Inverse scattering on non-compact manifolds with general metric

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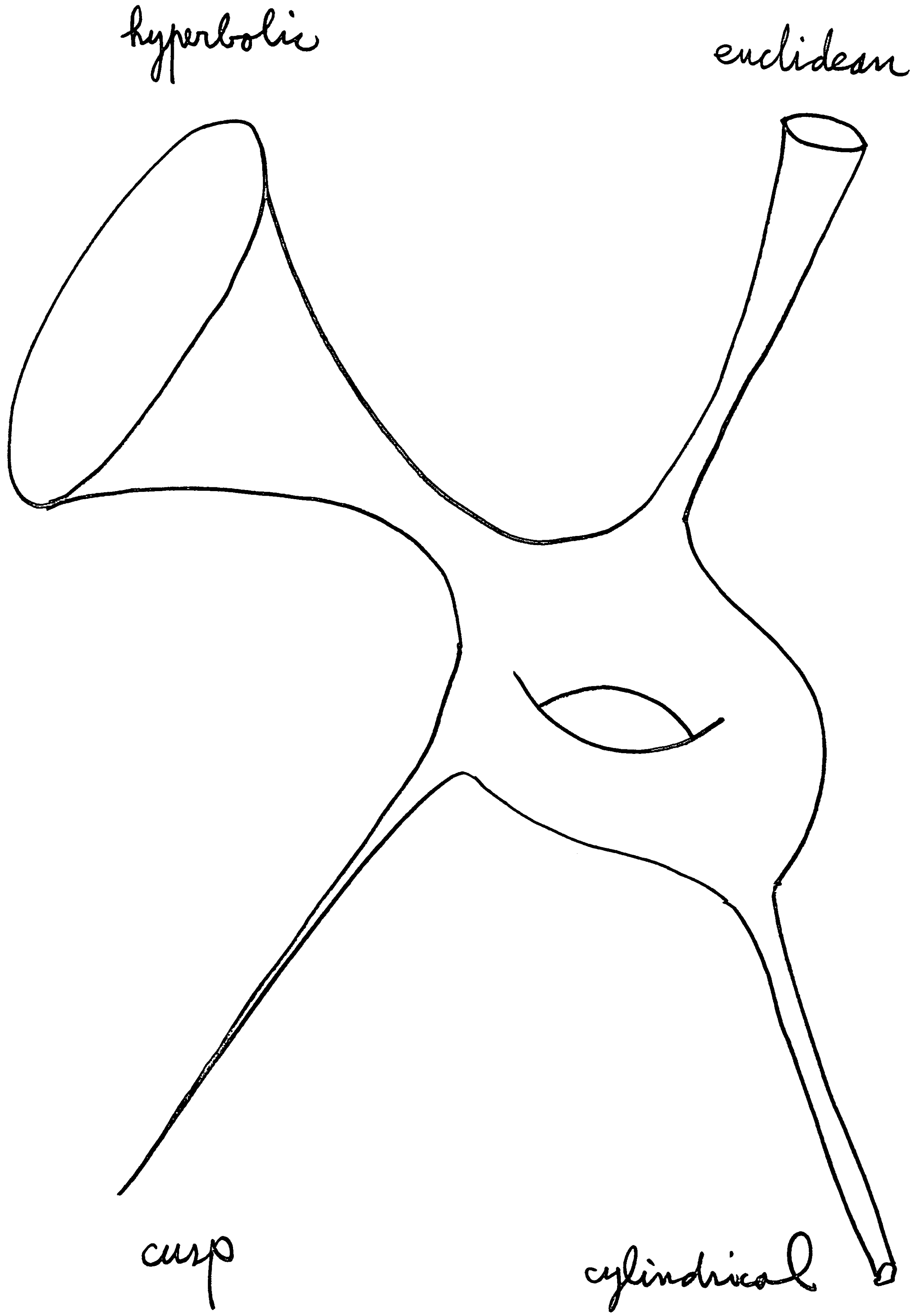

hyperbolic
euclidean
cusp
cylindrical

This talk is based on the recent preprint

Inverse scattering on non-compact manifolds with general metric,
M. Lassas and H. I. arXiv:2004.06431

We started the study of this topic about 2004.
Impressed by the work:

A. Sá Barreto, Radiation fields, scattering and inverse scattering on asymptotically hyperbolic manifolds, Duke Math. J. 129 (2005), 407-480.

This work was further extended by

C. Guillarmou and A. Sá Barreto, Scattering and inverse scattering on ACH manifolds, J. Reine Angew. Math. 622 (2208), 1-55.

Let us explain the problem taking the asymptotically Euclidean metric as an example.

Given an asymptotically Euclidean metric

$$g = g_{ij}dx^i dx^j, \quad g_{ij}(x) \sim \delta_{ij} \quad as \quad |x| \to \infty,$$

we conider the wave equation

$$\partial_t^2 u = \Delta_g u.$$

Then, as $t \to \pm\infty$, it behaves like solutions to the free wave equation

$$u(t) \sim u_0^{(\pm)}(t), \quad (\partial_t^2 - \Delta)u_0^{(\pm)} = 0.$$

Comparing their profiles at infinity, we can define the S-matrix.

What is the profile at infinity?

The reduced wave equation

$$(-\Delta_g - \lambda)v = 0,$$

has the solution behaving like as $r \to \infty$

$$v \simeq \frac{e^{i\sqrt{\lambda}r}}{r^{(n-1)/2}}a_+(\lambda,\omega) - \frac{e^{-\imath\sqrt{\lambda}r}}{r^{(n-1)/2}}a_-(\lambda,\omega), \quad \omega \in S^{n-1}.$$

The S-matrix

$$S(\lambda) : L^2(S^{n-1}) \ni a_-(\lambda,\omega) \to a_+(\lambda,\omega) \in L^2(S^{n-1})$$

is unitary. Then, $(\partial_t^2 - \Delta)e^{-i\sqrt{\lambda}t}v \sim 0$. As $t \to \pm\infty$, the amplitude of the wave equation is written by the S-matrix.
The inverse scattering problem means

Determination of the manifold and its metric from the S-matrix.

Sá Barreto's work is based on

- the boundary control method (BC-method) and
- the support theorem for the Radon transform on the hyperbolic space.

It means (roughly)

$$\mathcal{R}(s)f = 0 \text{ for } s > s_0 \Longrightarrow f = 0 \text{ for } |x| > s_0.$$

This is true for the hyperbolic space, however, not the case for the Euclidean space. One knows:

Sometimes, Euclidean space is harder than the hyperbolic space.

We (M. Lassas, Y. Kurylev and H. I.) took a different root and considered the case of

- cylindrical end (J. Funct. Anal. (2010), 2060-2118),

- hyperbolic end with cusp (J. Reine Ungew. Math. (2017), 53-103).

Then, we found that

these results are imbedded in a theorem which covers a general class of manifolds whose Laplacian has a continuous spectrum

(Contemp. Math. 615 (2014), 143-163).

The proof was essentially finished about 2014 for smooth manifolds. However, all of us were so busy that it took 5 years to complete it.

The class of manifolds

The first issue is

What should be the class of manifolds we consider?

As a model, we take the manifold of the form

$$\mathcal{M} = (0, \infty) \times M.$$

Here M = manifold at infinity is an $n-1$-dimensional compact Riemannian manifold, endowed with the metric

$$ds^2 = (dr)^2 + \rho(r)^2 h(r, x, dx),$$

where $h(r, x, dx)$ is the metric on M ($x \in M$) depending smoothly on $r > 0$ such that as $r \to \infty$

$$h(r, x, dx) = h(x, dx) + O(r^{-\gamma}), \quad \gamma > 0,$$

$h(x, dx)$ being a metric on M.

Here $\rho(r)$ behaves as $r \to \infty$

- $\rho(r) \to \infty$, i.e. the volume of the end diverges at infinity,
- $\rho(r) \to 0$, i.e. the end shrinks to a point.

For example

- $\rho(r) = e^r$, hyperbolic space (funnel)
- $\rho(r) = r$, Euclidean
- $\rho(r) = 1$, cylindrical end
- $\rho(r) = e^{-r}$, hyperbolic cusp

We restrict ourselves to the case in which (roughly)

$$C^{-1}e^{-cr} \leq \rho(r) \leq Ce^{cr}, \quad C, c > 0.$$

Because

Outside this region, the Laplacian may not have continuous spectrum.

We put

$$g = \det\big(\rho(r)h_{ij}(r,x)\big), \quad h(r,x,dx) = h_{ij}(r,x)dx^i dx^j,$$

and assume that, letting $' = \partial_r$,

$$\frac{g'}{4g} \sim \frac{(n-1)c_0}{2} \quad as \quad r \to \infty,$$

where

$$c_0 = \begin{cases} c & when \quad \rho(r) = e^{\pm cr}, \quad c > 0, \\ 0 & when \quad \rho(r) = r^{\beta}. \end{cases}$$

Then, the Laplacian has the continuous spectrum

$$\sigma_{ess}(-\Delta_g) = [E_0, \infty),$$

$$E_0 = \left(\frac{(n-1)c_0}{2}\right)^2.$$

Two main issues

There are two main issues.

- Asymptoitic behavior at infinity of solutions to the reduced wave equation
- Conic singularities of the manifold

The issues for the reduced wave equation

$$(-\Delta_g - \lambda)u = f.$$

The critical decay order at infinity : $\rho(r)^{-(n-1)/2}$

We expect : for $\lambda \in \sigma_{cont}(-\Delta_g)$

(1) Rellich type theorem

$$(-\Delta_g - \lambda)u = 0, \quad r > R, \quad u = o(\rho(r)^{-(n-1)/2})$$
$$\Longrightarrow u = 0, \quad r > R.$$

(2) Resolvent expansion

$$u_\pm = (-\Delta_g - \lambda \mp i0)^{-1} f$$

$$u_\pm \simeq \rho(r)^{-(n-1)/2} e^{\pm i\Phi(\lambda,r)} a_\pm(\lambda,\omega), \quad \omega \in M$$

(3) Generalized Fourier transform

$$f \to a_{\pm}(\lambda, \omega) =: \mathcal{F}_{\pm}(\lambda) f$$

(4) S-matrix

$$S(\lambda) : L^2(M) \ni a_{-}(\lambda, \omega) \to a_{+}(\lambda, \omega) \in L^2(M)$$

(5) Scattering from cusp

The slower the volume growth, the harder the analysis !

Are there thresholds for the rate of volume growth?

The issues for conic singularities

Example from number theory

$$\mathcal{M} = G \backslash \mathbf{C}_+$$

$G =$ discrete group

e.g. $G = SL(2, \mathbf{Z}) \Longrightarrow$ Modular surface

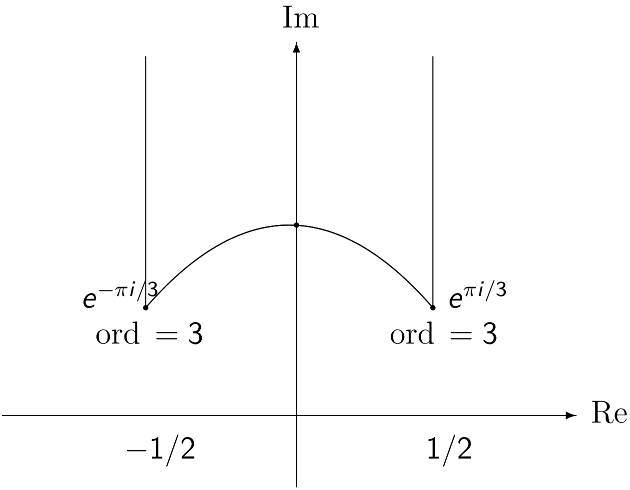


Figure : Fundamental domain for $PSL(2, \mathbf{Z})$

Simple example

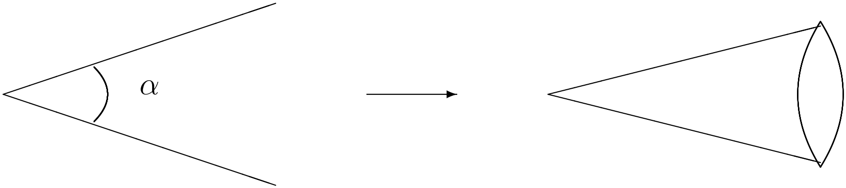


Figure : S_α

This cone has a singularity at the top.

If $\alpha/2\pi \in \mathbf{Q} \Longrightarrow$ Lift to the covering space
$\Longrightarrow$ One can compute smoothly.

If $\alpha/2\pi \notin \mathbf{Q} \Longrightarrow$?

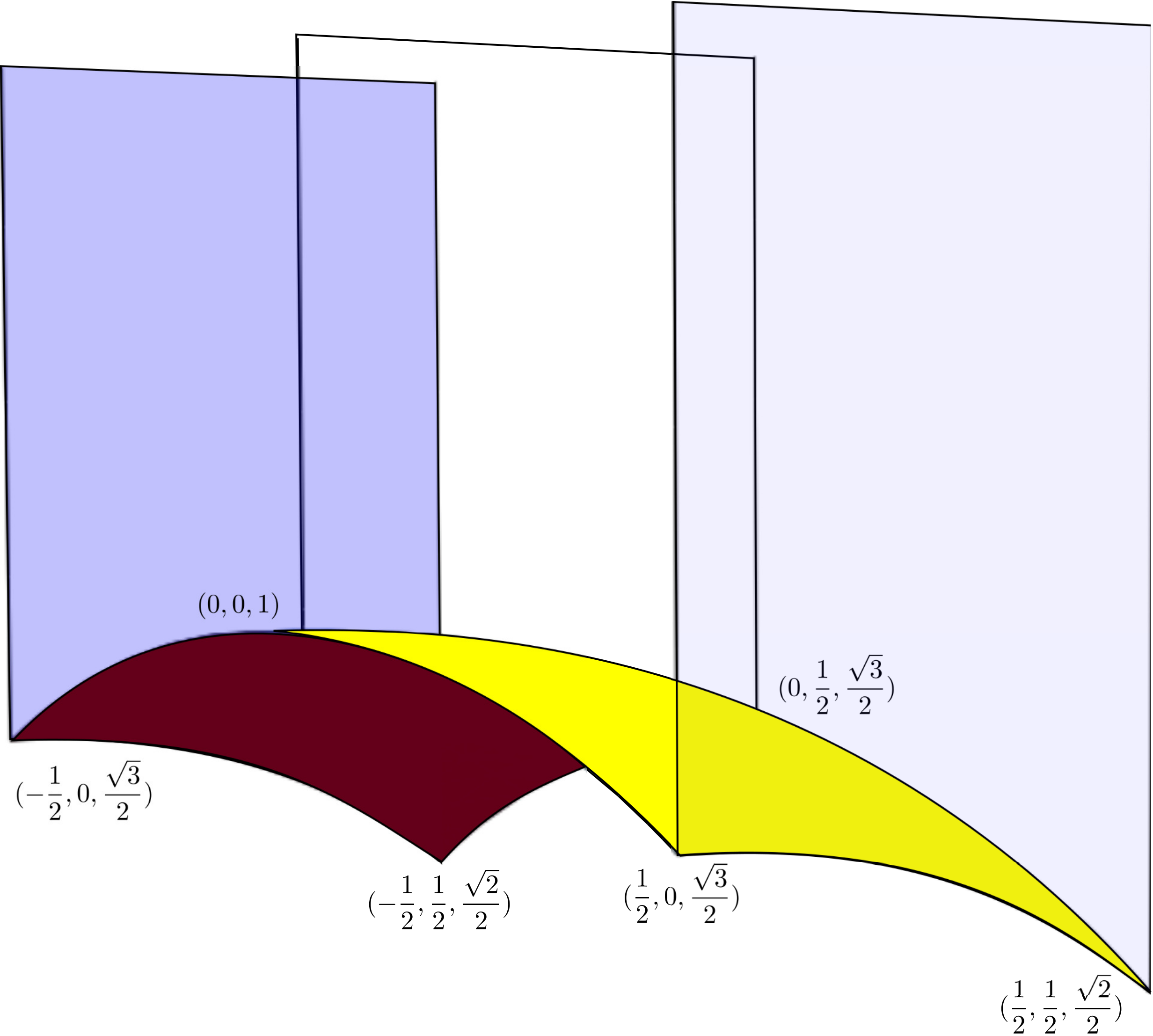
The point is :

- Passing to the polar coordinates, it has a smooth differential structure at the top.
- However, the Riemannian metric is singular at the top.

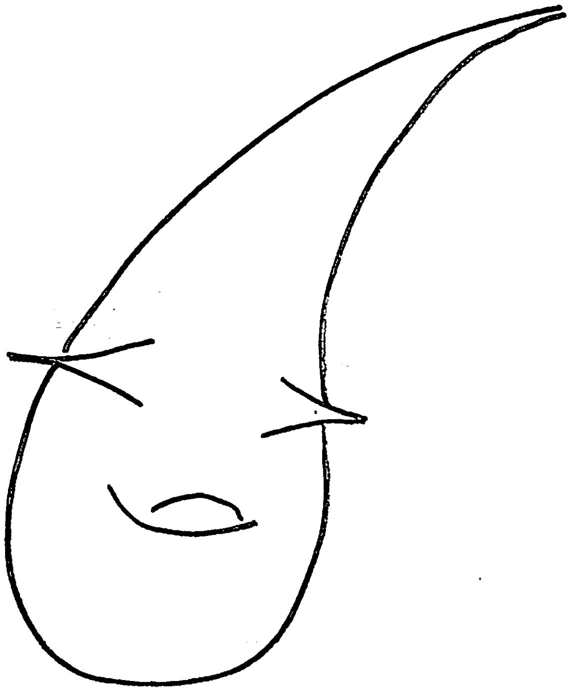
There is also a 3-dim. analogue

$$\mathcal{M} = SL(2, \mathbf{C})\backslash\mathbf{R}^3_+$$

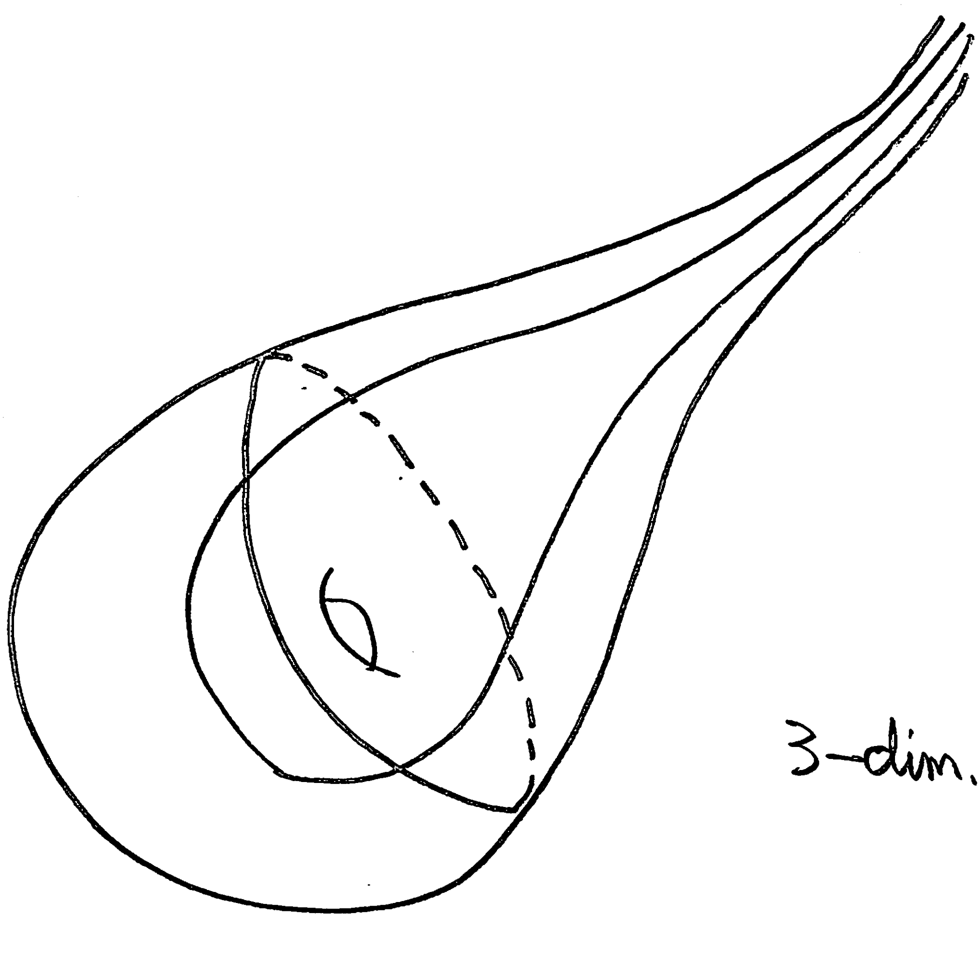
See the following figure:



$(0,0,1)$
$(0,\frac{1}{2},\frac{\sqrt{3}}{2})$
$(-\frac{1}{2},0,\frac{\sqrt{3}}{2})$
$(-\frac{1}{2},\frac{1}{2},\frac{\sqrt{2}}{2})$
$(\frac{1}{2},0,\frac{\sqrt{3}}{2})$
$(\frac{1}{2},\frac{1}{2},\frac{\sqrt{2}}{2})$



2-dim.



3-dim.

There are new problems for manifolds with singularities, and we are interested in the possibility of developing spectral theory and BC-method for them,

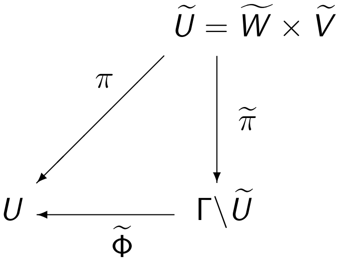
e.g. Y. Kurylev, M. Lassas and T. Yamaguchi, Uniqueness and stability in inverse spectral problems for collapsing manifolds, arXiv:1209.5875 (2012)

So, we tried to find a class of manifolds slightly bigger than orbifolds.

We are going to explain

- Conic chart
- Singular set
- CMGA = conic manifold with group action

Definition of Conic chart



U = a topological space
$(\widetilde{U}, \Gamma, \pi)$ a conic chart of $U \Longleftrightarrow$

- $\widetilde{U} = \widetilde{W} \times \widetilde{V}$, $\widetilde{W} = B^k(0, R_0)$, $\widetilde{V} = B^{n-k}(0, R_1)$, $R_0, R_1 > 0$, $0 \leq k \leq n$, $k \neq n-1$.
- Γ = a finite group acting on $\widetilde{U}$, leaving $\widetilde{W}$ invariant.
- $\pi = \widetilde{\Phi} \circ \widetilde{\pi}$, $\widetilde{\pi}$ = the canonical projection $\widetilde{U} \to \Gamma\backslash\widetilde{U}$, $\widetilde{\Phi}$ = a homeomorphism $\Gamma\backslash\widetilde{U} \to U$.

Singular set and regular set

The singular set consists of

- $\widetilde{W} \times \{0\}$
- The set of fixed points of Γ

More precisely letting (the isotropy group of x)

$$\Gamma(x) = \{\gamma \in \Gamma\,;\, \gamma \cdot x = x\}, \quad x \in \widetilde{U},$$

we put ($k = \dim \widetilde{W}$)

$$\widetilde{U}^{sing} = \begin{cases} \left\{x \in \widetilde{U}\,;\, \Gamma(x) \neq \{e\}\right\}, \quad k = 0, \\ \left(\widetilde{W} \times \{0\}\right) \cup \left\{x \in \widetilde{U}\,;\, \Gamma(x) \neq \{e\}\right\}, \quad 0 < k < n, \\ \emptyset, \quad k = n. \end{cases}$$

$$\widetilde{U}^{reg} = \widetilde{U} \setminus \widetilde{U}^{sing},$$

Conic manifolds with group action (CMGA)

CMGA roughly means a set of conic charts and smooth metric $\widetilde{g}$ on $\widetilde{U}^{reg}$ bahaving continuously near $\widetilde{W} \times \{0\}$.

A *conic manifold admitting group action*, abbreviated to CMGA, is a topological space M equipped with the following structure: *There exists a family of open covering $\{U_j\,;\, j \in J\}$ of M having the following properties (C-1) $\sim$ (C-4)*:

(C-1) *For any $j \in J$, U_j has a conic chart $(\widetilde{U}_j, \Gamma_j, \pi_j)$, $\widetilde{U}_j = \widetilde{W}_j \times \widetilde{V}_j$, where for some $0 \leq k \leq n$, $k \neq n-1$,*

$$\widetilde{W}_j = B^k(0, R_0) \subset \mathbb{R}^k, \quad \widetilde{V}_j = B^{n-k}(0, R_1) \subset \mathbb{R}^{n-k}. \qquad (1)$$

(C-2) *Define $Y_j : \widetilde{U}_j \to \widetilde{W}_j$, $Z_j : \widetilde{U}_j \to \widetilde{V}_j$ and $\widetilde{U}_j^{reg}$ by*

$$\widetilde{U}_j \ni x \to x = (y, z) = (Y_j(x), Z_j(x)),$$

$$\widetilde{U}_j^{reg} = \Big(\widetilde{W}_j \times \big(B^{n-k}(0,R_1) \setminus \{0\}\big)\Big) \cap \big\{x \in \widetilde{U}_j\,;\, \Gamma_j(x) = \{e\}\big\}. \quad (2)$$

Then :

(C-2-1) *The action of* $\gamma \in \Gamma_j$ *keeps the y-coordinates invariant, i.e.*

$$Y_j(\gamma \cdot x) = Y_j(x), \quad \forall x \in \widetilde{U}_j, \quad \forall \gamma \in \Gamma_j.$$

(C-2-2) *There exists a* Γ_j*-invariant* C^∞*-metric* $\widetilde{g}_j$ *on* $\widetilde{U}_j^{reg}$*, i.e.*

$$\gamma_* \widetilde{g}_j = \widetilde{g}_j \quad on \quad \widetilde{U}_j^{reg}, \quad \forall \gamma \in \Gamma_j.$$

(C-2-3) *In the spherical coordinates* $Z_j(x) = s\omega = z$, $s = s(x) = |z|$, $\omega = \omega(x) = \dfrac{z}{|z|}$ *such that* $s \in (0, R_1)$ *and* $\omega \in S^{n-k-1}$ *on*

$B^{n-k}(0, R_0) \setminus \{0\}$, $\widetilde{g}_j$ has the form

$$\widetilde{g}_j = \sum_{p,q=1}^{k} a_{pq}^{(j)}(y,s,\omega)dy^p dy^q$$
$$+ ds^2 + s^2 \sum_{\ell,m=1}^{n-k} b_{\ell m}^{(j)}(y,s,\omega)d\omega^\ell d\omega^m + s\sum_{p=1}^{k}\sum_{\ell=1}^{n-k} h_{p\ell}^{(j)}(y,s,\omega)dy^p d\omega^\ell. \tag{3}$$

The coefficients satisfy

$$\begin{cases} a_{pq}^{(j)}(y,s,\omega) \to \widehat{a}_{pq}^{(j)}(y), \\ b_{\ell m}^{(j)}(y,s,\omega) \to \widehat{b}_{\ell m}^{(j)}(y,\omega), \\ h_{p\ell}^{(j)}(y,s,\omega) \to 0, \end{cases} \tag{4}$$

uniformly in (y,ω) as $s \to 0$. Moreover, there exist constants

$C_1 \geq C_0 > 0$ and a positive continuous function $T_j(y)$ such that

$$C_0\, g_{S^{n-k-1}} \leq \sum_{\alpha,\beta=1}^{n-k} \widehat{b}_{\ell m}^{(j)}(y,\omega) d\omega^\ell d\omega^m \leq T_j(y)^2 g_{S^{n-k-1}}, \qquad (5)$$

where $g_{S^{n-k-1}}$ is the standard metric of S^{n-k-1} and $C_0 \leq T_j(y) \leq C_1$.
(C-2-4) For $e \neq \gamma \in \Gamma_j$, e being the unit of Γ_j,

$$\mathrm{cap}_2(\{x \in \widetilde{U}\,;\, \gamma \cdot x = x\}) = 0, \qquad (6)$$

where $\mathrm{cap}_2(E)$ denotes the 2-capacity of a subset $E \subset \mathbb{R}^n$.
(C-3) *If $U_j \cap U_k \neq \emptyset$, there exists $\ell \in J$ such that $U_\ell \subset U_j \cap U_k$.*
(C-4) *If $U_\ell \subset U_k$, there exist an injective homomorphism $\mathcal{I}_{k\ell} : \Gamma_\ell \to \Gamma_k$, and a C^∞ injective map $\widetilde{I}_{k\ell} : \widetilde{U}_\ell \to \widetilde{U}_k$.*

Recall that the 2-capacity of a subset $E \subset \mathbb{R}^n$ is defined by

$$\mathrm{cap}_2(E) = \inf \int_{\mathbb{R}^n} \left(|u|^2 + |\nabla u|^2\right) dx,$$

where the infimum is taken over all $u \in H^2(\mathbb{R}^n)$ such that $u \geq 1$ almost everywhere on a neighborhood of E.

Note : for a set A

$$(n-2)\text{-dim. Hausdorff measure of } A < \infty \implies 2\text{- capacity of } A = 0.$$

So, roughly, we are assuming that

$$\text{dim. of singular sets} \leq n-2.$$

The Laplacian is defined through the quadratic form.
We need to be careful about the regularity of $D(-\Delta_g)$ around singular sets.

Our manifold is

$$\mathcal{M} = (0,\infty) \times M,$$

where M is an $(n-1)$-dimensional CMGA.
The metric is

$$ds^2 = (dr)^2 + \rho(r)^2 h(r,x,dx),$$

where

$$h(r,x,dx) - h_M(x,dx) \in S^{-\gamma_0}, \gamma > 1.$$

Here S^κ is the set of $f \in C^\infty((0,\infty); C^2(M^{reg})$ such that

$$p_M(\partial_r^\ell f(r)) \leq C(1+r)^{\kappa-\ell}, \forall \ell \geq 0,$$

$$p_M(f) = \sup_{y,s,\theta} \sum_{|\alpha|+|\beta|+|\gamma|\leq 2} s^{-|\gamma|} |\partial_y^\alpha \partial_s^\beta \partial_\theta^\gamma f(y,s,\theta)|,$$

y : rectangular coordinates on $\widetilde{W}$,
(s,θ) : spherical coordinates on $\widetilde{V}$.

The conditions on $\rho(r)$.

- If $\rho(r) \to \infty$, we call M regular.
- If $\rho(r) \to 0$, we call M cusp.

We assume

$$\frac{\rho'(r)}{\rho(r)} - c_0 \in S^{-\alpha_0},$$

$$h(r, x, dx) - h_M(x, dx) \in S^{-\gamma_0},$$

$$\alpha_0 > 0, \quad \gamma_0 > 1.$$

We also assume for $\beta_0 > 0$

- for regular $\mathcal{M}$

$$\frac{\rho'(r)}{\rho(r)} \geq \frac{\beta_0}{r} \quad for \quad r > r_0,$$

- for cusp $\mathcal{M}$

$$\rho(r) \leq C(1 + r)^{-\beta_0}.$$

We need to add more assumptions later.

Note

$\rho(r) \geq Cr^{\beta}$ for the regular case.

We finally assume :

$$\mathcal{M} = \mathcal{K} \cup \big(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_N\big) \cup \big(\mathcal{M}_{N+1} \cup \cdots \cup \mathcal{M}_{N+N'}\big),$$

$$\begin{aligned} &\mathcal{K} : \text{relatively compact, open,} \\ &\mathcal{M}_i, 1 \leq i \leq N : \text{regular ends,} \\ &\mathcal{M}_i, N+1 \leq i \leq N+N' : \text{cusp.} \end{aligned}$$

We put

$$E_{0,i} = \left(\frac{(n-1)c_{0,i}}{2}\right)^2,$$

$$E_0 = \min_{1 \leq i \leq N+N'} E_{0,i}.$$

We then have :

Theorem

$$\sigma_{ess}(-\Delta_g) = [E_0, \infty).$$

Rellic type theorem

$$(-\Delta - \lambda)u = 0, \quad r > R, \quad in \quad \mathbf{R}^n,$$

$$u = o(r^{-(n-1)/2}), \quad r \to \infty$$

$$\Longrightarrow u = 0, \quad r > R.$$

Rellich, Vekua (1943).

The equation holds only near infinity.

$\Longleftarrow$ Important in inverse scattering.

Theorem

Fix one regular end, and assume that

$$\alpha_0 > 0, \quad \beta_0 > 1/3, \quad \gamma_0 > 0, \quad E > E_0,$$

$$(-\Delta_{\mathcal{M}} - E)u = 0, \quad r > R$$

$$\liminf_{r \to \infty} \int_{S(r)} (|u'|^2 + |u|^2) dS(r) = 0$$

$$\Longrightarrow u = 0, \quad r > R.$$

Note $dS(r) \sim \rho(r)^{n-1} dM$, and $\rho(r) \geq Cr^{\beta_0}$.

Corollary

If $\beta_{0,i} > 1/3$ for some regular end, then

$$\sigma_p(-\Delta_{\mathcal{M}}) \cap (E_{0,i}, \infty) = \emptyset.$$

Theorem

If $0 < \beta_{0,i} \leq 1/3$ for all regular ends, $\sigma_p(-\Delta_{\mathcal{M}}) \cap (E, \infty)$ is discrete with possible accumulation points at $\{E_{0,1}, \cdots, E_{0,N+N'}\}$ and ∞.

The idea of the proof

Transform to an ordinary differential equation with unbounded operator coefficients

$$-u''(t) + B(t)u(t) + V(t)u(t) - \lambda u(t) = 0,$$

where $B(t)$ = Laplace-Beltrami operator on M.
(cf. Kato (1959))

Function spaces

On each end $(0, \infty) \times M$ with metric

$$ds^2 = (dr)^2 + \rho(r)^2 h(r, x, dx),$$

we define the following function spaces:

$$\mathbf{h}(r) = L^2(M; \sqrt{h(r,x)}dx),$$

where $h(r, x)$ = determinant of the metric $h(r, x, dx)$ on M.

$$L^2(I) \ni f \iff \int_I \|f(r)\|^2_{\mathbf{h}(r)} \rho^{n-1}(r) dr < \infty.$$

$$\mathcal{B} \ni f \iff \|f\|_{\mathcal{B}} = \sum_{j=0}^{\infty} 2^{j/2} \|f\|_{L^2(I_j)} < \infty,$$

$$I_0 = (0, 1], \quad I_j = (2^{j-1}, 2^j], \quad j \geq 1.$$

$$\mathcal{B}^* \ni u \Longleftrightarrow \sup_{R>1} \frac{1}{R} \int_0^R \|v(r)\|^2_{\mathbf{h}(r)} \rho^{n-1}(r) dr < \infty.$$

$$L^{2,s} \ni f \Longleftrightarrow \int_0^\infty \|f(r)\|^2_{\mathbf{h}(r)} (1+r)^{2s} \rho^{n-1}(r) dr < \infty.$$

For $s > 1/2$

$$L^{2,s} \subset \mathcal{B} \subset L^{2,1/2} \subset L^2 \subset L^{2,-1/2} \subset \mathcal{B}^* \subset L^{2,-s}.$$

We extend these definitions to

$$\mathcal{M} = \mathcal{K} \cup \left(\cup_{i=1}^N \mathcal{M}_i \right) \cup \left(\cup_{i=1}^{N'} \mathcal{M}_{N+i} \right).$$

Limiting absorption principle

$$\mathcal{E} = \{E_{0,1}, \cdots, E_{0,N+N'}\} \cup \sigma_p(-\Delta_{\mathcal{M}})$$

$$R(z) = (-\Delta_{\mathcal{M}} - z)^{-1}$$

Recall that on each end $\mathcal{M}_i$

$$ds^2 = (dr)^2 + \rho_i(r)^2 h_i(r, x, dx),$$

$$\frac{\rho_i'(r)}{\rho_i(r)} = c_{0,i} + O(r^{-\alpha_{0,i}}), \quad \alpha_{0,i} > 1,$$

$$h_i(r, x, dx) = h_i(x, dx) + O(r^{-\gamma_{0,i}}), \quad \gamma_{0,i} > 1.$$

For regular end,

$$\rho_i(r) \geq C r^{\beta_{0,i}}.$$

Theorem

Assume $\beta_{0,i} > 0$. For $\lambda \in (E, \infty) \setminus \mathcal{E}$, $E = \inf \sigma_{ess}(-\Delta_{\mathcal{M}})$,

$$R(\lambda \pm i0) \in \mathbf{B}(\mathcal{B}; \mathcal{B}^*).$$

So, LAP holds for

$$\beta_{0,i} > 0.$$

Methods: Integration by parts

Alternatively, one can use Mourre theory (commutator method).

Resolvent asymptotics

We pick up one regular end $(0, \infty) \times M$ and observe the behavior as $r \to \infty$ of the resolvent $R(\lambda \pm i0) = (-\Delta_{\mathcal{M}} - \lambda \mp i0)^{-1}$.
Let

$$\phi(r, \lambda) = \sqrt{\lambda - \frac{(n-2)^2}{4}\left(\frac{\rho'(r)}{\rho(r)}\right)^2}$$

$$\Phi(r, \lambda) = \int_0^r \phi(t, \lambda) dt.$$

Note that as $r \to \infty$

$$\Phi(r, \lambda) \sim \sqrt{\lambda - E_0}\, r.$$

Recall that

$$\rho(r) \geq C r^{\beta_0}.$$

Theorem

Assume that

$$\beta_0 > 1/2.$$

Then, on $(0, \infty) \times M$

$$R(\lambda \pm i0)f \simeq \rho(r)^{-(n-1)/2} e^{\pm i\Phi(\lambda,r)} a(\lambda, x),$$

where $a(\lambda, \cdot) \in L^2(M)$.

Here $u \simeq v$ means

$$\frac{1}{R}\int_0^R \|u(r) - v(r)\|^2_{L^2(M)} \rho(r)^{(n-1)/2} dr \to 0.$$

Slowly growing ends

Λ = Laplace-Beltrami operator on M,
$0 = \lambda_0 \leq \lambda_1 \leq \cdots \to \infty$: eigenvalues
$\varphi_\ell(x)$: associated eigenvector

Theorem

Assume that

$$0 < \beta_0 \leq 1/2.$$

Then for $f \in \mathcal{B}$

$$R(\lambda \pm i0)f \simeq \sum_{\ell=0}^{\infty} c_\ell(\lambda, r)\rho(r)^{-(n-1)/2} e^{\pm i\varphi(\lambda,\lambda_\ell,r)} a_\ell \varphi_\ell(x), \qquad (7)$$

Here

$$\varphi(\lambda, \lambda_\ell, r) = \int_{r_0(\lambda,\lambda_\ell)}^{r} \alpha(\lambda, \lambda_\ell, s) ds,$$

$$\alpha(\lambda, \lambda_\ell, r) = \sqrt{\lambda - \left(\frac{(n-1)\rho'(r)}{2\rho(r)}\right)^2 - \frac{\lambda_\ell}{\rho(r)^2}}$$

$$r_0(\lambda, \lambda_\ell) = \left(\frac{2C(1+\lambda_\ell)}{\lambda}\right)^{1/2},$$

C = constant depending only on $\rho(r)$,

$$\chi(r) = \begin{cases} 0, & r < 1 \\ 1, & r > 2 \end{cases}$$

$$c_\ell(\lambda, r) = \left(\frac{\pi}{\sqrt{\lambda}}\right)^{1/2} \chi\Big(\frac{r}{r_0(\lambda, \lambda_\ell)}\Big).$$

So, on supp $c_\ell(\lambda, r)$

$$r \geq C_0 \lambda_\ell^{1/2}.$$

This means that in the expansion of the resolvent as $r \to \infty$, high energy states of the Laplace-Beltrami operator Λ of M appear behind low energy states.

This agress with the case of cylidrical ends

$$ds^2 = (dr)^2 + h(x, dx),$$

since in this case, we observe only a finite number, depending on the energy λ, of low lying eigenstates of the Laplace-Beltrami operator Λ.

Cusp

On cusp end, we have a similar expansion with λ_0 only.

The space at infinity

$$\mathbf{h} = \big(\oplus_{i=1}^{N} L^2(M_i)\big) \oplus \big(\oplus_{j=1}^{N'} \mathbf{C}\big)$$

Generalizd Fouier transform

Fr given $f \in \mathcal{B}$, observig $R(\lambda \pm i0)f$ near infinity of the end $(0, \infty) \times M_j$, we obtain

$$a_j^{(\pm)}(\lambda, x) \in L^2(M_j).$$

For the cusp,

$$a_j^{(\pm)}(\lambda) \in \mathbf{C}, \quad N+1 \leq j \leq N+N'.$$

Let

$$\varphi^{(out)}(\lambda) = (a_1^{(+)}(\lambda, x), \cdots, a_N^{(+)}(\lambda, x), a_{N+1}^{(+)}(\lambda), \cdots, a_{N+N'}^{(+)}(\lambda)) \in \mathbf{h},$$

$$\varphi^{(in)}(\lambda) = (a_1^{(-)}(\lambda, x), \cdots, a_N^{(-)}(\lambda, x), a_{N+1}^{(-)}(\lambda), \cdots, a_{N+N'}^{(-)}(\lambda)) \in \mathbf{h}.$$

Define

$$\mathcal{F}^{(+)}(\lambda) : f \to a^{(out)}(\lambda), \quad \mathcal{F}^{(-)}(\lambda) : f \to a^{(in)}(\lambda).$$

$$\big(\mathcal{F}^{(\pm)}f\big)(\lambda) = \mathcal{F}^{(\pm)}(\lambda)f.$$

$$\mathbf{H} = \Big(\oplus_{j=1}^{N} L^2((E_{0,j},\infty); L^2(M_j))\Big) \oplus \Big(\oplus_{j=N+1}^{N+N'} L^2((E_{0,j}; \mathbf{C})\Big)$$

$L^2_{ac}(\mathcal{M})$ = the absolutely continuous subspace for $-\Delta_{\mathcal{M}}$.
Then,

$$\mathcal{F}^{(\pm)} : L^2_{ac}(\mathcal{M}) \to \mathbf{H} : \textit{unitary},$$

$\mathcal{F}^{(\pm)}(\lambda)^* : \mathbf{h} \to \mathcal{B}^*$ is an eigenoperator:

$$(-\Delta_{\mathcal{M}} - \lambda)\mathcal{F}^{(\pm)}(\lambda)^*\phi = 0, \quad \phi \in \mathbf{h}.$$

Inversion formula holds: for $f \in L^2_{ac}(\mathcal{M})$,

$$f = \int_{E_0}^{\infty} \mathcal{F}^{(\pm)}(\lambda)^*(\mathcal{F}^{(\pm)}f)(\lambda)d\lambda.$$

Helmholtz equation and S-matrix

Define

$$\mathcal{N}(\lambda) = \{u \in \mathcal{B}^* \,;\, (-\Delta_{\mathcal{M}} - \lambda)u = 0\}$$

Then,

Theorem

$$\mathcal{N}(\lambda) = \mathcal{F}^{(\pm)}(\lambda)^* \mathbf{h}$$

Theorem

Given $\phi^{(in)} \in \mathbf{h}$, there exist unique $u \in \mathcal{N}(\lambda)$ and $\phi^{(out)} \in \mathbf{h}$ such that u has the asymptotic expansion at infinity having $\phi^{(in)}$ and $\phi^{(out)}$ as its profiles.

For example, when $\beta_{0,j} > 1/2$, this expansion means

$$\begin{aligned} u \simeq & C_+(\lambda)\rho(r)^{-(n-1)/2} e^{i\Phi_j(r,\lambda)} \phi_j^{(out)} \\ & - C_-(\lambda)\rho(r)^{-(n-1)/2} e^{-i\Phi_j(r,\lambda)} \phi_j^{(in)} \end{aligned}$$

on each end $\mathcal{M}_j$.

The operator

$$S(\lambda) : \mathbf{h} \ni \phi^{(in)} \to \phi^{(out)} \in \mathbf{h}$$

is unitary, and called the S-matrix.

$$S(\lambda) = \big(S_{\alpha\beta}(\lambda)\big), \quad 1 \leq \alpha, \beta \leq N + N'$$

is an $(N + N') \times (N + N')$ operator-valued matrix.

Inverse scattering from regular ends

We can now enter into the inverse scattering.

We need one more assumption on the singular set. Define

$$\Lambda(x) = \lim_{r\to 0} \frac{\mathrm{vol}_{\mathcal{M}}(B_{\mathcal{M}}(x,r)\cap \mathcal{M}^{reg})}{\mathrm{vol}_{\mathbf{R}^n}(B(0,r))}$$

(Note $\Lambda(x) = 1$ for $x \in \mathcal{M}^{reg}$).
Assume that

$$\Lambda(x) \neq 1 \quad if \quad x \in \mathcal{M}^{sing}.$$

For two-dimensional case, $\Lambda(x) < 1$ means that the angle of the cone is smaller than 2π, so the conic singularities are really thin ($\Lambda(x) < 1$), or thick ($\Lambda(x) > 1$) cone compared with the regular case $\Lambda(x) = 1$.

We are given two such manifolds $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$

$$\mathcal{M}^{(1)} = \mathcal{K}^{(1)} \cup \left(\cup_{i=1}^{N_1} \mathcal{M}_i^{(1)}\right) \cup \left(\cup_{i=N_1+1}^{N_1+N_1'} \mathcal{M}_i^{(1)}\right),$$

$$\mathcal{M}^{(2)} = \mathcal{K}^{(2)} \cup \left(\cup_{i=1}^{N_2} \mathcal{M}_i^{(2)}\right) \cup \left(\cup_{i=N_2+1}^{N_2+N_2'} \mathcal{M}_i^{(2)}\right).$$

Note that the numbers of ends are not assumed to be equal a-priori.

We pick up two regular ends $\mathcal{M}_i^{(1)}$, $\mathcal{M}_i^{(2)}$.
If $\beta_{0,i} > 1/3$, we do not need a new assumption.
If $0 < \beta_{0,i} \leq 1/3$, we assume that $\mathcal{M}_i$ is pure cusp, i.e.

$$ds^2 = (dr)^2 + \rho_i(r)^2 h_i(x, dx) \quad on \quad \mathcal{M}_i.$$

Theorem

Given two such manifolds $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$, assume that for some i, the regurar ends $\mathcal{M}_i^{(1)}$ and $\mathcal{M}_i^{(2)}$ are isometric, and the diagonal entries $S_{ii}^{(1)}(\lambda)$ and $S_{ii}^{(2)}(\lambda)$ of the S-matrices coincide for all energies $\lambda \in (E_{0,i}, \infty)$. Then, $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ are isometric.
More precisely, there is Φ s.t.
(1) $\Phi : \mathcal{M}^{(1)} \to \mathcal{M}^{(2)}$ is a homeomorphism.
(2) $\Phi(\mathcal{M}_{sing}^{(1)}) = \mathcal{M}_{sing}^{(2)}$.
(3) $\Phi : \mathcal{M}_{reg}^{(1)} \to \mathcal{M}_{reg}^{(2)}$ is a Riemannian isometry.

Inverse scattering from cusp

Theorem

For 2-dim. hyperbolic manifolds, the scattering matrix from cusp does not determine the manifold.

S. Zelditch, Comm. in P.D.E. (1992), pp. 221-260.

This is because the cusp gives only one dimensional contribution to the continuous spectrum, and does not have enough information to determine the whole manifold.

To define the usual (or physical) S-matrix, ones uses the $\mathcal{B}^*$-solutions to the Helmholtz equation. To get more information, we must use bigger classes of solutions, and

extend the notion of S-matrix

$\Longrightarrow$ generalized S-matrix

Consider a pure cusp end with metric

$$ds^2 = (dr)^2 + \rho(r)^2 h(x, dx).$$

$0 \leq \lambda_0 \leq \lambda_1 \leq \cdots$: eigenvalues of $-\Delta_M$
$e_\ell(x)$: the associated eigenvector.
Expand the solution of the equation

$$(-\Delta_{\mathcal{M}} - \lambda)u = 0 \quad on \quad (0, \infty) \times M \tag{8}$$

by $e_\ell(x)$. Then by the WKB methd, we obtain two solutions $u_{\ell,\pm}$ of the equation

$$-u'' - \frac{(n-1)\rho'(r)}{\rho(r)}u' + \left(\frac{\lambda_\ell^2}{\rho(r)^2} - \lambda\right)u = 0,$$

which behaves like

$$u_{\ell.\pm} \sim \rho(r)^{-(n-2)/2} e^{\pm\Phi(r,\lambda_\ell)}, \quad r \to \infty$$

$$\Phi(r,B)=\int_{r_0}^{r}\sqrt{\frac{B}{\rho^2}-\lambda+\frac{(n^2-2n)}{4}\left(\frac{\rho'}{\rho}\right)^2+\frac{(n-2)}{2}\left(\frac{\rho'}{\rho}\right)'}dr$$

Then,

$$\Phi(r,B)\sim\sqrt{B}r^c,$$

or

$$\Phi(r,B)\sim\sqrt{B}e^{cr}$$

with $c>0$. So,

- $\rho(r)^{(n-2)/2}u_+$ is (super)-exponentially increasing,
- $\rho(r)^{(n-2)/2}u_-$ is (super)-exponentially decreasing.

Let a_ℓ, b_ℓ be Fourier coefficients of the solution u to (8):

$$(u(r,\cdot),e_\ell)_{L^2(M)}=a_\ell u_{\ell,+}+b_\ell u_{\ell,-}.$$

We inroduce *Space of sequences*

$$\mathbf{A}_{\pm} \ni \{c_{\ell,\pm}\}_{\ell=0}^{\infty} \Longleftrightarrow \sum_{\ell=0}^{\infty} |c_{\ell,\pm}|^2 |u_{\ell,\pm}(r)|^2 < \infty, \quad \forall r > 0.$$

By the behavior of $u_{\ell,\pm}$, we have

- $\{c_{\ell,+}\}$ is a (super)-exponentially decaying sequence,
- $\{c_{\ell,-}\}$ is a (super)-exponentially growing sequence.

Generalized incoming solution

$$\Psi^{(in)} = \sum_{\ell=0}^{\infty} a_\ell u_{\ell,+}(r) e_\ell(x), \quad \{a_\ell\}_{\ell=0}^{\infty} \in \mathbf{A}_+,$$

Generalized outgoing solution

$$\Psi^{(out)} = \sum_{\ell=0}^{\infty} b_\ell u_{\ell,-}(r) e_\ell(x), \quad \{b_\ell\}_{\ell=0}^{\infty} \in \mathbf{A}_-.$$

Using them as profiles at the cusp, we can define the generalized S-matrix : $S_{gen}(\lambda) : \{a_\ell\} \to \{b_\ell\}$.

$S_{gen}(\lambda)$ is an infinite matrix, whose $(0,0)$ entry is the usual S-matrix for the cusp.

Theorem

Given two such manifolds $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$, assume that the cusp ends $\mathcal{M}_\kappa^{(1)}$ and $\mathcal{M}_\kappa^{(2)}$ are isometric, and the (κ, κ) components of the generalized S-matrix coincide for all energies. Then, $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ are isometric. More precisely, there is Φ s.t.
(1) $\Phi : \mathcal{M}^{(1)} \to \mathcal{M}^{(2)}$ is a homeomorphism.
(2) $\Phi(\mathcal{M}^{(1)}_{sing}) = \mathcal{M}^{(2)}_{sing}$.
(3) $\Phi : \mathcal{M}^{(1)}_{reg} \to \mathcal{M}^{(2)}_{reg}$ is a Riemannian isometry.

Redution to the Boundary Control method

When $\mathcal{M}$ is a C^∞-manifold, pick up one end $\mathcal{M}_i$ and consider a surface

$$S_0 = \mathcal{M}_i \cap \{r = r_0\}.$$

Split $\mathcal{M}$ into two parts

$$\mathcal{M} = \mathcal{M}^{(ext)} \cup \mathcal{M}^{(int)}, \quad \mathcal{M}^{(ext)} = \mathcal{M}_i \cap \{r > r_0\}.$$

Consider the boundary value problem in the *interior domain*

$$\begin{cases} (-\Delta_{\mathcal{M}} - \lambda)u = 0, \quad in \quad \mathcal{M}^{int}, \\ u = f \quad on \quad S_0 \end{cases} \tag{9}$$

one can define the D-N map :

$$\Lambda(\lambda) : f \to \frac{\partial}{\partial \nu} u\Big|_{S_0}.$$

Then :

The S-matrix $S_{ii}(\lambda)$ and D-N map $\Lambda_i(\lambda)$ determine each other.

This argument does not work well when $\mathcal{M}$ has singularities, since we use the potential theory in the proof.

Source-to-Solution map

Take a relatively compact open set $\mathcal{O} \subset \mathcal{M}$, and consider stationary source-to-solution operator

$$U_{\mathcal{O},\pm}(\lambda) : L^2(\mathcal{O}) \ni F \to (-\Delta_{\mathcal{M}} - \lambda \mp i0)^{-1} F \in L^2(\mathcal{O})$$

(F is extended to be 0 outside $\mathcal{O}$.) It has a physical meaning. Consider the wave equation with time-periodic source:

$$(\partial_t^2 - \Delta_{\mathcal{M}})u = e^{-ikt} F.$$

Then, as $t \to \pm\infty$, the solution also becomes time-periodic

$$u(t) \sim e^{-ikt} u_{\pm}, \quad u_{\pm} = (-\Delta_{\mathcal{M}} - k^2 \mp i0)^{-1} F.$$

This is called the limiting amplitude principle.
Then, the S-matrix determines the source-to-solution map.

Theorem

Assume that two ends $\mathcal{M}_i^{(1)}$ and $\mathcal{M}_i^{(2)}$ are isometric. If $S_{ii}^{(1)}(\lambda) = S_{ii}^{(2)}(\lambda)$ for some energy λ, then $U_{\mathcal{O},\pm}^{(1)}(\lambda) = U_{\mathcal{O},\pm}^{(2)}(\lambda)$.

Then, starting from $\mathcal{O} \subset \mathcal{M}^{reg}$, you can apply the boundary control method to determine the manifold and its metric around $\mathcal{O}$.

So, you can reconstruct $\mathcal{M}^{reg}$.

To determine $\mathcal{M}^{sing}$, recall that its codimension is at least 2. So, you can approach it from almost any direction.

A new problem here is that one needs to extend Tatar's uniqueness theorem for the singular metric. It can be done.

Remaining problem

Rellich-type theorem for the case $0 < \beta_0 \leq 1/3$.

$$ds^2 = (dr)^2 + \rho(r)^2 h(r, x, dx),$$

$$\rho(r) \geq C r^{\beta_0},$$

$$(-\Delta_g - \lambda) u = 0, \quad r > R,$$

$$u = o(\rho(r)^{-(n-1)/2})$$

$$\stackrel{?}{\Longrightarrow} u = 0 \quad r > R$$

Finally

Thank you for your attention!