Stability estimates for inverse problems for PDE with unknown boundaries

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INTRODUCTION - THE SECOND ORDER ELLIPTIC CASE

Assume Ω bounded domain, $\partial\Omega \in C^{1,\alpha}$, $\partial\Omega = \Gamma^{(a)} \cup \Gamma^{(i)}$ and $\Gamma^{(i)} = \partial\Omega \setminus \mathrm{Int}_{\partial\Omega}\Gamma^{(a)}$

Given: A (Symmetric, elliptic, Lipschitz) and $\psi \not\equiv 0$ s.t.

$$\psi = 0 \text{ , on } \Gamma^{(i)},$$

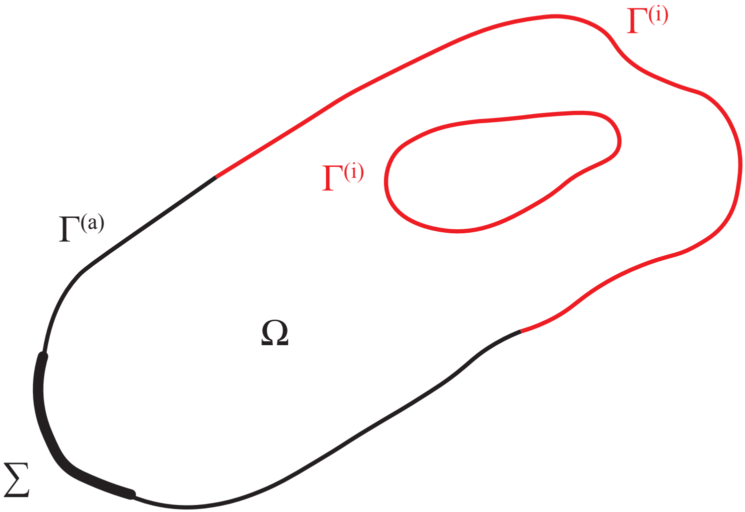
Let u be solution to

$$\begin{cases} \mathrm{div}\,(A\nabla u) = 0, & \text{in } \Omega, \\ \\ u = \psi, & \text{on } \partial\Omega, \end{cases}$$

Assume to know

$$A\nabla u \cdot \nu, \text{ on } \Sigma \subset \Gamma^{(a)},$$

Determine: $\Gamma^{(i)}$


$\Gamma^{(i)}$
$\Gamma^{(i)}$
$\Gamma^{(a)}$
Ω
Σ

- **Stability Issue**: continuous dependence of $\Gamma^{(i)}$ from the Cauchy data

$$u\ ,\ A\nabla u\cdot\nu \text{ on } \Sigma$$

- We prove a **logarithmic (i.e. optimal) stability estimate.**

Some references

- **second order elliptic equations**:
  Beretta, V., (1998); Alessandrini, Beretta, Rosset, V. , (2000); J.Cheng, Y. C. Hon, M. Yamamoto, (2001); Inglese, Mariani (2004); Bacchelli, V. (2006); Sincich, (2010).
- **3D elasticity systems (log-log estimate)**:
  Morassi, Rosset, (2004), (2009).
- **plate equation and generalized plane stress problem**:
  Morassi, Rosset, V. (2012), (2019), (2020).
- **parabolic equations**:
  V. (1997); Francini (2000); Canuto, Rosset, V., (2002); V. (2008); H. Kawakami, M. Tsuchiya, (2013).
- **wave equation**:
  V. (2015).
- **optimality of log estimates**:
  Alessandrini, (1997) (elliptic case); Di Cristo, Rondi, V., (2006) (parabolic case).

Strategy in the 2nd order elliptic I.P.

In [Alessandrini, Beretta, Rosset, V. , (2000)] , in order to prove optimal stability estimate we have used:

- Stability estimates for Cauchy problem and smallness propagation estimates
- Finite vanishing property at the interior and at the boundary

Let P be an elliptic operator of order 2. We say that P enjoys a **finite vanishing property at the interior if**
(Aronszajin's Theorem, 1962)

for any $x_0 \in \Omega$ and any non identically vanishing solution u to

$$Pu = 0 \qquad \text{in} \quad \Omega$$

we have

$$\|u\|_{L^2(B_r(x_0))} \geq Cr^N, \forall r \in (0, r_0).$$

where $N, C, r_0 > 0$ may depend on u

Similarly, we say that P enjoys a **finite vanishing property at the boundary**, for instance, w.r.t. Dirichlet conditions, if (Adolfsson–Escauriaza Theorem, 1997)

for any non identically vanishing u that satisfies

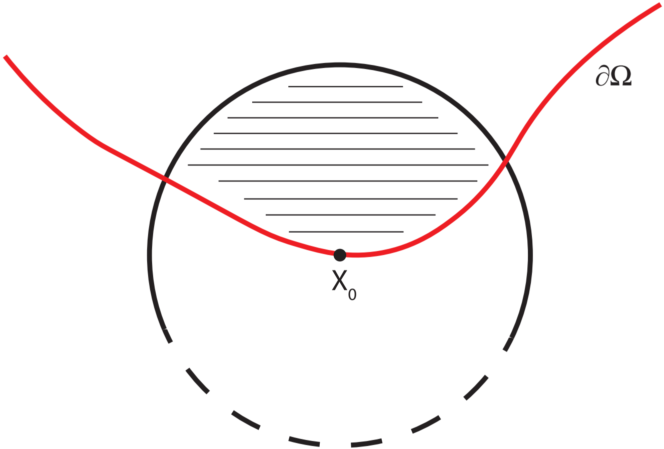
$$\begin{cases} Pu = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma, \end{cases} .$$

where Γ is an open portion (in the induced topology) of $\partial\Omega$, $x_0 \in \Gamma$ we have

$$\|u\|_{L^2(B_r(x_0)\cap\Omega)} \geq Cr^N, \forall r \in (0, r_0).$$

and, consequently

$$\|\nabla u\|_{L^2(B_r(x_0)\cap\Omega)} \geq \widetilde{C}r^{N-1}, \forall r \in (0, r_0).$$


∂Ω
X0

Sketch of proof of stability estimate

u_j, $j = 1, 2$ solutions to

$$\begin{cases} \text{div}\left(A(x)\nabla u_j\right) = 0, & \text{in } \Omega_j, \\ u_j = \psi_j, & \text{on } \partial\Omega_j. \end{cases}$$

($\psi_j = \psi$ on $\Gamma^{(a)}$, $\psi_1 = 0$ on $\Gamma_1^{(i)}$ and $\psi_2 = 0$ on $\Gamma_2^{(i)}$)
Assume

$$\|A\nabla u_1 \cdot \nu - A\nabla u_2 \cdot \nu\|_{L^2(\Sigma)} \leq \varepsilon$$

Set

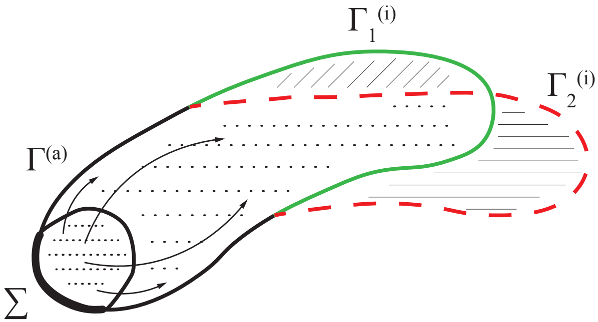
$$G = \text{ the connected component of } \Omega_1 \cap \Omega_2 \text{ s.t. } \overline{G} \supset \Gamma^{(a)}.$$

MAIN STEPS
(I) Estimate of

$$u_j \quad \text{in} \quad \Omega_j \setminus G \quad j = 1, 2$$

(II) From (I) we estimate $d_{\mathcal{H}}(\overline{\Omega}_1, \overline{\Omega}_2)$ (Hausdorff distance).

STEP (I)



Stability Estimate for Cauchy Problem : : : :
Smallness Propagation Estimates

Energy Estimate for u_1, u_2

STEP (II)

Proposition

If

$$\int_{\Omega_j \setminus G} u_j^2 \leq \eta^2(\varepsilon)$$

then

$$d_{\mathcal{H}}\left(\overline{\Omega}_1, \overline{\Omega}_2\right) \leq C\eta^s(\varepsilon),$$

where s and C depend on

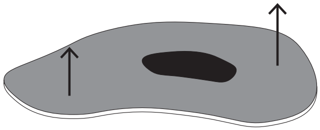
$$\|\psi\|_{H^{1/2}} \,/\, \|\psi\|_{L^2}$$

Proof. By Quantitative Estimates of Strong Unique Continuation (at Interior and at the Boundary)

$$\eta^2(\varepsilon) \geq \int_{\Omega_j \setminus G} u_j^2 \geq C\left(d_{\mathcal{H}}\left(\overline{\Omega}_1, \overline{\Omega}_2\right)\right)^C.$$

DETERMINATION OF A RIGID INCLUSION IN A THIN ISOTROPIC ELASTIC PLATE

Thin elastic plate: $\Omega \times \left[-\frac{h}{2}, \frac{h}{2}\right]$, having middle plane Ω, D rigid inclusion



$$\mathcal{L}w := \mathrm{div}\left(\mathrm{div}\left(\mathbf{P}\nabla^2 w\right)\right) = 0, \qquad \text{in } \Omega \setminus \overline{D}.$$

where w is the **transversal displacement** and

$$\underbrace{\mathbf{P}}_{\textit{plate tensor}} = \frac{h^3}{12} \underbrace{\mathbf{C}}_{\textit{elasticity tensor}}$$

$$C_{ijkl} = C_{klij} = C_{klji}, \quad i, j, k, l = 1, 2$$

$$\underbrace{\mathbf{C}A \cdot A \geq \gamma |A|^2}_{\textit{Ellipticity}},$$

for every 2x2 symmetric matrix A.

Assuming that the plate is made by isotropic material we have

$$\mathbf{P}A = B\left[(1-\nu)A^{sym} + \nu tr(A) I_2\right]$$

for every 2×2 matrix A, where

$$B(x) = \frac{h^3}{12}\left(\frac{E(x)}{1-\nu^2(x)}\right), \text{ (bending stiffness)}$$

$$E(x) = \frac{\mu(x)(2\mu(x)+3\lambda(x))}{\mu(x)+\lambda(x)}, \text{ (Young's modulus)}$$

$$\nu(x) = \frac{\lambda(x)}{2(\mu(x)+\lambda(x))} \text{ (Poisson's coefficient).}$$

the Lamé parameters λ, μ satisfy

$$\mu(x) \geq \alpha_0 \quad 2\mu(x) + 3\lambda(x) \geq \gamma_0$$

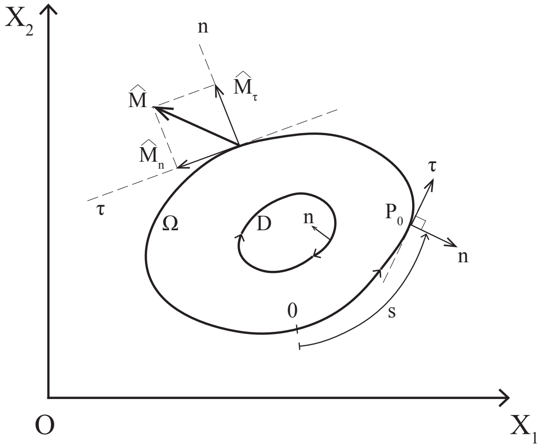
Direct Problem:

$D \Subset \Omega$ rigid inclusion, D, Ω simply connected bounded domain of class $C^{1,1}$ (at least)

$$(P)\begin{cases} \mathcal{L}w = 0, & \text{in } \Omega \setminus \overline{D}, \\ (\mathbf{P}\nabla^2 w)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \\ div(\mathbf{P}\nabla^2 w) \cdot n + \partial_s((\mathbf{P}\nabla^2 w)n \cdot \tau) = \partial_s(\widehat{M}_\tau), & \text{on } \partial\Omega, \\ w = 0, & \text{on } \partial D, \\ \partial_n w = 0, & \text{on } \partial D, \end{cases}$$

n outward normal to $\partial(\Omega \setminus D)$, $\widehat{M}_\tau$ and $\widehat{M}_n$ are, respectively, the twisting and bending component of the assigned couple field $\widehat{M}$.

Here, $\Gamma^{(a)} = \partial\Omega$ **and** $\Gamma^{(i)} = \partial D$.



$$\widehat{M} = \widehat{M}_\tau n + \widehat{M}_n \tau = \widehat{M}_2 e_1 + \widehat{M}_1 e_2, \qquad \text{on } \partial\Omega$$

$$\tau = e_2 \times n$$

If $\widehat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$, $\int_{\partial\Omega} \widehat{M}_\alpha = 0$, $\alpha = 1, 2$, then problem (P) has a unique solution weak solution $w \in H^2(\Omega \setminus \overline{D})$ satisfying

$$\|w\|_{H^2(\Omega\setminus\overline{D})} \leq C\|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}.$$

INVERSE PROBLEM

Determine an **unknown** rigid inclusion D from the additional measurement of the Dirichlet data $\{w, \partial_n w\}$ taken on an open portion Σ of $\partial\Omega$, that is from the Cauchy data on Σ :

$$(Cauchy) \begin{cases} w|_{\Sigma}, \\ \partial_n w|_{\Sigma} \\ (\mathbf{P}\nabla^2 w)n \cdot n|_{\Sigma} = -\widehat{M}_n \\ div(\mathbf{P}\nabla^2 w) \cdot n + \partial_s((\mathbf{P}\nabla^2 w)n \cdot \tau)|_{\Sigma} = \partial_s(\widehat{M}_\tau) \end{cases}$$

APPLICATIONS

Non-destructive testing for quality assessment of materials

Hypotheses and a priori assumptions

HYPOTHESES (Concerning the Data)

- $\partial\Omega$ of class $C^{2,1}$ with constants r_0 , M_0 ; $\partial\Omega \cap B_{r_0}(P_0) \subset \Sigma$, for some $P_0 \in \Sigma$
- $|\Omega| \leq M_1$
- $\mathrm{supp}(\widehat{M}) \subset \Sigma$, $\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2)$, $\left(\widehat{M}_n, \partial_s(\widehat{M}_\tau)\right) \not\equiv 0$ and $\dfrac{\|\widehat{M}\|_{L^2}}{\|\widehat{M}\|_{H^{-1/2}}} \leq F$
- Σ of class $C^{3,1}$ with constants r_0 , M_0

A PRIORI ASSUMPTION (Concerning the Solution)

- $D \Subset \Omega$
- $\text{dist}(D, \partial\Omega) \geq r_0$
- ∂D of class $C^{6,\alpha}$ with constants r_0 , M_0 , $\alpha \in (0, 1)$

Theorem (Stability, Morassi, Rosset, V. (2019))

Let $w_i \in H^2(\Omega \setminus \overline{D_i})$ *be the solutions to (P),* $i = 1, 2$*.*
If, given $\varepsilon > 0$*, we have*

$$\left\{ \|w_1 - w_2\|_{L^2(\Sigma)} + \|\partial_n(w_1 - w_2)\|_{L^2(\Sigma)} \right\} \leq \varepsilon,$$

then we have

$$d_{\mathcal{H}}(\overline{D_1}, \overline{D_2}) \leq C(|\log \varepsilon|)^{-\eta},$$

for every ε*,* $0 < \varepsilon < 1$*, where* $C > 0$*,* η*,* $0 < \eta \leq 1$*, are constants only depending on the a priori data.*

$d_{\mathcal{H}}(\overline{D_1}, \overline{D_2})$ is the Hausdorff distance between $\overline{D_1}$ and $\overline{D_2}$.

Main tool of the proof

Theorem (Optimal three spheres inequality at the boundary)

If $x_0 \in \partial D$ *and*

$$\mathcal{L}w = 0, \quad \text{in } \Omega \setminus \overline{D},$$

there exist $C > 1$ *such that, for every* $r_1 < r_2 < r_3 < dist(x_0, \partial\Omega)$,

$$\|w\|_{L^2(B_{r_2}(x_0)\cap(\Omega\setminus\overline{D}))} \leq C\left(\frac{r_3}{r_2}\right)^C \|w\|^{\theta}_{L^2(B_{r_1}(x_0)\cap(\Omega\setminus\overline{D}))} \|w\|^{1-\theta}_{L^2(B_{r_3}(x_0)\cap(\Omega\setminus\overline{D}))}$$

where

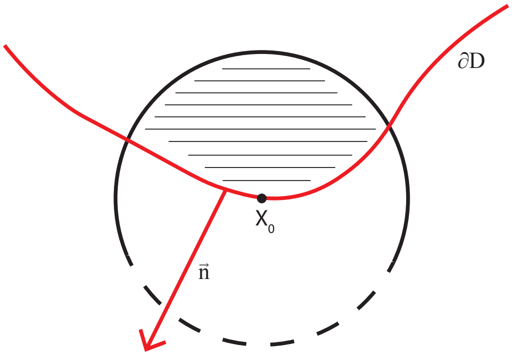
$$\theta = \frac{\log\left(\frac{r_3}{Cr_2}\right)}{\log\left(\frac{r_3}{r_1}\right)}.$$

Alessandrini, Rosset, V., ARMA, 2019

Corollary (finite vanishing rate at the boundary)

Under the above hypotheses, there exist C, N such that

$$\int_{B_r(x_0)\cap(\Omega\setminus\overline{D})} w^2 \geq Cr^N$$



In the interior, similar results hold true. In particular we have

Theorem (finite vanishing rate in the interior)

If $x_0 \in \Omega \setminus \overline{D}$ *and* $B_r(x_0) \Subset \Omega \setminus \overline{D}$ *there exist* C, N *such that*

$$\int_{B_r(x_0)} \left|\nabla^2 w\right|^2 \geq Cr^N$$

First qualitative result:

Taira Shirota, A remark on the unique continuation theorem for certain fourth order elliptic equations, Proc. Japan Acad. 36 (1960), 571–573.

Basic steps of the stability proof

Similarly to 2nd order case:

a) Stability estimates of continuation from Cauchy data:

$$\max\left\{\int_{D_1\setminus\overline{D}_2}|\nabla^2 w_2|^2, \int_{D_2\setminus\overline{D}_1}|\nabla^2 w_1|^2\right\}\leq\omega(\varepsilon)$$

b) by the Three Sphere Inequality in the interior and at the boundary,

$$d_{\mathcal{H}}(\overline{D_1},\overline{D_2})\leq\left(\max\left\{\int_{D_1\setminus\overline{D}_2}|\nabla^2 w_2|^2, \int_{D_2\setminus\overline{D}_1}|\nabla^2 w_1|^2\right\}\right)^{\delta}\leq(\omega(\varepsilon))^{\delta}$$

Another result of finite rate vanishing at the boundary

Let $x_0 \in \partial D$

$$\begin{cases} \mathcal{L}w = 0, & \text{in } B_{R_0}(x_0) \cap (\Omega \setminus \overline{D}), \\ w = 0, & \text{on } B_{R_0}(x_0) \cap \partial D, \\ (\mathbf{P}\nabla^2 w)n \cdot n = 0, & \text{on } B_{R_0}(x_0) \cap \partial D, \end{cases}$$

Theorem (Rosset, Morassi, V. (in preparation))

Under the above hypotheses, there exist C, N such that

$$\int_{B_r(x_0)\cap(\Omega\setminus\overline{D})} w^2 \geq C r^N$$

GENERALIZED PLANE STRESS PROBLEM

Here, $u = u_1 e_1 + u_2 e_2$ represents the in-plane displacement field. Let us consider the two-dimensional system

$$\partial_\beta N_{\alpha\beta} = 0, \quad \text{in } \mathcal{U} := B_{R_0}(x_0) \cap (\Omega \setminus \overline{D}) \tag{1}$$

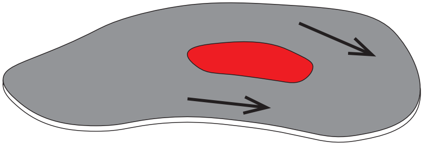
where

$$N_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta}, \quad \epsilon_{\alpha\beta} = \frac{1}{2}\left(\partial_\beta u_\alpha + \partial_\alpha u_\beta\right),$$

$x_0 \in \partial D$ and $\mathcal{U}$ is simply connected (i.e. R_0 small enough), $\mathbf{C}$ is the elasticity tensor of the (isotropic) material

$$\mathbf{C}A = \frac{hE(x)}{1-\nu^2(x)}\left[(1-\nu)A^{sym} + \nu tr(A) I_2\right]$$

for every 2×2 matrix A,



By using the Airy's function (1863), a **finite vanishing rate at the boundary** can be proved for (1) w. r. t. Neumann Condition

$$N_{\alpha\beta} n_\beta = 0, \quad \text{on } B_{R_0}(x_0) \cap \partial D \tag{2}$$

Airy's function

$$\begin{cases} \partial_1 N_{11} + \partial_2 N_{12} = 0, & \text{in } \mathcal{U} := B_{R_0}(x_0) \cap (\Omega \setminus \overline{D}), \\ \partial_1 N_{21} + \partial_2 N_{22} = 0, & \text{in } \mathcal{U} := B_{R_0}(x_0) \cap (\Omega \setminus \overline{D}), \end{cases}$$

We have that

$$-N_{12} dx_1 + N_{11} dx_2, \quad -N_{22} dx_1 + N_{21} dx_2$$

are exact forms. Hence exists $\widetilde{\varphi}_1$and $\widetilde{\varphi}_2$ such that
($\bigstar$) $\partial_1 \widetilde{\varphi}_1 = -N_{12}$, $\partial_2 \widetilde{\varphi}_1 = N_{11}$ and $\partial_1 \widetilde{\varphi}_2 = -N_{22}$, $\partial_2 \widetilde{\varphi}_2 = N_{21}$.
The symmetry of $N_{\alpha\beta}$ implies $N_{12} = N_{21}$, hence

$$\partial_1 \widetilde{\varphi}_1 = -\partial_2 \widetilde{\varphi}_2,$$

and, again, the differential form

$$-\widetilde{\varphi}_2 dx_1 + \widetilde{\varphi}_1 dx_2,$$

is exact so that there exists φ (**Airy's function**) such that

$$\partial_1 \varphi = -\widetilde{\varphi}_2, \quad \partial_2 \varphi = \widetilde{\varphi}_1$$

By $(\bigstar)$ and the definition of $N_{\alpha\beta}$ we have

$$\begin{cases} \epsilon_{11} = \frac{1}{hE}\left(\partial^2_{22}\varphi - \nu\partial^2_{11}\varphi\right), \\ \epsilon_{12} = -\frac{1+\nu}{hE}\partial^2_{12}\varphi, \\ \epsilon_{22} = \frac{1}{hE}\left(\partial^2_{11}\varphi - \nu\partial^2_{22}\varphi\right) \end{cases}$$

Now, since $\epsilon_{\alpha\beta} = \frac{1}{2}\left(\partial_\beta u_\alpha + \partial_\alpha u_\beta\right)$ we have

$$\partial^2_{22}\epsilon_{11} - 2\partial^2_{12}\epsilon_{12} + \partial^2_{11}\epsilon_{22} = 0$$

hence

$$\mathrm{div}\left(\mathrm{div}\left(\mathbf{L}\nabla^2\varphi\right)\right) = 0, \qquad \text{in } \mathcal{U}$$

where

$$L_{\alpha\beta\gamma\delta} = \frac{1+\nu}{hE}\delta_{\alpha\gamma}\delta_{\beta\delta} - \frac{\nu}{hE}\delta_{\alpha\beta}\delta_{\gamma\delta}$$

By using the weak formulation of (1), (2) and by choosing the indeterminate constants, we have also

$$\varphi = \partial_n \varphi = 0, \qquad \text{on } B_{R_0}(x_0) \cap \partial D.$$

We have

Theorem (Morassi, Rosset, V. (2020))

If ∂D is of $C^{6,\alpha}$ class and u is not constant in $B_{R_0}(x_0) \cap (\Omega \setminus \overline{D})$ then there exists C, N positive such that for every $r < R_0/2$, we have

$$\int_{B_r(x_0)\cap(\Omega\setminus\overline{D})} |\nabla u|^2 \geq C r^N$$

Theorem above is the main tool for the proof of optimal stability estimate for identification of cavities in the Generalized Plane Stress problem in linear elasticity.

SKETCH OF THE PROOF OF THREE SPHERES INEQUALITY

AT THE BOUNDARY FOR THE PLATE EQUATION

a) The plate equation can be rewritten in the form

$$\Delta^2 w = -2\frac{\nabla B}{B} \cdot \nabla \Delta w + q_2(w) \qquad \text{in } \Omega \setminus \overline{D},$$

where q_2 is a second order operator. Assume $x_0 \equiv 0$ and let $\Gamma = \partial D \cap B_R$ a small portion of ∂D

b) Flattening Γ by a conformal mapping the resulting equation preserves the same structure:

$$\begin{cases} \Delta^2 u = a \cdot \nabla \Delta u + p_2(u), & \text{in } B_1^+, \\ u(x,0) = u_y(x,0) = 0, & \forall x \in (-1,1) \end{cases}$$

where u is the solution in the new coordinates and p_2 is a second order operator.

c) We use the following reflection of u,

$$\overline{u}(x,y) = \begin{cases} u(x,y), & \text{in } B_1^+ \\ v(x,y), & \text{in } B_1^- \end{cases}$$

where

$$v(x,y) = -[u(x,-y) + 2yu_y(x,-y) + y^2\Delta u(x,-y)]$$

which has the advantage of ensuring that $\overline{u} \in H^4(B_1)$ if $u \in H^4(B_1^+)$

Poritsky, Trans. Amer. Math. Soc. 59 (1946), 248–279

John, Bull. Amer. Math. Soc. 63 (1957), 327–344

d) Then we apply the Carleman estimate

$$\sum_{k=0}^{3} \tau^{6-2k} \int \rho^{2k+\epsilon-2-2\tau} |D^k U|^2 dxdy \leq C \int \rho^{6-\epsilon-2\tau} (\Delta^2 U)^2 dxdy,$$

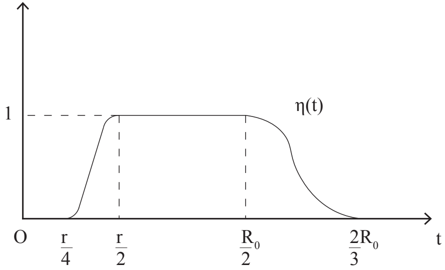
for every $\tau \geq \overline{\tau}$ and $\mathrm{supp} U \subset B_{\widetilde{R}_0} \setminus \{0\}$, where $0 < \varepsilon < 1$ is fixed and

$$\rho(x, y) \sim \sqrt{x^2 + y^2} \text{ as } (x, y) \to (0, 0)$$

$U = \xi \bar{u}$ where $\xi := \eta(\sqrt{x^2 + y^2})$ is a cut-off function

$$0 \le \eta \le 1 \ , \ \left| \frac{d^k \eta}{dt^k}(t) \right| \le C r^{-k} \text{ in } \left(\frac{r}{4}, \frac{r}{2} \right) ,$$

$$\eta = \begin{cases} 1, & \text{in } \left[\frac{r}{2}, \frac{R_0}{2} \right] \\ 0, & \text{in } \left(0, \frac{r}{4}\right) \cup \left(\frac{2}{3} R_0, 1\right) . \end{cases}$$



e) Nevertheless we still have a problem:
the term $\Delta^2 v$ on the right-hand side of the Carleman estimate involves derivatives of the forth order of v or, by definition of v , derivatives of u up to the **sixth order**, **hence cannot be absorbed in a standard way by the left hand side.**

In order to overcome this obstruction...

1) Using the structure of the equation and the expression of the reflection u, we rewrite in a suitable way $\Delta^2 v$:

For every $(x, y) \in B_1^-$, we have

$$\Delta^2 v(x, y) = H(x, y) + (P_2(v))(x, y) + (P_3(u))(x, -y),$$

where

$$H(x, y) = 6\frac{a_1}{y}(v_{yx}(x, y) + u_{yx}(x, -y)) + \\ + 6\frac{a_2}{y}(-v_{yy}(x, y) + u_{yy}(x, -y)) - \frac{12a_2}{y}u_{xx}(x, -y),$$

where a_1, a_2 are the components of the vector a. Moreover, for every $x \in (-1, 1)$,

$$v_{yx}(x, 0) + u_{yx}(x, 0) = 0\ ,\ -v_{yy}(x, 0) + u_{yy}(x, 0) = 0\ ,\ u_{xx}(x, 0) = 0$$

To handle the singularity of these terms as $y \to 0$

2) We use Hardy's inequality: If $f(0) = 0$ then

$$\int_0^{+\infty} \frac{f^2(t)}{t^2} dt \leq 4 \int_0^{+\infty} (f'(t))^2 dt.$$

After a few steps we have

$$\sum_{k=0}^{3} \tau^{6-2k} \int_{B^+_{R_0}} \rho^{2k+\epsilon-2-2\tau} |D^k(\xi u)|^2 + $$

$$+ \sum_{k=0}^{3} \tau^{6-2k} \int_{B^-_{R_0}} \rho^{2k+\epsilon-2-2\tau} |D^k(\xi v)|^2 \leq$$

$$\leq \underbrace{C \int_{B^-_{R_0}} \rho^{6-\epsilon-2\tau} \xi^2 |H(x,y)|^2}_{I_1} +$$

$$+ \underbrace{C \int_{B^-_{R_0}} \rho^{6-\epsilon-2\tau} \xi^2 \sum_{k=0}^{2} |D^k v|^2}_{I_2} + \underbrace{C \int_{B^+_{R_0}} \rho^{6-\epsilon-2\tau} \xi^2 \sum_{k=0}^{3} |D^k u|^2}_{I_3}.$$

I_2 and I_3 can be absorbed easily by the right hand side.

$$I_1 = \int_{B^-_{R_0}} \rho^{6-\epsilon-2\tau}\xi^2|H(x,y)|^2 \leq C(J_1 + J_2 + J_3),$$

where, for instance

$$J_1 = \int_{-R_0}^{R_0} \left(\int_{-\infty}^{0} \left| y^{-1}(v_{yx}(x,y) + (u_{yx}(x,-y))\rho^{\frac{6-\epsilon-2\tau}{2}}\xi \right|^2 dy \right) dx$$

and J_2, J_3 are similar. To estimate J_1, J_2 and J_3 we use Hardy's inequality:

$$J_j \leq C\int_{B^-_{R_0}} \rho^{6-\epsilon-2\tau}\xi^2|D^3v|^2 + C\tau^2\int_{B^-_{R_0}} \rho^{4-\epsilon-2\tau}\xi^2|D^2v|^2 +$$
$$+ C\int_{B^+_{R_0}} \rho^{6-\epsilon-2\tau}\xi^2|D^3u|^2 + C\tau^2\int_{B^+_{R_0}} \rho^{4-\epsilon-2\tau}\xi^2|D^2u|^2 + C\mathcal{I},$$

where $\mathcal{I}$ is a sum of integrals over $B^\pm_{r/2} \setminus B^\pm_{r/4}$ and $B^\pm_{2R_0/3} \setminus B^\pm_{R_0/2}$

SOME OPEN PROBLEMS ABOUT FINITE RATE VANISHING PROPERTY AT THE BOUNDARY

- **Could the assumption $\Gamma \in C^{6,\alpha}$ be reduced?**
- **The case of isotropic Kirchhoff Love plate with other boundary conditions, in particular, Neumann condition**
- **The case $\left|\Delta^2 u\right| \leq C \sum_{k=0}^{3} \left|D^k u\right|$**
- **The case of three dimensional Lamé system**

Thanks!