Non-uniqueness results for the anisotropic Calderón problem at fixed energy

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The anisotropic Calderón problem

- Let (M, g) be a smooth, connected, compact and orientable Riemannian manifold with smooth boundary ∂M.
- Let Γ_D, Γ_N be non-empty open subsets of ∂M.
- Consider the Dirichlet problem at a fixed frequency $\lambda \notin \sigma(-\Delta_g)$

$$\begin{cases} -\Delta_g u = \lambda u & \text{on } M, \\ u = \psi & \text{on } \partial M. \end{cases} \tag{1}$$

where, in local coordinates,

$$\Delta_g u := \frac{1}{\sqrt{|g|}} \partial_i(\sqrt{|g|} g^{ij} \partial_j u) , \quad |g| := \det(g_{ij}) .$$

The Dirichlet-to-Neumann (DN) map

For all $\psi \in H^{1/2}(\partial M)$ with supp $\psi \subset \Gamma_D$,

$$\Lambda_{g,\Gamma_D,\Gamma_N}(\lambda)(\psi) := (\partial_\nu u)_{|\Gamma_N},$$

where

- $u \in H^1(M)$ is the unique solution of (1).
- $(\partial_\nu u)_{|\Gamma_N}$ is the normal derivative of u along Γ_N.

Three sub-cases of particular interest:

- **Full data**: $\Gamma_D = \Gamma_N = \partial M$, with DN map $=: \Lambda_g(\lambda)$.
- **Local data**: $\Gamma_D = \Gamma_N := \Gamma$, where Γ is any non-empty proper open subset of ∂M, with DN map $=: \Lambda_{g,\Gamma}(\lambda)$.
- **Data on disjoint sets**: Γ_D and Γ_N with $\Gamma_D \cap \Gamma_N = \emptyset$, with DN map $=: \Lambda_{g,\Gamma_D,\Gamma_N}(\lambda)$.

Some gauge invariances of the DN map

Gauge invariances

- If $\dim\ M \geq 2$, for all $\phi \in \mathrm{Diff}(M)$ such that $\phi_{|\Gamma_D \cup \Gamma_N} = Id$
$$\Lambda_{\phi^* g, \Gamma_D, \Gamma_N}(\lambda) = \Lambda_{g, \Gamma_D, \Gamma_N}(\lambda). \tag{2}$$
- If $\dim\ M = 2$ and $\lambda = 0$, then for all $c \in C^\infty(M)$ such that $c > 0$ and $c_{|\Gamma_N} = 1$,
$$\Lambda_{cg, \Gamma_D, \Gamma_N}(0) = \Lambda_{g, \Gamma_D, \Gamma_N}(0). \tag{3}$$

The anisotropic Calderón problem

Let M be a smooth compact connected orientable manifold with smooth boundary ∂M and let g, $\tilde{g}$ be smooth Riemannian metrics on M. Let λ be a fixed frequency that does not belong to $\sigma(-\Delta_g) \cup \sigma(-\Delta_{\tilde{g}})$. Let Γ_D, Γ_N be non empty open subsets of ∂M. If

$$\Lambda_{g,\Gamma_D,\Gamma_N}(\lambda) = \Lambda_{\tilde{g},\Gamma_D,\Gamma_N}(\lambda),$$

then is it true that

$$g = \tilde{g},$$

- up to the gauge invariances (2) in dimension ≥ 2
- up to the gauge invariances (2) and (3) in dimension 2 and $\lambda = 0$?

A brief non-exhaustive survey of some known results

The most comprehensive results are known for *zero frequency* $\lambda = 0$, assuming full data ($\Gamma_D = \Gamma_N = \partial M$) or local data ($\Gamma_D = \Gamma_N := \Gamma$).

Some uniqueness results in the case of local data

- If $\dim M = 2$ and (M, g) is smooth, then g is uniquely determined by $\Lambda_{g,\Gamma}(0)$ up to the gauge invariances (2) - (3), [Lee, Uhlmann](1993). For bounded Lipschitz domains of $\mathbb{R}^2$, with a reconstruction procedure, see [Nachman](1996).
- If $\dim M \geq 3$ and (M, g) is real analytic, then g is uniquely determined by $\Lambda_{g,\Gamma}(0)$ up to the gauge invariance (2), [Lee, Uhlmann] (1993), [Lassas, Uhlmann] (2001).
- If $\dim M \geq 3$ and (M, g) is Einstein (and thus analytic in its interior), then g is uniquely determined by $\Lambda_{g,\Gamma}(0)$ up to the gauge invariance (2), [Guillarmou, Sá Barreto] (2009).

If the background metric is not analytic, the general anisotropic Calderón problem in dimension $n \geq 3$ is still an open problem, whether one is dealing with full or local data. However, some important results exist for special classes of manifolds and metrics.

Definition

A manifold (M, g) is *conformally transversally anisotropic* if

$$M \subset\subset \mathbb{R} \times M_0, \quad g = c(e \oplus g_0),$$

where (M_0, g_0) is a given $(n-1)$-dimensional smooth compact connected Riemannian manifold with boundary, e is the Euclidean metric on $\mathbb{R}$ and c is a smooth strictly positive function in the cylinder $\mathbb{R} \times M_0$.

Theorem

If ∂M_0 is strictly convex and for all $x \in M_0$, the exponential map exp_x is a diffeomorphism from its maximal domain of definition in $T_x M_0$ onto M_0, then the conformal factor c is uniquely determined from the DN map for local data, [Dos Santos Ferreira, Kenig, Kurylev, Lassas, Salo, Sjöstrand, Vasy, Uhlmann](2009, 2013, 2016).

Finally, some results are known in the case of data on disjoint sets. For example:

Theorem

If $\Gamma_D \cap \Gamma_N = \emptyset$, then g is uniquely determined (up to the gauge invariance (2)) from $\Lambda_{g,\Gamma_D,\Gamma_N}(\lambda)$ at all frequencies λ (and under some technical assumptions), [Lassas, Oksanen] (2014), [Kurylev, Lassas, Oksanen] (2016).

Singular metrics

For metrics $g = (g_{ij})$ with measurable bounded coefficients satisfying the uniform ellipticity condition:

$$\sum_{i,j} g^{ij}(x)\xi_i\xi_j \geq c|\xi|^2 \text{ for a.e. } x \in M \text{ and } \xi \in \mathbb{R}^n, \quad c > 0,$$

with the DN map defined in a distributional sense, we have

Uniqueness results

- If M is a bounded domain of $\mathbb{R}^2$ and g is L^∞, there is uniqueness [Astala, Lassas, Päivärinta] (2006).
- If dim $M \geq 3$ and $g = c(x)\ Id$ is conformally flat, then the DN map determines $c(x)$ in the following cases:
  - $c \in C^{1,\frac{1}{2}+\epsilon}$, [Brown] (1996).
  - c Lipschitz with Lipschitz constant close to 1, [Haberman, Tataru] (2013).
  - c Lipschitz, [Caro, Rogers] (2016).
  - c with 3/2 derivatives, local data, [Krupchyk, Uhlmann](2016).

Singular metrics

Non-uniqueness results

Counterexamples to uniqueness to the global Calderón problem have been obtained for a class of metrics that are highly singular on a given closed hypersurface lying inside the manifold. The interior of the hypersurface is said to be "cloaked". [Greenleaf, Kurylev, Lassas and Uhlmann], (2003, 2009).

Non-uniqueness results

We have obtained counterexamples for metrics which are smooth in M. For disjoint data ($\Gamma_D \cap \Gamma_N = \emptyset$), these are also smooth on ∂M. For local data ($\Gamma_D = \Gamma_N \subset \partial M$), these are Hölder continuous on ∂M.

Main idea

Use a basic link between the Calderón problem for metrics in the conformal class $[g]$ of a fixed metric g and the Calderón problem for some related Schrödinger operators $-\Delta_g + V$.

This link relies on the transformation law of Laplace-Beltrami operators under conformal rescalings of the metric:

$$-\Delta_{c^4 g} u = c^{-(n+2)} \left(-\Delta_g + q_{g,c}\right) \left(c^{n-2} u\right), \quad q_{g,c} = c^{-n+2} \Delta_g c^{n-2},$$

$$q_{g,c} = \frac{n-2}{4(n-1)} \left(Scal_g - c^4 Scal_{c^4 g}\right).$$

The DN map for Schrödinger operators

DN map for Schrödinger operators

- Let (M, g) be a fixed Riemannian manifold as above. Let $V \in L^\infty(M)$ be a function on M. Let $\lambda \notin \sigma(-\Delta_g + V)$. Let Γ_D, Γ_N be non-empty open subsets of ∂M.
- For $\psi \in H^{\frac{1}{2}}(\partial M)$ with *supp* $\psi \subset \Gamma_D$, the DN map is defined by :

$$\Lambda_{g,V,\Gamma_D,\Gamma_N}(\lambda)(\psi) = (\partial_\nu u)_{|\Gamma_N},$$

where $u \in H^1(M)$ is the unique solution of

$$\begin{cases} (-\Delta_g + V)u = \lambda u, & \text{on } M, \\ u = \psi, & \text{on } \partial M. \end{cases}$$

The basic lemma

Lemma

Let $c \in C^\infty$, $c > 0$ on M, with $c = 1$ on $\Gamma_D \cup \Gamma_N$. Let $\lambda \notin \sigma(-\Delta_{c^4 g})$.

1. *If $\Gamma_D \cap \Gamma_N = \emptyset$, then*

$$\Lambda_{c^4 g, \Gamma_D, \Gamma_N}(\lambda) = \Lambda_{g, V_{g,c,\lambda}, \Gamma_D, \Gamma_N}(\lambda), \tag{4}$$

   where

$$V_{g,c,\lambda} = q_{g,c} + \lambda(1 - c^4), \quad q_{g,c} = c^{-n+2} \Delta_g c^{n-2}. \tag{5}$$

2. *If $\Gamma_D \cap \Gamma_N \neq \emptyset$ and $\partial_\nu c = 0$ on Γ_N, then (4)-(5) also holds.*

If we can find $c \in C^\infty$, $c > 0$, $c \neq 1$ on M such that $c = 1$ on $\Gamma_D \cup \Gamma_N$ and $V_{g,c,\lambda} = 0$ (and $\partial_\nu c = 0$ on Γ_N in case 2), then

$$\Lambda_{c^4 g,\Gamma_D,\Gamma_N}(\lambda) = \Lambda_{g,\Gamma_D,\Gamma_N}(\lambda)\,.$$

The condition $V_{g,c,\lambda} = 0$ is equivalent in terms of the conformal factor c to the non-linear elliptic pde

$$\Delta_g c^{n-2} + \lambda(c^{n-2} - c^{n+2}) = 0\,. \tag{6}$$

Non uniqueness in the case of local data

Main Lemma for local data

In the case $\Gamma_D = \Gamma_N := \Gamma$, (when $\lambda = 0$ and $n \geq 3$), in order to get

$$\Lambda_{c^4 g, \Gamma_D, \Gamma_N}(0) = \Lambda_{g, \Gamma_D, \Gamma_N}(0).$$

the conformal factor c must satisfy the same elliptic PDE as above, with $\lambda = 0$,

$$\begin{cases} \Delta_g c^{n-2} = 0, & \text{on } M, \\ c = 1, & \text{on } \Gamma, \end{cases} \tag{7}$$

together with $\partial_\nu c = 0$ on Γ.

- Idea: Construct a metric g such that $-\Delta_g$ does not satisfy the unique continuation principle (otherwise $c \equiv 1$).
- Rk: It is impossible to take $\Gamma = \partial M$, otherwise 0 would be a Dirichlet eigenvalue of the operator $-\Delta_g$ with eigenfunction $u = c - 1$.

The unique continuation principle (UCP)

The (UCP) for local Cauchy data

We say that a partial differential equation $P(x, D)u = 0$ on a domain Ω with smooth boundary satisfies the *unique continuation principle* if $P(x, D)u = 0$ in Ω and $u_{|\Gamma} = \partial_\nu u_{|\Gamma} = 0$, where Γ is a nonempty open set of $\partial\Omega$, implies the equality $u = 0$ on Ω.

Theorem [Hörmander, Tataru]

In dimension $n \geq 3$, the unique continuation principle holds for a second order uniformly elliptic operator if the coefficients of its principal part are locally Lipschitz continuous, while in dimension $n = 2$, the unique continuation principle holds if the coefficients of the principal part are L^∞.

Counterexamples to the UCP

Two classical counterexamples

In dimension $n = 3$, if the coefficients of the principal part are only *Hölder continuous* of order $\rho < 1$, there exist examples of non-unique continuation by [Pliś], (1963), for an elliptic pde in general form, and later by [Miller], (1972), for an elliptic pde in divergence form (the latter counterexample was improved by [Mandache], (1996)).

Our basic idea

We construct a metric g on a suitable manifold M such that the Laplace-Beltrami operator Δ_g coincides with Miller's elliptic operator and the conformal factor c is very close to Miller's solution.

Miller's counterexample

Miller constructed a smooth solution $u(t, x, y)$ of a uniformly elliptic equation in divergence form:

$$div\ (\mathcal{A}\ \nabla u) = 0, \tag{8}$$

where $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + a_1(t, x, y) + A_1(t) & a_2(t, x, y) \\ 0 & a_2(t, x, y) & 1 + a_3(t, x, y) + A_3(t) \end{pmatrix}. \tag{9}$$

This matrix $\mathcal{A}$ has its eigenvalues in $[\alpha, \alpha^{-1}]$ with ellipticity constant $\alpha \in (0, 1)$.

Miller's Theorem

Theorem (Miller (1972))

There exists an example of non-unique continuation on the half-space $E = [0, +\infty) \times \mathbb{R}^2$ *for a uniformly elliptic equation*

$$\partial_t^2 u + \partial_x((1 + a_1 + A_1)\partial_x u) + \partial_x(a_2 \partial_y u) + \partial_y(a_2 \partial_x u) + \partial_y((1 + a_3 + A_3)\partial_y u) = 0$$

1. *The solution* $u(t, x, y)$ *is* C^∞ *on* E, *identically zero for* $t \geq 1$, *but not identically zero in any open subset of* $[0, 1) \times \mathbb{R}^2$.
2. *The coefficients* $a_1(t, x, y)$, $a_2(t, x, y)$, $a_3(t, x, y)$ *are* C^∞ *on* E *and are identically zero for* $t \geq 1$.
3. *The coefficients* $A_1(t)$, $A_3(t)$ *are Hölder continuous on* $[0, \infty)$, C^∞ *on* $[0, 1)$, *and identically zero for* $t \geq 1$.
4. *All functions* u, a_1, a_2, a_3 *are periodic in* x *and* y *with period* 2π.

Construction of the Riemannian manifold

- Since the solution $u(t,x,y)$ is periodic in (x,y) with period 2π, [Giannotti], (2004), Miller's solution can be considered as a solution to an elliptic equation on the toroidal cylinder

$$M = [0,1] \times T^2 \,.$$

- We equip the manifold $M = [0,1] \times T^2$ with the Riemannian metric:

$$g = Ddt^2 + (1 + a_3 + A_3)dx^2 - 2a_2 dxdy + (1 + a_1 + A_1)dy^2, \quad (10)$$

where $D = det\ \mathcal{A}$. We have $\sqrt{|g|}\ (g^{-1}) = \mathcal{A}$, and Miller's solution satisfies

$$\Delta_g u = 0.$$

Properties of g and Miller's solution

- The boundary ∂M of M has two connected components:

$$\partial M = \Gamma_0 \cup \Gamma_1, \quad \Gamma_0 = \{0\} \times T^2, \quad \Gamma_1 = \{1\} \times T^2.$$

The metric

$$g = Ddt^2 + (1 + a_3 + A_3)dx^2 - 2a_2 dxdy + (1 + a_1 + A_1)dy^2$$

is smooth inside the manifold, but only Hölder continuous on the end Γ_1.

- Since the solution $u(t, x, y)$ is smooth on $E = [0, +\infty) \times \mathbb{R}^2$ and is identically zero for $t \geq 1$, all the derivatives of u are also identically zero at $t = 1$. In particular, one has:

$$u_{|\Gamma_1} = 0 \ , \ \partial_\nu u_{|\Gamma_1} = 0.$$

Definition and properties of the conformal factors

- We set
$$c_\epsilon(t,x,y) = 1 + \epsilon u(t,x,y),$$
and choose $\epsilon_0 > 0$ sufficiently small to ensure that $c_\epsilon(t,x,y) \geq \frac{1}{2}$ on M for all $\epsilon \in (0, \epsilon_0)$.
- These conformal factors c_ϵ are smooth on M, are not identically 1 on M, and satisfy :
$$\Delta_g c_\epsilon = 0 \text{ in } M \ , \ c_{\epsilon|\Gamma_1} = 1 \ , \ \partial_\nu c_{\epsilon|\Gamma_1} = 0. \tag{11}$$

Non-Uniqueness in the Calderon problem for local data and Hölder continuous metrics

We conclude:

Theorem

There exist an infinite number of smooth positive conformal factors c_ϵ which are not identically 1 on M, such that

$$\Lambda_{c_\epsilon^4 g,\Gamma_1} = \Lambda_{g,\Gamma_1}.$$

It remains to check that the metrics g and $c_\epsilon^4 g$ are not isometric:

- Assume that for all $0 < \epsilon_1 \leq \epsilon_0$, there exists $\epsilon \in (0, \epsilon_1)$ and a diffeomorphism $\phi_\epsilon : M \longrightarrow M$ s. t. $\phi_{\epsilon|\Gamma_1} = Id$ and $\phi_\epsilon^* g = c_\epsilon^4 g$. Since ϕ_ϵ is a diffeomorphism, $Vol_g(M) = Vol_{\phi_\epsilon^* g}(M) = Vol_{c_\epsilon^4 g}(M)$. Hence :

$$\int_M [(1 + \epsilon u)^6 - 1] \sqrt{|g|}\, dx = 0 \text{ for all } \epsilon > 0.$$

- The term of order 2 of this polynomial in the variable ϵ must be equal to 0, i.e $\int_M u^2 \sqrt{|g|}\, dx = 0$, which is not possible since u is not identically 0. So there exists $0 < \epsilon_1 \leq \epsilon_0$ such that g and $c_\epsilon^4 g$ are not isometric for all $\epsilon \in (0, \epsilon_1)$.

Non-uniqueness for disjoint sets - A new gauge invariance

From the basic lemma above, we obtain:

Corollary

Let $\lambda \notin \sigma(-\Delta_g)$ and let $\Gamma_D, \Gamma_N \subset \partial M$ be such that $\Gamma_D \cap \Gamma_N = \emptyset$. If there exists a smooth strictly positive function c satisfying

$$\begin{cases} \Delta_g c^{n-2} + \lambda(c^{n-2} - c^{n+2}) = 0, & on\ M, \\ c = 1, & on\ \Gamma_D \cup \Gamma_N, \end{cases} \tag{12}$$

then the conformally rescaled Riemannian metric $c^4 g$ satisfies

$$\Lambda_{c^4 g, \Gamma_D, \Gamma_N}(\lambda) = \Lambda_{g, \Gamma_D, \Gamma_N}(\lambda).$$

Solving the nonlinear elliptic PDE

- Setting $w = c^{n-2}$, the condition (12) can be written as the nonlinear Dirichlet problem:

$$\begin{cases} \Delta_g w + \lambda(w - w^{\frac{n+2}{n-2}}) = 0 & \text{on } M, \\ w = \eta & \text{on } \partial M, \end{cases} \tag{13}$$

  where η is any smooth positive function such that $\eta = 1$ on $\Gamma_D \cup \Gamma_N$.

- To find solutions of (13) with $w \neq 1$ on M, we make the crucial assumption

$$\overline{\Gamma_D \cup \Gamma_N} \neq \partial M,$$

  and we use the well-known technique of lower and upper solutions.

Upper and lower solutions

Upper and lower solutions

- An upper solution $\overline{w}$ is a function in $C^2(M) \cap C^0(\overline{M})$ satisfying

$$\Delta_g \overline{w} + \lambda(\overline{w} - \overline{w}^{\frac{n+2}{n-2}}) \leq 0 \text{ on } M, \quad \text{and} \quad \overline{w}_{|\partial M} \geq \eta. \tag{14}$$

- A lower solution $\underline{w}$ is a function in $C^2(M) \cap C^0(\overline{M})$ satisfying

$$\Delta_g \underline{w} + \lambda(\underline{w} - \underline{w}^{\frac{n+2}{n-2}}) \geq 0 \text{ on } M, \quad \text{and} \quad \underline{w}_{|\partial M} \leq \eta. \tag{15}$$

We shall use the well-known result :

Lemma

Assume we can find a lower solution $\underline{w}$ and an upper solution $\overline{w}$ satisfying $\underline{w} \leq \overline{w}$ on M.

Then there exists a solution $w \in C^\infty(\overline{M})$ of (13) such that $\underline{w} \leq w \leq \overline{w}$.

Theorem

For all $\lambda \notin \sigma(-\Delta_g)$ and for all smooth positive functions η such that $\eta \neq 1$ on ∂M, there exists a positive solution $w \in C^\infty(\overline{M})$ of (13) satisfying $w \neq 1$ on M.

Proof

Assume for instance that $\lambda \geq 0$.

- If $\eta \gneq 1$, then $\underline{w} = 1$ is a lower solution and $\overline{w} = \max \eta$ is an upper solution of (13). Moreover, they clearly satisfy $\underline{w} \leq \overline{w}$.
- Likewise, if $0 < \eta \lneq 1$, then $\underline{w} = \min \eta$ is a lower solution and $\overline{w} = 1$ is an upper solution of (13). They still satisfy $\underline{w} \leq \overline{w}$.
- Finally, if $0 < \min \eta < 1 < \max \eta$, then $\underline{w} = \min \eta$ is a lower solution and $\overline{w} = \max \eta$ is an upper solution of (13). Moreover, they satisfy $\underline{w} \leq \overline{w}$. □

At this stage, we have found conformal factors c^4 such that

$$\Lambda_{c^4 g, \Gamma_D, \Gamma_N}(\lambda) = \Lambda_{g, \Gamma_D, \Gamma_N}(\lambda).$$

These conformal factors c^4, which satisfy a nonlinear elliptic PDE, can be viewed as a natural gauge invariance of the anisotropic Calderón problem with data on disjoint sets.

We can also construct another large class of counterexamples to uniqueness modulo this gauge invariance for a particular class of cylindrical Riemannian manifolds.

Cylindrical Riemannian manifolds

The model

Let us consider the following cylindrical Riemannian manifold equipped with a warped product metric:

$$M = [0,1]_x \times K_\omega, \quad g = f^4(x)[dx^2 + g_K].$$

- K is an arbitrary $(n-1)$-dimensional closed manifold.
- $f = f(x)$ is a smooth positive function on $[0,1]$ and g_K is a smooth Riemannian metric on K.

∂M has two connected components, $\partial M = \Gamma_0 \cup \Gamma_1$.

Non uniqueness modulo the gauge

We have the following result :

Theorem

Let $(M = [0,1] \times K, g)$ be as above and let Γ_D, Γ_N belong to different connected components of ∂M. Let $\lambda \in \mathbb{R}$ be a fixed frequency. Then there exists an infinite number of smooth positive conformal factors c and $\tilde{c}$ on M which aren't gauge related in the above sense, such that

$$\Lambda_{c^4 g, \Gamma_D, \Gamma_N}(\lambda) = \Lambda_{\tilde{c}^4 g, \Gamma_D, \Gamma_N}(\lambda).$$

We remark that this non-uniqueness result holds when $\Gamma_D = \Gamma_0$ and $\Gamma_N = \Gamma_1$, hence when $\overline{\Gamma_D \cup \Gamma_N} = \partial M$.

Strategy, 1

The proof of the last theorem relies on the following non uniqueness result for anisotropic Calderón problem for Schrödinger operators:

Theorem

Let $M = [0,1] \times K$ be a cylindrical manifold having two ends equipped with a warped product metric $g = f^4(x)[dx^2 + g_K]$, $V = V(x) \in L^\infty(M)$ and $\lambda \in \mathbb{R}$ not belonging to the Dirichlet spectrum of $-\Delta_g + V$. Then there exists an infinite family of potentials $\tilde{V}$ that satisfy

$$\Lambda_{g,V,\Gamma_D,\Gamma_N}(\lambda) = \Lambda_{g,\tilde{V},\Gamma_D,\Gamma_N}(\lambda),$$

whenever Γ_D and Γ_N are open sets that belong to different connected components of ∂M.

Remark: The family of potentials $\tilde{V}$ is explicit in terms of g and V.

Strategy, 2

Assume this last theorem is true, then we can easily prove the non uniqueness for the anisotropic DN map modulo the gauge invariance:

Main steps of the proof

- Start from $V \neq \tilde{V}$ such that $\Lambda_{g,V,\Gamma_D,\Gamma_N}(\lambda) = \Lambda_{g,\tilde{V},\Gamma_D,\Gamma_N}(\lambda)$.
- Construct conformal factors c and $\tilde{c}$ such that $V_{g,c,\lambda} = V$ and $V_{g,\tilde{c},\lambda} = \tilde{V}$, and $c = \tilde{c} = 1$ on $\Gamma_D \cup \Gamma_N$.
- Then $\Lambda_{c^4 g,\Gamma_D,\Gamma_N}(\lambda) = \Lambda_{\tilde{c}^4 g,\Gamma_D,\Gamma_N}(\lambda)$.
- Finally, if $c^4 g$ and $\tilde{c}^4 g$ are gauge related, we can prove that $V = \tilde{V}$.

It remains to prove the non uniqueness result for the DN map for the Schrödinger operators.

The global DN map $\Lambda_{g,V}(\lambda)$: first simplifications

- ∂M has two components: $\partial M = \Gamma_0 \cup \Gamma_1$ where $\Gamma_0 \simeq \Gamma_1 \simeq K$.
- For any $s \in \mathbb{R}$, $H^s(\partial M) = H^s(\Gamma_0) \oplus H^s(\Gamma_1)$. We use the vector notation

$$\varphi = \begin{pmatrix} \varphi^0 \\ \varphi^1 \end{pmatrix}, \quad \forall \varphi \in H^s(\partial M) = H^s(\Gamma_0) \oplus H^s(\Gamma_1).$$

- The DN map is a linear operator from $H^{1/2}(\partial M)$ to $H^{-1/2}(\partial M)$ and thus has the structure of an operator-valued 2×2 matrix

$$\Lambda_{g,V}(\lambda) = \begin{pmatrix} \Lambda_{g,V,\Gamma_0,\Gamma_0}(\lambda) & \Lambda_{g,V,\Gamma_1,\Gamma_0}(\lambda) \\ \Lambda_{g,V,\Gamma_0,\Gamma_1}(\lambda) & \Lambda_{g,V,\Gamma_1,\Gamma_1}(\lambda) \end{pmatrix},$$

where $\Lambda_{g,V,\Gamma_j,\Gamma_k}(\lambda)$ are operators from $H^{1/2}(K)$ to $H^{-1/2}(K)$.
- For smooth enough boundary data ψ, we have

$$\Lambda_{g,V}(\lambda) \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} = \begin{pmatrix} (\partial_\nu u)_{|\Gamma_0} \\ (\partial_\nu u)_{|\Gamma_1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{f^4(0)}}(\partial_x u)_{|x=0} \\ \frac{1}{\sqrt{f^4(1)}}(\partial_x u)_{|x=1} \end{pmatrix}.$$

Decomposition of the global DN map on angular harmonics

Decomposition

- Let $(Y_k)_{k\geq 0}$ be a Hilbert basis of eigenfunctions of the Laplace-Beltrami operator $-\triangle_K$ associated to the eigenvalues $(\mu_k)_{k\geq 0}$.
- Write $\psi = (\psi^0, \psi^1) \in H^{1/2}(\Gamma_0) \times H^{1/2}(\Gamma_1)$ as

$$\psi^0 = \sum_{k\geq 0} \psi^0_k Y_k, \quad \psi^1 = \sum_{k\geq 0} \psi^1_k Y_k.$$

- We look for the unique solution u of the Dirichlet problem of the form

$$u = \sum_{k\geq 0} u_k(x) Y_k(\omega).$$

Decomposition of the global DN map on angular harmonics

For any $k \geq 0$, the function $v_k = f^{n-2} u_k$ is the unique solution of the boundary value problem given by

$$\begin{cases} -v_k'' + [q_f + (V - \lambda) f^4] v_k = -\mu_k v_k, \text{ on } [0,1], \\ v_k(0) = f^{n-2}(0)\psi_k^0, \quad v_k(1) = f^{n-2}(1)\psi_k^1, \end{cases} \tag{16}$$

where $q_f = \frac{(f^{n-2})''}{f^{n-2}}$.

The restriction $\Lambda_{g,V}^k(\lambda)$ of the global DN map onto each harmonic $< Y_k >$ has the structure of a 2×2 matrix and satisfies for all $k \geq 0$

$$\Lambda_{g,V}^k(\lambda) \begin{pmatrix} \psi_k^0 \\ \psi_k^1 \end{pmatrix} = \begin{pmatrix} \frac{(n-2)f'(0)}{f^{n+1}(0)} v_k(0) - \frac{v_k'(0)}{f^n(0)} \\ -\frac{(n-2)f'(1)}{f^{n+1}(1)} v_k(1) + \frac{v_k'(1)}{f^n(1)} \end{pmatrix}.$$

The radial ODE

We can express the global Dirichlet to Neumann map on each harmonic using the Weyl-Titchmarsh formalism.

- Consider the boundary value problem

$$\begin{cases} -v'' + [q_f + (V-\lambda)f^4]v = -\mu v, \text{ on } [0,1], \\ v(0) = 0, \quad v(1) = 0. \end{cases} \tag{17}$$

- Since $q_f + (V-\lambda)f^4 \in L^1([0,1])$, we can define for all $\mu \in \mathbb{C}$ the fundamental systems of solutions

$$\{c_0(x,\mu), s_0(x,\mu)\}, \quad \{c_1(x,\mu), s_1(x,\mu)\},$$

of (17) by imposing the Cauchy conditions

$$\begin{array}{llll} c_0(0,\mu)=1, & c_0'(0,\mu)=0, & s_0(0,\mu)=0, & s_0'(0,\mu)=1, \\ c_1(1,\mu)=1, & c_1'(1,\mu)=0, & s_1(1,\mu)=0, & s_1'(1,\mu)=1. \end{array}$$

The characteristic and Weyl-Titchmarsh functions

The characteristic function

The characteristic function is defined by

$$\Delta_{g,V}(\mu) = W(s_0, s_1).$$

The Weyl-Titchmarsh functions

The Weyl solutions Ψ and Φ are the unique solutions of (17) having the form

$$\Psi(x,\mu) = c_0(x,\mu) + M_{g,V}(\mu)s_0(x,\mu),$$
$$\Phi(x,\mu) = c_1(x,\mu) - N_{g,V}(\mu)s_1(x,\mu),$$

which satisfy the boundary conditions at $x = 1$ and $x = 0$ respectively. The Weyl-Titchmarsh functions are thus given by

$$M_{g,V}(\mu) = -\frac{W(c_0, s_1)}{\Delta_{g,V}(\mu)}, \quad N_{g,V}(\mu) = -\frac{W(c_1, s_0)}{\Delta_{g,V}(\mu)}.$$

The final expression of the global DN map

Recall that for all $k \geq 0$

$$\Lambda_{g,V}^{k}(\lambda)\begin{pmatrix}\psi_k^0\\ \psi_k^1\end{pmatrix} = \begin{pmatrix} \frac{(n-2)f'(0)}{f^{n+1}(0)} v_k(0) - \frac{v_k'(0)}{f^n(0)} \\ -\frac{(n-2)f'(1)}{f^{n+1}(1)} v_k(1) + \frac{v_k'(1)}{f^n(1)} \end{pmatrix}.$$

Final expression

Writing v_k with the fundamental systems of solutions $c_j(x,\mu)$ and $s_j(x,\mu)$, we get for $\Lambda_{g,V}^{k}(\lambda)$

$$\Lambda_{g,V}^{k}(\lambda) = \begin{pmatrix} \frac{(n-2)f'(0)}{f^3(0)} - \frac{M_{g,V}(\mu_k)}{f^2(0)} & -\frac{f^{n-2}(1)}{f^n(0)\Delta_{g,V}(\mu_k)} \\ -\frac{f^{n-2}(0)}{f^n(1)\Delta_{g,V}(\mu_k)} & -\frac{(n-2)f'(1)}{f^3(1)} - \frac{N_{g,V}(\mu_k)}{f^2(1)} \end{pmatrix}.$$

As a consequence, assume for instance that $\Gamma_D \subset \Gamma_0$ and $\Gamma_N \subset \Gamma_1$.

The knowledge of the partial DN map $\Lambda_{g,V,\Gamma_D,\Gamma_N}(\lambda)$ is equivalent to that of

$$-\sum_k \left[\frac{f^{n-2}(0)}{f^n(1)\Delta_{g,V}(\mu_k)} \right] \psi_k Y_k(\omega),$$

for all $\omega \in \Gamma_N$ and for all $\psi \in H^{1/2}(K)$ with supp $\psi \subset \Gamma_D$.

Characteristic function and isospectrality

Recall that

$$\begin{cases} -v'' + [q_f + (V-\lambda)f^4]v = -\mu v, \text{ on } [0,1], \\ v(0) = 0, \quad v(1) = 0, \end{cases} \tag{18}$$

where $q_f = \frac{(f^{n-2})''}{f^{n-2}}$. We can prove the following lemma:

Lemma

Let $g = f^4(x)[dx^2 + g_K]$ be a fixed metric and $V = V(x), \tilde{V} = \tilde{V}(x) \in L^\infty(M)$. Let $\lambda \in \mathbb{R}$ not belong to the Dirichlet spectra of $-\Delta_g + V$ and $-\Delta_g + \tilde{V}$. Then

$$\Delta_{g,V}(\mu) = \Delta_{g,\tilde{V}}(\mu), \quad \forall \mu \in \mathbb{C},$$

if and only if

$$q_f + (V-\lambda)f^4 \text{ and } q_f + (\tilde{V}-\lambda)f^4 \text{ are isospectral for } (18).$$

Isospectral potentials

Pöschel and Trubowitz gave a complete description of isospectral potentials for the Schrödinger operators with Dirichlet boundary conditions (18). Precisely, for each eigenfunction ϕ_l, $l \geq 1$ of (18), we can find a one parameter family of explicit potentials isospectral to $Q(x) = q_f + (V - \lambda)f^4 \in L^2([0,1])$ by the formula

$$Q_{l,t}(x) = Q(x) - 2\frac{d^2}{dx^2}\log\theta_{l,t}(x), \qquad \forall t \in \mathbb{R},$$

where

$$\theta_{l,t}(x) = 1 + (e^t - 1)\int_x^1 \phi_l^2(s)ds.$$

Given V, we get one-parameter families of isospectral potentials $\tilde{V}$

$$\tilde{V}_{l,t}(x) = V(x) - \frac{2}{f^4(x)}\frac{d^2}{dx^2}\log\theta_{l,t}(x), \quad \forall l \geq 1, \quad \forall t \in \mathbb{R}. \tag{19}$$

Main Theorem

Theorem

Let $M = [0,1] \times K$ *be a cylindrical manifold having two ends equipped with a warped product metric* $g = f^4(x)[dx^2 + g_K]$, $V = V(x) \in L^\infty(M)$ *and* $\lambda \in \mathbb{R}$ *not belong to the Dirichlet spectrum of* $-\triangle_g + V$. *Then the family of potentials* $\tilde{V}_{l,t}$ *defined in (19) for all* $l \geq 1$ *and* $t \in \mathbb{R}$ *satisfies*

$$\Lambda_{g,V,\Gamma_D,\Gamma_N}(\lambda) = \Lambda_{g,\tilde{V}_{l,t},\Gamma_D,\Gamma_N}(\lambda),$$

whenever Γ_D *and* Γ_N *are open sets that belong to different connected components of* ∂M.

Some perspectives

- Global data?
- Models with more than two boundary components?
- Is it necessary for the boundary of the manifold to be compatible with the separation of variables?
- Extensions to operators acting on sections of vector bundles?

Some references

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Thank you very much for your attention!