Quantum Optics in Random Media

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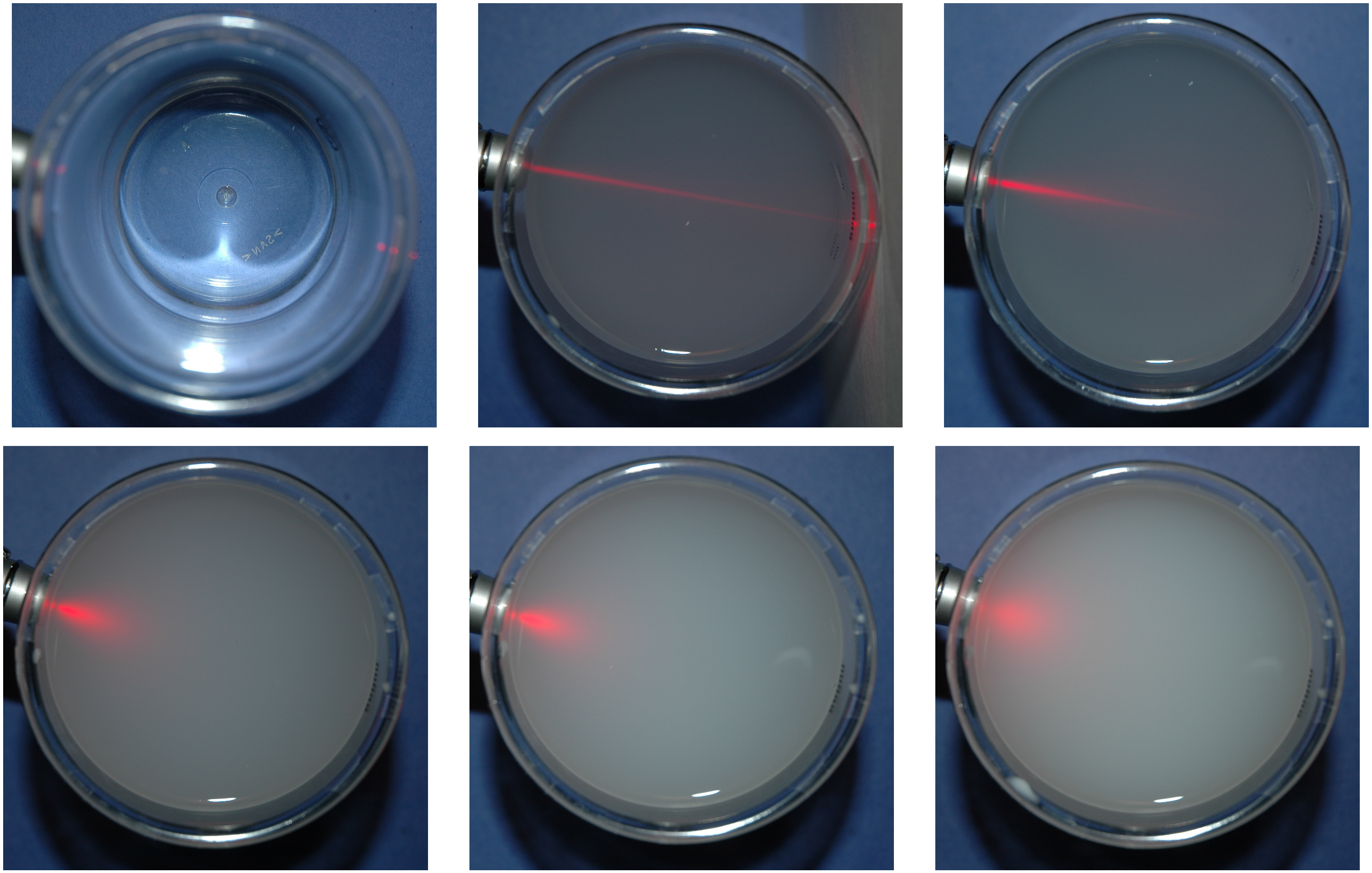
Joint work with Joe Kraisler

Light scattering



Radiative transport

The propagation of light in disordered media can be described by the radiative transport equation. The specific intensity $I(\mathbf{x}, \hat{\mathbf{k}}, t)$ obeys

$$\frac{1}{c}\frac{\partial I}{\partial t} + \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} I = \sigma \int d\hat{\mathbf{k}}' \left[A(\hat{\mathbf{k}}', \hat{\mathbf{k}}) I(\mathbf{x}, \hat{\mathbf{k}}') - A(\hat{\mathbf{k}}, \hat{\mathbf{k}}') I(\mathbf{x}, \hat{\mathbf{k}}) \right] \ .$$

The phase function A is normalized so that $\int A(\hat{\mathbf{k}}, \hat{\mathbf{k}}') d\hat{\mathbf{k}}' = 1$ for all $\hat{\mathbf{k}}$.

At large distances and long times, the specific intensity is well approximated by the solution to the diffusion equation

$$\frac{1}{c}\frac{\partial u}{\partial t} = D \Delta u \ ,$$

where $I(\mathbf{x}, \hat{\mathbf{k}}, t) = u(\mathbf{x}, t) + \ell^* \hat{\mathbf{k}} \cdot \nabla u(\mathbf{x}, t)$ and $D = 1/3c\ell^*$ is related to the lowest-order angular moment of A.

The RTE can be derived from the high-frequency asymptotics of wave propagation in random media.

Motivation

- Radiative transport of light is based on classical theories of light propagation

- Are there quantum effects in multiple light scattering?

  — spontaneous emission in random media

  — transport of entangled states

- Possible applications to imaging and communications

Quantum optics

- Quantum theory of the interaction of light and matter

- New physics

  — nonclassical states of light

  — entanglement

- One atom interacting with one photon

- Many-atom problems are exponentially hard

Overview

- Crash course in quantum optics

- New tools for many-body problems

  — real-space formulation of QED

  — PDEs with random coefficients

- Case studies

  — spontaneous emission in random media

  — two-photon transport

Quantization of the field

We consider a scalar model of the electromagnetic field (without polarization). The field u obeys the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u \ .$$

We expand the solution into Fourier modes of the form

$$u(\mathbf{x}, t) = \sum_{\mathbf{k}} u_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}$$

and find that

$$\ddot{u}_{\mathbf{k}} + \omega_k^2 u_{\mathbf{k}} = 0 \ .$$

This corresponds to independent harmonic oscillator modes with frequency $\omega_k = c|\mathbf{k}|$.

The oscillators are quantized by promoting the $u_{\mathbf{k}}$ to operators in the usual manner. The Hamiltonian of the quantized field is given by

$$H = \sum_{\mathbf{k}} \hbar\omega_k a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \ .$$

The creation and annihilation operators obey the bosonic commutation relations

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'} \ , \quad [a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0 \ .$$

Here $a^\dagger_{\mathbf{k}}$ and $a_{\mathbf{k}}$ are defined by

$$a^\dagger_{\mathbf{k}} |n_k\rangle = \sqrt{n_k + 1}\, |n_k + 1\rangle \ , \quad a_{\mathbf{k}} |n_k\rangle = \sqrt{n_k}\, |n_k - 1\rangle \ .$$

Photons are collective excitations of the quantized field. There are n_k photons in the state $|n_k\rangle$, each with energy $\hbar\omega_k$.

Two-level atom

We consider a two level atom with Hamiltonian

$$H_A = \hbar\Omega\sigma^\dagger\sigma \ ,$$

where Ω is the transition frequency of the atom. Here σ is the lowering operator and $\sigma^\dagger$ is the raising operator for the atomic states. That is, $\sigma = |0\rangle\langle 1|$, where $|0\rangle$ is the ground state and $|1\rangle$ is the excited state. Note that σ obeys the fermionic anticommutation relation $\{\sigma, \sigma^\dagger\} = 1$.

If the atom is initially in its excited state it will remain there forever; it is an eigenstate.

The atom has an electric dipole moment which couples to the field according to the interaction Hamiltonian

$$H_I = \hbar g \sum_{\mathbf{k}} \left(a_{\mathbf{k}}^{\dagger} \sigma + a_{\mathbf{k}} \sigma^{\dagger} \right) \ ,$$

where the coupling constant g is proportional to the dipole moment. The first term corresponds to loss of a photon by the atom and the gain of a photon by the field; the second term has the opposite effect.

The coupling to the field causes the excited state to decay; this is called spontaneous emisssion.

Many atoms

We consider a system of two-level atoms. The atoms are taken to be sufficiently well separated that interatomic interactions may be neglected.

Spontaneous emission involves transfer of the photon from one atom to another. This leads to entanglement of the atoms.

The system is described by the Hamiltonian $H = H_F + H_A + H_I$.

The Hamiltonian of the field is of the form

$$H_F = \sum_{\mathbf{k}} \hbar\omega_k a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}$$

The Hamiltonian of the atoms is given by

$$H_A = \sum_j \hbar \Omega \sigma_j^\dagger \sigma_j \ .$$

The operators σ_j and $\sigma_j^\dagger$ obey the anticommutation relations

$$\begin{aligned} \{\sigma_j, \sigma_{j'}^\dagger\} &= \delta_{jj'} \ , \\ \{\sigma_j, \sigma_{j'}\} &= 0 \ . \end{aligned}$$

The interaction Hamiltonian is of the form

$$H_I = \sum_j \sum_{\mathbf{k}} \hbar g_{\mathbf{k}} \left(a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger \right) \left(e^{i\mathbf{k}\cdot\mathbf{x}_j} \sigma_j + e^{-i\mathbf{k}\cdot\mathbf{x}_j} \sigma_j^\dagger \right) \ ,$$

where $\mathbf{x}_j$ is the position of the jth atom.

The number of equations grows exponentially with the number of atoms.

Real-space quantization

In order to treat the atoms and the field on the same footing, we introduce a real-space representation of the fields. To this end, we define the operator $\phi(\mathbf{x})$ as the Fourier transform of $a_{\mathbf{k}}$:

$$\phi(\mathbf{x}) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}} \ .$$

Evidently, ϕ is a Bose field with commutation relations

$$[\phi(\mathbf{x}), \phi^\dagger(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}') \ ,$$
$$[\phi(\mathbf{x}), \phi(\mathbf{x}')] = 0 \ .$$

H_F becomes

$$H_F = \hbar c \int d^3x (-\Delta)^{1/2} \phi^\dagger(\mathbf{x}) \phi(\mathbf{x}) \ .$$

This follows from the facts that $H_F = \hbar c \sum_{\mathbf{k}} |\mathbf{k}| a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$ and $|\mathbf{k}|$ is the Fourier multiplier of $(-\Delta)^{1/2}$.

The operator $(-\Delta)^{1/2}$ is non-local and defined by the Fourier integral

$$(-\Delta)^{1/2} f(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} |\mathbf{k}| \tilde{f}(\mathbf{k}) \ .$$

It also has the spatial representation

$$(-\Delta)^{1/2} f(\mathbf{x}) = \frac{1}{\pi^2} P \int d^3y \, \frac{f(\mathbf{x}) - f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^4} \ .$$

We do not attribute any physical significance to the nonlocality of H_F.

To facilitate the treatment of random media, we introduce a continuum model of the atomic degrees of freedom. The atomic Hamiltonian then becomes

$$H_A = \hbar\Omega \int d^3x \rho(\mathbf{x}) \sigma^\dagger(\mathbf{x}) \sigma(\mathbf{x}) \ ,$$

where ρ is the number density of atoms. In addition, the atomic operators are replaced by a Fermi field σ which obeys the anticommutation relations

$$\{\sigma(\mathbf{x}), \sigma^\dagger(\mathbf{x})\} = \frac{1}{\rho(\mathbf{x})} \delta(\mathbf{x} - \mathbf{x}') \ ,$$
$$\{\sigma(\mathbf{x}), \sigma(\mathbf{x})\} = 0 \ .$$

The interaction Hamiltonian is now

$$H_I = \hbar g \int d^3x \rho(\mathbf{x}) \left(\phi^\dagger(\mathbf{x}) \sigma(\mathbf{x}) + \phi(\mathbf{x}) \sigma^\dagger(\mathbf{x}) \right) \ .$$

One-photon states

The total Hamiltonian is given by

$$H = \hbar \int d^3x \Big[c(-\Delta)^{1/2} \phi^\dagger(\mathbf{x})\phi(\mathbf{x}) + \Omega\rho(\mathbf{x})\sigma^\dagger(\mathbf{x})\sigma(\mathbf{x}) + g\rho(\mathbf{x}) \left(\phi^\dagger(\mathbf{x})\sigma(\mathbf{x}) + \phi(\mathbf{x})\sigma^\dagger(\mathbf{x}) \right) \Big] \ ,$$

Consider a one-photon state of the form

$$|\mathbf{\Psi}\rangle = \int d^3x \left[\psi(\mathbf{x},t)\phi^\dagger(\mathbf{x}) + \rho(\mathbf{x})a(\mathbf{x},t)\sigma^\dagger(\mathbf{x}) \right] |0\rangle \ ,$$

where $|0\rangle$ is the combined vacuum state of the field and the ground state of the atoms. Here $a(\mathbf{x},t)$ denotes the probability amplitude for exciting an atom and $\psi(\mathbf{x},t)$ is the amplitude for creating a photon.

The amplitudes obey the normalization condition

$$\int d^3x \left(|\psi(\mathbf{x},t)|^2 + \rho(\mathbf{x})|a(\mathbf{x},t)|^2 \right) = 1 \ .$$

The dynamics of the state $|\mathbf{\Psi}\rangle$ is governed by the Schrodinger equation

$$i\hbar\partial_t|\mathbf{\Psi}\rangle = H|\mathbf{\Psi}\rangle \ .$$

We find that a and ψ obey

$$\begin{aligned} i\partial_t\psi &= c(-\Delta)^{1/2}\psi + g\rho(\mathbf{x})a \ , \\ i\partial_t a &= g\psi + \Omega a \ . \end{aligned}$$

Single atom and spontaneous emission

Consider a single atom at the origin with $\rho(\mathbf{x}) = \delta(\mathbf{x})$. We assume that the atom is initially in its excited state and that there are no photons present in the field. The probability that the atom remains in its excited state decays exponentially at long times ($\Gamma t \gg 1$):

$$|a(0,t)|^2 = e^{-\Gamma t} ,$$

where

$$\Gamma = \frac{g^2\Omega^2}{\pi c^3} .$$

This agrees with the classic result of Wigner and Weisskopf from the 1930s. Notably, they did not make use of the PDE point of view.

Likewise the one-photon probability density is

$$|\tilde{\psi}(0,t)|^2 = \frac{|g|^2}{(\Omega - \delta\omega)^2 + \Gamma^2/4} ,$$

where $\delta\omega$ is the Lamb shift, which must be renormalized. The above has the form of a Lorentzian spectral line. This should be compared to the scattering cross section for a point scatterer in classical optics.

Random media

In a random medium, we wish to determine $\langle |a(\mathbf{x},t)|^2\rangle$ and $\langle |\psi(\mathbf{x},t)|^2\rangle$, where $\langle \cdots \rangle$ denotes statistical averaging.

$$i\partial_t \psi = c(-\Delta)^{1/2}\psi + g\rho(\mathbf{x})a \ ,$$
$$i\partial_t a = g\psi + \Omega a \ .$$

The atomic density $\rho(\mathbf{x})$ is taken to be a random field of the form $\rho(\mathbf{x}) = \rho_0(1+\eta(\mathbf{x}))$, where ρ_0 is constant. The density fluctuation η is a statistically homogeneous and isotropic random field with correlations

$$\langle \eta(\mathbf{x})\rangle = 0 \ ,$$
$$\langle \eta(\mathbf{x})\eta(\mathbf{y})\rangle = C(|\mathbf{x}-\mathbf{y}|) \ .$$

The solutions a and ψ oscillate rapidly on the scale of the wavelength. We are interested in high-frequency asymptotics. Here the propagation distance is long compared to the wavelength, the propagation time is large compared to the period, and ρ is slowly varying.

We introduce slow space and time coordinates $\mathbf{x} \to \mathbf{x}/\epsilon$ and $t \to t/\epsilon$, where ϵ is small. The rescaled amplitudes $a_\epsilon(\mathbf{x}, t) = a(\mathbf{x}/\epsilon, t/\epsilon)$ and $\psi_\epsilon(\mathbf{x}, t) = \psi(\mathbf{x}/\epsilon, t/\epsilon)$ satisfy

$$
\begin{aligned}
i\epsilon\partial_t\psi_\epsilon &= \epsilon c(-\Delta)^{1/2}\psi_\epsilon + g\rho_0\left(1 + \sqrt{\epsilon}\eta(\mathbf{x}/\epsilon)\right)a_\epsilon \ , \\
i\epsilon\partial_t a_\epsilon &= g\psi_\epsilon + \Omega a_\epsilon \ .
\end{aligned}
$$

We consider the high-frequency limit $\epsilon \to 0$ and rescale η so that the randomness is sufficiently weak with $C = O(\epsilon)$.

The high-frequency, weak disorder regime is precisely the setting in which radiative transport theory holds for classical wave fields.

Wigner transform

The Wigner transform is defined by

$$W_\epsilon(\mathbf{x},\mathbf{k},t) = \int \frac{d^3x'}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}'} \mathbf{U}_\epsilon(\mathbf{x}-\epsilon\mathbf{x}'/2,t) \otimes \mathbf{U}^*_\epsilon(\mathbf{x}+\epsilon\mathbf{x}'/2,t) \ ,$$

where $\mathbf{U}_\epsilon(\mathbf{x},t) = (\psi_\epsilon(\mathbf{x},t), a_\epsilon(\mathbf{x},t))$. W can be thought of as a phase-space probability density. The probability densities $|\psi_\epsilon(\mathbf{x},t)|^2$ and $|a_\epsilon(\mathbf{x},t)|^2$ are related to the Wigner transform by

$$|\psi_\epsilon(\mathbf{x},t)|^2 = \int d^3k W^\epsilon_{11}(\mathbf{x},\mathbf{k},t) \ ,$$

$$|a_\epsilon(\mathbf{x},t)|^2 = \int d^3k W^\epsilon_{22}(\mathbf{x},\mathbf{k},t) \ .$$

The diagonal elements of W are real-valued, but not generally non-negative. However, in the high-frequency limit $\epsilon \to 0$, W becomes nonnegative.

Radiative transport

The average Wigner transform can be decomposed into modes:

$$W_0(\mathbf{x},\mathbf{k},t) = a_+(\mathbf{x},\mathbf{k},t)A_+(\mathbf{k}) + a_-(\mathbf{x},\mathbf{k},t)A_-(\mathbf{k}) \ .$$

It can be seen that in the high-frequency limit, the modes $a_\pm$ obey a kinetic equation of the form

$$\frac{1}{c}\partial_t a_\pm + \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} a_\pm + \sigma_\pm a_\pm = \int d\hat{\mathbf{k}}' A(\mathbf{k},\mathbf{k}') a_\pm(\mathbf{x},\mathbf{k}',t) \ .$$

The absorption coefficients $\sigma_\pm$ depend on the parameters ρ_0, g, k, Ω. The phase function A is related to density correlations. This is a long story.

The diffusion approximation for $a_\pm$ is constructed in the standard way.

Key ingredients

- The Wigner transform can be shown to obey an evolution equation

$$\epsilon i \partial_t W_\epsilon = L[\eta] W_\epsilon$$

- We consider W_ϵ in the high-frequency limit $\epsilon \to 0$ and introduce a multiscale expansion of the form

$$W_\epsilon(\mathbf{x}, \mathbf{k}, t) = W_0(\mathbf{x}, \mathbf{k}, t) + \sqrt{\epsilon} W_1(\mathbf{x}, \mathbf{X}, \mathbf{k}, t) + \epsilon W_2(\mathbf{x}, \mathbf{X}, \mathbf{k}, t) + \cdots \ ,$$

where $\mathbf{X} = \mathbf{x}/\epsilon$ is a fast variable and W_0 is taken to be deterministic.

- By separating terms of order $O(1)$, $O(\sqrt{\epsilon})$ and $O(\epsilon)$, we obtain a hierarchy of kinetic equations. By averaging over η and introducing a suitable closure, we get the required kinetic equations.

Spontaneous emission

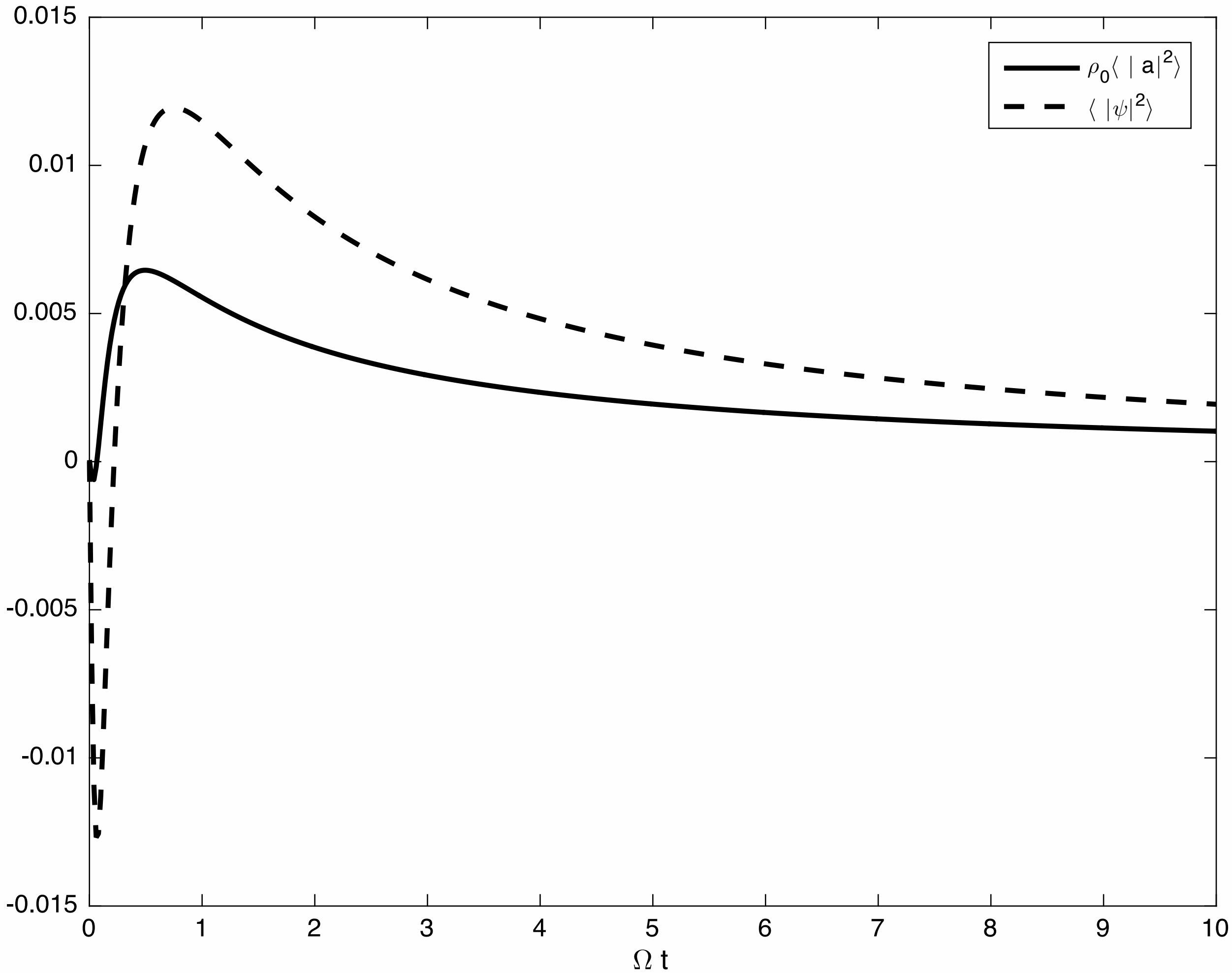
We suppose that the atoms are initially excited in a volume of linear dimensions l_s and that there are no photons present in the field:

$$a(\mathbf{x},0) = \left(\frac{1}{\pi l_s^2}\right)^{3/4} e^{-|\mathbf{x}|^2/2l_s^2} \ ,$$
$$\psi(\mathbf{x},0) = 0 \ .$$

The kinetic equations are solved in the diffusion approximation for an infinite medium. We assume isotropic scattering with $A = 1/(4\pi)$ and set $\Omega l_s/c = 1$, $\rho_0(g/\Omega)^2 = 1$.

At long times ($\Omega t \gg 1$)

$$\langle |a(\mathbf{x},t)|^2\rangle = \frac{C_1}{t^{3/2}} + O\left(\frac{1}{t^{5/2}}\right) \ ,$$
$$\langle |\psi(\mathbf{x},t)|^2\rangle = \frac{C_2}{t^{3/2}} + O\left(\frac{1}{t^{5/2}}\right) \ .$$


0.015
0.01
0.005
0
-0.005
-0.01
-0.015
0
1
2
3
4
5
6
7
8
9
10
Ω t
$\rho_0\langle\,|a|^2\rangle$
$\langle\,|\psi|^2\rangle$

Two-photon states

Recall the Hamiltonian

$$H = \hbar \int d^3x \Big[c(-\Delta)^{1/2} \phi^\dagger(\mathbf{x})\phi(\mathbf{x}) + \Omega\rho(\mathbf{x})\sigma^\dagger(\mathbf{x})\sigma(\mathbf{x}) + g\rho(\mathbf{x}) \left(\phi^\dagger(\mathbf{x})\sigma(\mathbf{x}) + \phi(\mathbf{x})\sigma^\dagger(\mathbf{x}) \right) \Big] \; ,$$

Consider a two-photon state of the form

$$|\Psi\rangle = \int d^3x_1 d^3x_2 \Big[\psi_2(\mathbf{x}_1, \mathbf{x}_2, t)\phi^\dagger(\mathbf{x}_1)\phi^\dagger(\mathbf{x}_2) + \rho(\mathbf{x}_1)\psi_1(\mathbf{x}_1, \mathbf{x}_2, t)\phi^\dagger(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_2) + \rho(\mathbf{x}_1)\rho(\mathbf{x}_2)a(\mathbf{x}_1, \mathbf{x}_2, t)\sigma^\dagger(\mathbf{x}_1)\sigma^\dagger(\mathbf{x}_2) \Big] |0\rangle \; .$$

Here a denotes the amplitude for exciting two atoms, ψ_2 is the amplitude for creating two photons, and ψ_1 is the amplitude for jointly exciting an atom and creating a photon. Here a is antisymmetric (fermionic) and ψ_2 is symmetric (bosonic).

Note that there can be entanglement of both the photons and the atoms.

The dynamics of the state $|\mathbf{\Psi}\rangle$ is governed by the Schrodinger equation

$$i\hbar\partial_t|\mathbf{\Psi}\rangle = H|\mathbf{\Psi}\rangle \ .$$

We find that a, ψ_1 and ψ_2 obey

$$\begin{aligned}
i\partial_t\psi_2 &= c(-\Delta_{\mathbf{x}_1})^{1/2}\psi_2 + c(-\Delta_{\mathbf{x}_2})^{1/2}\psi_2 + \frac{g}{2}\left(\rho(\mathbf{x}_1)\psi_1 + \rho(\mathbf{x}_2)\tilde{\psi}_1\right) \ , \\
i\partial_t\psi_1 &= \left[c(-\Delta_{\mathbf{x}_2})^{1/2} + \Omega\right]\psi_1 - 2g\rho(\mathbf{x}_2)a + 2g\psi_2 \ , \\
i\partial_t a &= \frac{g}{2}(\tilde{\psi}_1 - \psi_1) + 2\Omega a \ ,
\end{aligned}$$

where $\tilde{\psi}_1(\mathbf{x}_1, \mathbf{x}_2) = \psi_1(\mathbf{x}_2, \mathbf{x}_1)$.

For the case of a single atom, we obtain a modified exponential decay of the population of the excited state, which describes the process of stimulated emission.

Radiative transport

In a random medium, the average Wigner transform can be expanded into modes in a manner similar to the one-photon problem. It can be seen that in the high-frequency limit, the modes $a_i(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2)$, where $i = 1, 2, 3, 4$ obey kinetic equations of the form

$$\frac{1}{c}\partial_t a_i + \widehat{\mathbf{k}}_1 \cdot \nabla_{\mathbf{x}_1} a_i + \widehat{\mathbf{k}}_2 \cdot \nabla_{\mathbf{x}_2} a_i + \sigma_i a_i = \mathcal{T}_i a_i \ .$$

The coefficients σ_i depend on the parameters ρ_0, g, k, Ω. The transport operator $\mathcal{T}_i$ is related to density correlations.

The diffusion approximation for a_i is constructed in the standard way.

We suppose that two photons are present in the field and that the atoms are initially in their ground states:

$$\psi_2(\mathbf{x}_1, \mathbf{x}_2, 0) = e^{-|\mathbf{x}_1-\mathbf{y}_0|^2/2l_s^2} e^{-|\mathbf{x}_2-\mathbf{y}_1|^2/2l_s^2} + e^{-|\mathbf{x}_1-\mathbf{y}_1|^2/2l_s^2} e^{-|\mathbf{x}_2-\mathbf{y}_0|^2/2l_s^2} ,$$

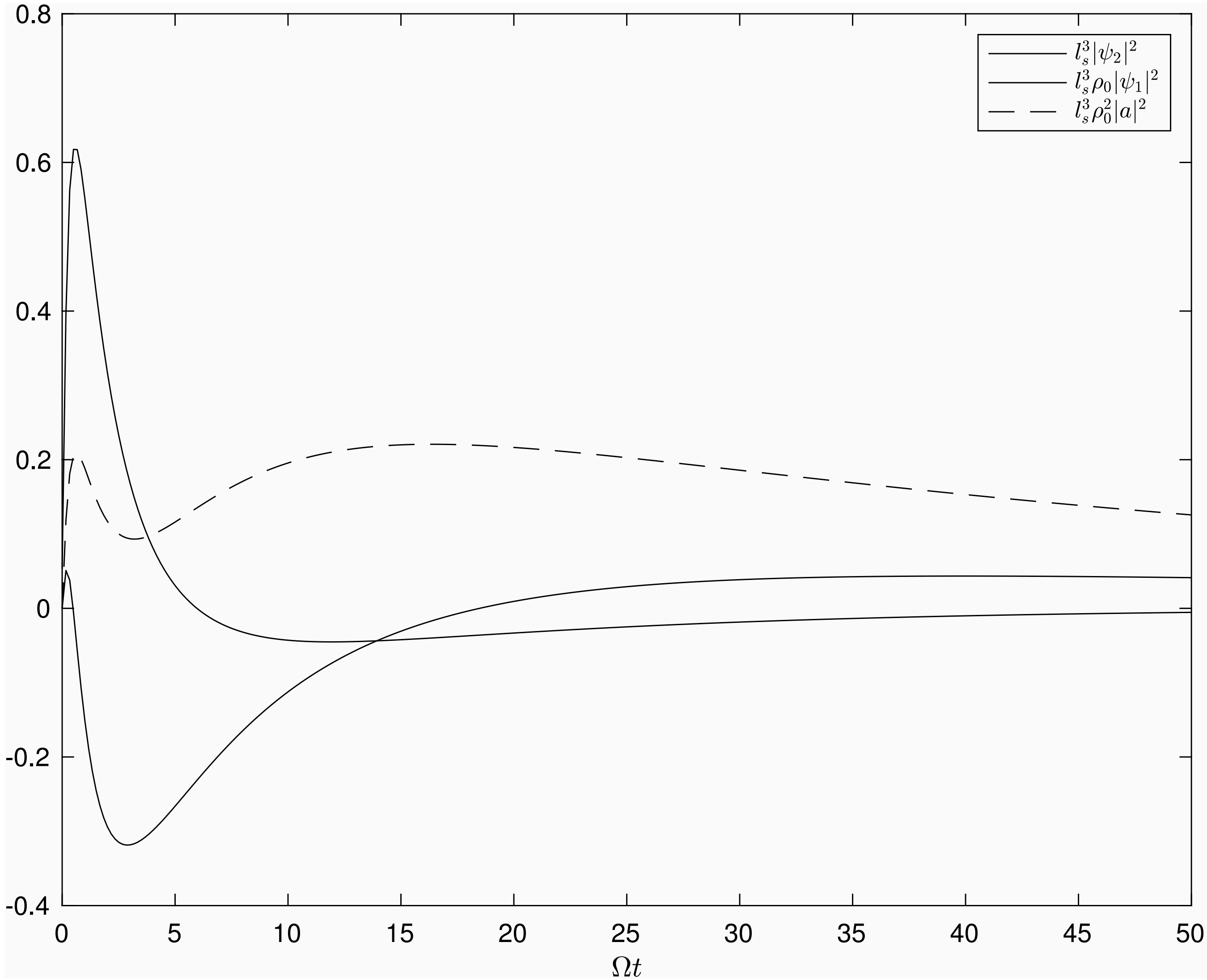
$$\psi_1(\mathbf{x}_1, \mathbf{x}_2, 0) = 0 ,$$

$$a(\mathbf{x}_1, \mathbf{x}_2, 0) = 0 .$$

We note that the initial two-photon state is entangled (not separable).

Entanglement cannot be created by scattering an unentangled incident state.

The kinetic equations are solved in the diffusion approximation for an infinite medium. We assume isotropic scattering and set $\Omega l_s/c = 1$, $\rho_0 (g/\Omega)^2 = 1$.



$l_s^3|\psi_2|^2$
$l_s^3\rho_0|\psi_1|^2$
$l_s^3\rho_0^2|a|^2$
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
5
10
15
20
25
30
35
40
45
50
Ωt

Conclusions

- Many-body problems in quantum optics present fundamental mathematical challenges

  — computational cost grows exponentially with the number of atoms

  — this is more than a technical problem; it is a problem of principle

- Real-space quantization

- Kinetic equations

- Applications to spontaneous emission in random media and transport of entangled two-photon states.

Open problems

- Entanglement measures

- Imaging with entangled two-photon states

- Localization for $(-\Delta)^{1/2} + V(x)$

- Resonances for $(-\Delta)^{1/2} + V(x)$

THANK YOU!