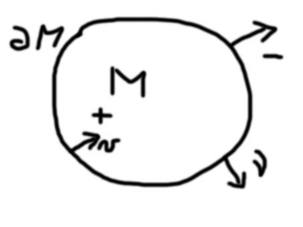
3) Proof of injectivity and factorization for Loop Groups.

Setting

- (M, g) is a compact Riemannian manifold with boundary ∂M.
- $SM = \{(x, v) \in TM : \ |v| = 1\}$ is the unit sphere bundle with boundary $\partial(SM)$.
- Outfllux and influx boundaries:

$$\partial_{\pm} SM = \{(x, v) \in \partial(SM) \ : \ \pm\langle v, \nu\rangle \leq 0\},$$

  where ν is the the outer unit normal vector.
- ∂M is strictly convex (positive definite second fundamental form).



We let $\tau(x, v)$ be the first time when a geodesic starting at (x, v) leaves M.

Definition. We say (M, g) is non-trapping if $\tau(x, v) < \infty$ for all $(x, v) \in SM$.

By Morse theory a non-trapping manifold with strictly convex boundary is contractible (Serre 1951).

Most of the time will assume that M is simple: it is non-trapping and it has no conjugate points.

Examples: Strictly convex domains in the plane and small C^2 perturbations of them.

Non-abelian X-ray

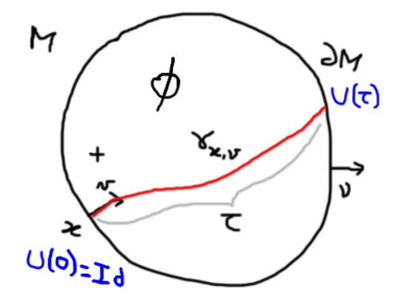
Let $\Phi \in C^{\infty}(M, \mathbb{C}^{n\times n})$ be a matrix field.

Given a unit-speed geodesic $\gamma : [0, \tau] \to M$ with endpoints $\gamma(0), \gamma(\tau) \in \partial M$, we consider the matrix ODE

$$\dot{U} + \Phi(\gamma(t))U = 0, \qquad U(0) = \mathrm{Id}.$$

We define the scattering data of Φ on γ to be $C_{\Phi}(\gamma) := U(\tau)$.

When Φ is scalar, we obtain $\log U(\tau) = -\int_0^{\tau} \Phi(\gamma(t))\, dt$, the classical X-ray/Radon transform of Φ along the curve γ.



- The collection of all such data makes up the *scattering data* or *non-Abelian X-ray transform* of Φ, viewed as a map

$$C_\Phi : \partial_+ SM \to GL(n, \mathbb{C}).$$

- **Geometric Inverse Problem:** recover Φ from C_Φ.

Injectivity in 2D

Theorem 1 (P-Salo-Uhlmann 2011, P-Salo 2020)
If (M, g) is a simple surface, the map $\Phi \mapsto C_\Phi$ is injective.

Earlier work on this problem:

- Vertgeim (1992), Sharafutdinov (2000);
- Finch-Uhlmann (2001), R. Novikov (2002) and G. Eskin (2004) for Euclidean domains in the plane.

Additional partial results by Zhou (2017), Monard-P (2017) and P-Salo (2018).

A simple observation:

If $G \subset GL(n, \mathbb{C})$ is matrix Lie group with Lie algebra $\mathfrak{g}$ and Φ takes values in $\mathfrak{g}$, then

$$C_\Phi : \partial_+ SM \to G.$$

The 2011 P-Salo-Uhlmann result gave injectivity in the unitary case; i.e. $\mathfrak{g} = \mathfrak{u}(n)$ consists of skew-hermitian matrices and $G = U(n)$ is the unitary group.

Injectivity in higher dimensions

Theorem 2 (P-Salo-Uhlmann-Zhou 2016)
Let (M, g) *be a compact connected manifold of dimension* ≥ 3 *with strictly convex boundary and suppose* (M, g) *admits a smooth strictly convex function. Then* $\Phi \mapsto C_\Phi$ *is injective.*

This theorem uses completely different techniques and exploits the template laid out by Stefanov-Uhlmann-Vasy in their recent proof of lens rigidity.

Watch Andras' talk!

Polarimetric Neutron Tomography (PNT)

The non-Abelian X-ray transform arises naturally when trying to reconstruct a magnetic field from spin measurements of neutrons.

In this case

$$\Phi(x) = \begin{bmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{bmatrix} \in \mathfrak{so}(3)$$

where $B(x) = (B_1, B_2, B_3)$ is the magnetic field.

The scattering data takes values $C_\Phi : \partial_+ SM \to SO(3)$.

Cf. [Desai, Lionheart et al., Nature Sc. Rep. 2018] and [Hilger et al., Nature Comm. 2018].

The experiment

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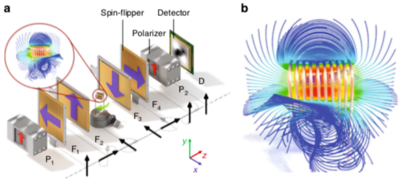

Fig. 1 Tensor tomography. **a** Schematic drawing of the setup used for tensor tomography with spin-polarized neutrons, comprising spin polarizers (P), spin flippers (F) and a detector (D). **b** Selected magnetic field lines around an electric coil (calculation, see text and Methods)

From Hilger et al., Nature Comm. 2018.

- Data produced: $C_\Phi(x, v) \in SO(3)$.
- This is done with an ingenious sequence of spin flippers placed before and after the magnetic field being measured.
- The material containing the magnetic field can also be rotated so as to produce parallel beams from different angles.

But we face the usual problems:

- No explicit reconstruction formula.
- Measurements are noisy.

Thus we have observations $(X_i, V_i) \in \partial_+ SM$ and

$$Y_i = C_\Phi(X_i, V_i) + \varepsilon_i, \quad 1 \le i \le N, \quad (\varepsilon_i)_{jk} \sim^{\text{i.i.d.}} \mathcal{N}(0, \sigma^2).$$

We will assume $(X_i, V_i) \sim^{\text{i.i.d}} \lambda$, where λ is the probability measure given by the standard area form of $\partial_+ SM$ (independent of ε_i).

We let P_Φ^N be the joint probability law of the data

$$D_N = (Y_i, (X_i, V_i))_{i=1}^N.$$

Statistical algorithm for inversion.

We adopt a Bayesian inference approach.

- We think of Φ as a random parameter with distribution given by a normal (or Gaussian) prior $\pi(\Phi)$ (e.g. a Matérn process).
- Using the observations we compute the posterior $\pi(\Phi|D_N)$ using Bayes rule; namely

$$\pi(\Phi|D_N) \propto f_{D_N}(D_N|\Phi)\,\pi(\Phi),$$

$$\text{Posterior} \propto \text{Likelihood} \times \text{Prior}.$$

- From the posterior we extract the posterior mean $\bar{\Phi}_N = E^{\pi}(\Phi|D_N)$.

Since the noise is Gaussian the log-likelihood (up to additive constant) is

$$\ell(\Phi) := -\frac{1}{2\sigma^2}\sum_{i=1}^{N}\|Y_i - C_\Phi(X_i, V_i)\|^2.$$

Then you hope for:

"as $N \to \infty$, $\bar{\Phi}_N$ will approach the true Φ_0 we wish to reconstruct"

This actually works!

Theorem 3 (Consistency, Monard-Nickl-P 2019)

Assume $\Phi_0 : M \to \mathfrak{so}(n)$ is sufficiently smooth. The estimator $\bar{\Phi}_N$ is consistent in the sense that in $P^N_{\Phi_0}$-probability

$$\|\bar{\Phi}_N - \Phi_0\|_{L^2} \to 0$$

as the sample size $N \to \infty$.

Proof of injectivity of $\Phi \mapsto C_\Phi$ in 2D

The proof is in two stages.

- New idea: use a factorization theorem from Loop Groups to reduce the case of $GL(n, \mathbb{C})$ to the unitary case of $U(n)$.
- The unitary case (P-Salo-Uhlmann 2011) is handled with energy identities (aka Pestov identities) and the existence of scalar holomorphic integrating factors.
- Scalar (fibrewise) holomorphic integrating factors exist thanks to the surjectivity of I_0^* (Pestov-Uhlmann 2005).

Since M is a topologically a disc after taking global isothermal coordinates with $g = e^{2\lambda}(dx_1^2 + dx_2^2)$ we have

$$SM = M \times S^1$$

and the geodesic vector field X may be written as

$$X = e^{-\lambda}\left(\cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2} + \left(-\frac{\partial\lambda}{\partial x_1}\sin\theta + \frac{\partial\lambda}{\partial x_2}\cos\theta\right)\frac{\partial}{\partial\theta}\right).$$

In the Euclidean case ($\lambda = 0$) this reduces to

$$X = \cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2} = v \cdot \nabla_x$$

which is the standard form of the transport operator.

The space $L^2(SM)$ decomposes orthogonally into vertical Fourier modes

$$L^2(SM) = \bigoplus_{k \in \mathbb{Z}} H_k$$

where H_k is the eigenspace of $-i\frac{\partial}{\partial \theta}$ corresponding to the eigenvalue k. Let $\Omega_k = C^\infty(SM) \cap H_k$.

A smooth function $u \in C^\infty(SM)$ has a Fourier expansion:

$$u(x, \theta) = \sum_{k \in \mathbb{Z}} u_k(x, \theta) = \sum_{k \in \mathbb{Z}} \tilde{u}_k(x) e^{ik\theta}.$$

A function $u \in C^\infty(SM)$ is fibre-wise holomorphic if $u_k = 0$ for $k < 0$.

The geodesic vector field has the remarkable mapping property

$$X : \Omega_m \mapsto \Omega_{m-1} \oplus \Omega_{m+1}.$$

In fact $X = \eta_+ + \eta_-$, where $\eta_\pm : \Omega_m \to \Omega_{m\pm 1}$ are Cauchy-Riemann type operators (Guillemin and Kazhdan 80).

In the Euclidean case:

$$X = e^{i\theta}\partial + e^{-i\theta}\bar{\partial}.$$

A Loop Group factorization.

Theorem 4 (Pressley-Segal 86)

Given a smooth map $R : S^1 \to GL(n, \mathbb{C})$, there are smooth maps $U : S^1 \to U(n)$ and $F : S^1 \to GL(n, \mathbb{C})$ such that $R = FU$ and F is the boundary value of a holomorphic map $\{z : |z| < 1\} \mapsto GL(n, \mathbb{C})$.

This is one of several factorization theorems including Birkhoff's factorization (1909) equivalent to the classification of holomorphic vector bundles over S^2.

Key Lemma. *Let (M, g) be a compact non-trapping surface with strictly convex boundary. Let $\mathbb{A} \in C^\infty(SM, \mathbb{C}^{n\times n})$ and assume $\mathbb{A} \in \oplus_{k\geq -1}\Omega_k$. Let $R : SM \to GL(n, \mathbb{C})$ be a smooth function solving $XR + \mathbb{A}R = 0$ (always exists) and consider the Loop Group factorization $R = FU$. Then*

$$\mathbb{B} := F^{-1}XF + F^{-1}\mathbb{A}F$$

is skew-hermitian and $\mathbb{B} \in \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$. In other words $\mathbb{B}$ determines a pair (B, Ψ) with $B \in \Omega^1(M, \mathfrak{u}(n))$ and $\Psi \in C^\infty(M, \mathfrak{u}(n))$.

Thus with the holomorphic gauge F we can move our problem to the unitary case.

Proof of the Key Lemma.

Differentiate $R = FU$ along the geodesic flow to obtain

$$0 = XR + \mathbb{A}R = (XF)U + FXU + \mathbb{A}FU.$$

Writing $\mathbb{B} := F^{-1}XF + F^{-1}\mathbb{A}F$, it follows that

$$\mathbb{B} = -(XU)U^{-1}.$$

Since U is unitary, $(XU)U^{-1}$ is skew-hermitian and so is $\mathbb{B}$.

The mapping property

$$X : \oplus_{k \geq 0}\Omega_k \to \oplus_{k \geq -1}\Omega_k$$

and holomorphicity give $\mathbb{B} \in \oplus_{k \geq -1}\Omega_k$. The lemma follows.

$\square$

Final Message.

- We can now handle any Lie group and any pair (A, Φ), where A is a $\mathfrak{g}$-valued connection.
- We have stability estimates (so far in the unitary case).
- We have a Bayesian algorithm for reconstruction with a consistency theorem backing it up.
- There is now a good understanding of the non-Abelian X-ray for simple surfaces, but many challenges remain, most notably questions regarding the range and uncertainty quantification for the statistical algorithm (Bernstein von-Mises type theorems).