Reduced order modeling for inverse problems

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Joint work with:

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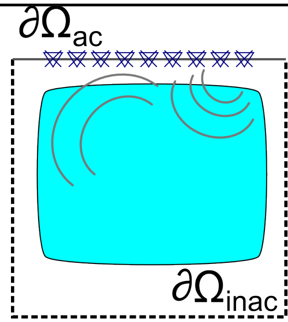
Goal of talk

- Describe two approaches for building data driven reduced order models (ROM) for linear PDE's.
  - Parabolic PDE and describe ROM construction in Laplace (frequency) domain.
  - Hyperbolic PDE and describe causal ROM construction in the time domain.
- Give some details on how we used the ROM for inversion.



- Find coefficient $q(\boldsymbol{x})$ of $A = -\nabla \cdot [q(\boldsymbol{x})\nabla]$ in parabolic PDE

$$\begin{aligned} \partial_t \mathbf{u}(t,\boldsymbol{x}) + A\mathbf{u}(t,\boldsymbol{x}) &= 0 \\ \mathbf{u}(0,\boldsymbol{x}) &= \boldsymbol{b}(\boldsymbol{x}) = (b^{(1)}(\boldsymbol{x}),\ldots,b^{(m)}(\boldsymbol{x})) \\ \partial_n \mathbf{u}(t,\boldsymbol{x}) &= 0 \ \text{ on } \partial\Omega_{\text{ac}}, \quad \mathbf{u}(\text{t},\boldsymbol{x}) = 0 \ \text{ on } \partial\Omega_{\text{inac}} \end{aligned}$$

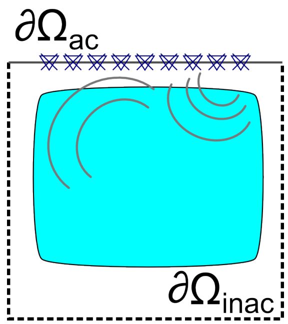
- "Sensor functions" $\left(b^{(s)}(\boldsymbol{x})\right)_{s=1}^{m}$ are supported near $\partial\Omega_{\text{ac}}$

Data: matrix $\boldsymbol{D}(t) = \int_\Omega d\boldsymbol{x}\, \boldsymbol{b}^T(\boldsymbol{x})\mathbf{u}(t,\boldsymbol{x}) = \langle \boldsymbol{b}, \mathbf{u}(t,\cdot)\rangle$

Work in frequency domain with $2n$ matrices:

$$\widehat{\boldsymbol{D}}(\omega_j) = \int_0^\infty dt\, e^{-\omega_j t}\boldsymbol{D}(t), \quad \partial_\omega \widehat{\boldsymbol{D}}(\omega_j), \quad j = 1,\ldots,n$$

Two inverse problems



- Find coefficient $q(\boldsymbol{x})$ of symmetrized A in hyperbolic PDE*

$$\partial_t^2 \mathbf{u}(t,\boldsymbol{x}) + A\mathbf{u}(t,\boldsymbol{x}) = 0$$

$$\mathbf{u}(0,\boldsymbol{x}) = \boldsymbol{b}(\boldsymbol{x}), \quad \partial_t \mathbf{u}(0,\boldsymbol{x}) = 0$$

$$\partial_n \mathbf{u}(t,\boldsymbol{x}) = 0 \text{ on } \partial\Omega_{\text{ac}}, \quad \mathbf{u}(\text{t},\boldsymbol{x}) = 0 \text{ on } \partial\Omega_{\text{inac}}$$

Data: 2n matrices $\boldsymbol{D}(t_j) = \int_\Omega d\boldsymbol{x}\, \boldsymbol{b}^T(\boldsymbol{x})\mathbf{u}(t_j,\boldsymbol{x}) = \left\langle \boldsymbol{b}, \mathbf{u}(t_j,\cdot)\right\rangle$

at time instants $t_j = j\tau$, for $j = 0,\ldots,2n-1$

*Acoustics: $q = c =$ wavespeed, $\mathbf{u} =$pressure/c and $A = -c\Delta(c\cdot)$

Parabolic PDE: ROM in the frequency domain

- Laplace transform (droping hats) of field at frequency ω_j is

$$\mathbf{u}_j(\boldsymbol{x}) := \mathbf{u}(\omega_j, \boldsymbol{x}) = (\omega_j I + A)^{-1} \boldsymbol{b}(\boldsymbol{x})$$

and we know

$$\begin{aligned} \boldsymbol{D}(\omega_j) &= \left\langle \boldsymbol{b}, \mathbf{u}_j \right\rangle = \left\langle \boldsymbol{b}, (\omega_j I + A)^{-1} \boldsymbol{b} \right\rangle \\ \partial_\omega \boldsymbol{D}(\omega_j) &= -\left\langle \boldsymbol{b}, (\omega_j I + A)^{-2} \boldsymbol{b} \right\rangle, \quad j = 1, \ldots, n \end{aligned}$$

- **ROM** is a pair $\boldsymbol{A}^{\text{ROM}} \in \mathbb{R}^{nm \times nm}$ and $\boldsymbol{b}^{\text{ROM}} \in \mathbb{R}^{nm \times m}$ that interpolates data:

$$\begin{aligned} \boldsymbol{D}(\omega_j) &= {\boldsymbol{b}^{\text{ROM}}}^T (\omega_j \boldsymbol{I} + \boldsymbol{A}^{\text{ROM}})^{-1} \boldsymbol{b}^{\text{ROM}} \\ \partial_\omega \boldsymbol{D}(\omega_j) &= -{\boldsymbol{b}^{\text{ROM}}}^T (\omega_j \boldsymbol{I} + \boldsymbol{A}^{\text{ROM}})^{-2} \boldsymbol{b}^{\text{ROM}}, \quad j = 1, \ldots, n \end{aligned}$$

- Galerkin space: range $\boldsymbol{U}(\boldsymbol{x})$, $\boldsymbol{U}(\boldsymbol{x}) := \left(\mathbf{u}_1(\boldsymbol{x}), \ldots, \mathbf{u}_n(\boldsymbol{x})\right)$

$$\mathbf{u}(\omega, \boldsymbol{x}) \approx \boldsymbol{U}(\boldsymbol{x})\boldsymbol{g}(\omega) \quad \text{s.t.} \quad \boldsymbol{U}^T\Big[(\omega I + \boldsymbol{A})\boldsymbol{U}\boldsymbol{g}(\omega) - \boldsymbol{b}\Big] = 0$$

where $\boldsymbol{U}^T\boldsymbol{U} = \Big(\big\langle \mathbf{u}_j, \mathbf{u}_l \big\rangle\Big)_{1 \le j,l \le n}$

At $\omega = \omega_j$: $\mathbf{u}(\omega_j, \boldsymbol{x}) = \mathbf{u}_j(\boldsymbol{x}) = \boldsymbol{U}(\boldsymbol{x})\boldsymbol{e}_j \rightsquigarrow \boldsymbol{g}(\omega_j) = \boldsymbol{e}_j$

Here $\boldsymbol{e}_j$ = matrix with j^{th} block equal to $\boldsymbol{I}_m$ and zero elsewhere.

- With $nm \times nm$ matrices: $\boldsymbol{M} = \boldsymbol{U}^T\boldsymbol{U}$ and $\boldsymbol{S} = \boldsymbol{U}^T\boldsymbol{A}\boldsymbol{U}$ $\rightsquigarrow$

$$(\omega \boldsymbol{M} + \boldsymbol{S})\boldsymbol{g}(\omega) = \boldsymbol{U}^T\boldsymbol{b}, \qquad \forall \omega. \tag{1}$$

We don't know approximation space but can get $\boldsymbol{M}$ and $\boldsymbol{S}$ and thus $\boldsymbol{g}(\omega)$ from data.

- Diagonal of $\boldsymbol{M}$ is easy:

$$M_{jj} = \left\langle \mathbf{u}_j, \mathbf{u}_j \right\rangle = \left\langle (\omega_j I + A)^{-1}\boldsymbol{b}, (\omega_j I + A)^{-1}\boldsymbol{b} \right\rangle$$
$$= \left\langle \boldsymbol{b}, (\omega_j I + A)^{-2}\boldsymbol{b} \right\rangle = -\partial_\omega \boldsymbol{D}(\omega_j)$$

- Eq. (1) for $\omega = \omega_j$ s.t. $\boldsymbol{g}(\omega_j) = \boldsymbol{e}_j$ multiplied on left by $\boldsymbol{e}_l^T \rightsquigarrow$

$$\omega_j M_{lj} + S_{jl} = \boldsymbol{e}_l^T(\omega_j \boldsymbol{M} + \boldsymbol{S})\boldsymbol{e}_j = \langle \mathbf{u}_l, \boldsymbol{b} \rangle = \boldsymbol{D}(\omega_l)$$

- Taking $l = j \rightsquigarrow \quad S_{jj} = \boldsymbol{D}(\omega_j) + \omega_j \partial_\omega \boldsymbol{D}(\omega_j)$

- If $l \neq j$, we obtain similarly, using symmetries,

$$\omega_l M_{lj} + S_{jl} = \boldsymbol{D}(\omega_j) \rightsquigarrow$$

$$M_{jl} = \frac{\boldsymbol{D}(\omega_l) - \boldsymbol{D}(\omega_j)}{\omega_j - \omega_l}, \quad S_{jl} = \frac{\omega_j \boldsymbol{D}(\omega_j) - \omega_l \boldsymbol{D}(\omega_l)}{\omega_j - \omega_l}, \quad j \neq l$$

• Let columns in $\boldsymbol{V}(\boldsymbol{x})$ be orthonormal basis of Galerkin space given by Gram-Schmidt

$$\boldsymbol{U}(\boldsymbol{x}) = \boldsymbol{V}(\boldsymbol{x})\boldsymbol{R} \quad \rightsquigarrow \quad \boldsymbol{M} = \boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{R}^T\boldsymbol{R}$$

Upper triangular $\boldsymbol{R}$ given by Cholesky factorization of $\boldsymbol{M}$

• The ROM:

$$\boldsymbol{A}^{\text{ROM}} = \boldsymbol{V}^T\boldsymbol{A}\boldsymbol{V} = \boldsymbol{R}^{-T}\boldsymbol{U}^T\boldsymbol{A}\boldsymbol{U}\boldsymbol{R}^{-1} = \boldsymbol{R}^{-T}\boldsymbol{S}\boldsymbol{R}^{-1}$$

$$\boldsymbol{b}^{\text{ROM}} = \boldsymbol{V}^T\boldsymbol{b} = \boldsymbol{R}^{-T}\boldsymbol{U}^T\boldsymbol{b} = \boldsymbol{R}^{-T}\begin{pmatrix} \boldsymbol{D}(\omega_1) \\ \vdots \\ \boldsymbol{D}(\omega_n) \end{pmatrix}$$

Galerkin eq. $(\omega \boldsymbol{M} + \boldsymbol{S})\boldsymbol{g}(\omega) = \boldsymbol{U}^T\boldsymbol{b}$ with $\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}$ & $\boldsymbol{M} = \boldsymbol{R}^T\boldsymbol{R}$

$$\rightsquigarrow \quad \boldsymbol{R}^T(\omega \boldsymbol{I} + \underbrace{\boldsymbol{R}^{-T}\boldsymbol{S}\boldsymbol{R}^{-1}}_{\boldsymbol{A}^{\text{ROM}}})\boldsymbol{R}\boldsymbol{g}(\omega) = \boldsymbol{R}^T \underbrace{\boldsymbol{V}^T\boldsymbol{b}}_{\boldsymbol{b}^{\text{ROM}}}$$

- ROM Galerkin equation:

$$(\omega \boldsymbol{I} + \boldsymbol{A}^{\text{ROM}})\mathbf{u}^{\text{ROM}}(\omega) = \boldsymbol{b}^{\text{ROM}}, \qquad \mathbf{u}^{\text{ROM}}(\omega) = \boldsymbol{R}\boldsymbol{g}(\omega)$$

Data driven ROM satisfies discrete equivalent of the PDE

- Using $\boldsymbol{g}(\omega_j) = \boldsymbol{e}_j$, for $j = 1, \ldots, n$,

$$\mathbf{u}_j^{\text{ROM}} := \mathbf{u}^{\text{ROM}}(\omega_j) = \boldsymbol{R}\boldsymbol{e}_j = \boldsymbol{V}^T\boldsymbol{U}\boldsymbol{e}_j = \boldsymbol{V}^T\mathbf{u}_j$$

and therefore

$$\mathbf{u}_j(\boldsymbol{x}) = \boldsymbol{V}\boldsymbol{V}^T\mathbf{u}_j(\boldsymbol{x}) = \boldsymbol{V}(\boldsymbol{x})\mathbf{u}_j^{\text{ROM}}$$

ROM data interpolation

- For $j = 1, \ldots, n$ we have

$$\boldsymbol{D}(\omega_j) = \left\langle \boldsymbol{b}, \mathbf{u}_j \right\rangle = \underbrace{\langle \boldsymbol{b}, \boldsymbol{V} \rangle}_{\boldsymbol{b}^{\text{ROM}^T}} \underbrace{(\omega_j \boldsymbol{I} + \boldsymbol{A}^{\text{ROM}})^{-1} \boldsymbol{b}^{\text{ROM}}}_{\mathbf{u}_j^{\text{ROM}}}$$

- For derivative:

$$\partial_\omega \boldsymbol{D}(\omega_j) = -\left\langle \boldsymbol{b}, (\omega_j I + \boldsymbol{A})^{-2} \boldsymbol{b} \right\rangle = -\left\langle \mathbf{u}_j, \mathbf{u}_j \right\rangle$$

$$= -\left\langle \boldsymbol{V} \mathbf{u}_j^{\text{ROM}}, \boldsymbol{V} \mathbf{u}_j^{\text{ROM}} \right\rangle$$

$$= -\mathbf{u}_j^{\text{ROM}^T} \langle \boldsymbol{V}, \boldsymbol{V} \rangle \mathbf{u}_j^{\text{ROM}}$$

$$= -\mathbf{u}_j^{\text{ROM}^T} \mathbf{u}_j^{\text{ROM}}$$

• For any orthogonal matrix $\boldsymbol{L} \in \mathbb{R}^{nm\times nm}$ the data is also interpolated by

$$\widetilde{\boldsymbol{A}}^{\text{ROM}} := \boldsymbol{L}^T \boldsymbol{A}^{\text{ROM}} \boldsymbol{L} \quad \text{and} \quad \widetilde{\boldsymbol{b}}^{\text{ROM}} := \boldsymbol{L}^T \boldsymbol{b}^{\text{ROM}}$$

• We use $\boldsymbol{L}$ that makes $\widetilde{\boldsymbol{A}}^{\text{ROM}}$ block tridiagonal and $\widetilde{\boldsymbol{b}}^{\text{ROM}}$ zero except in the first block

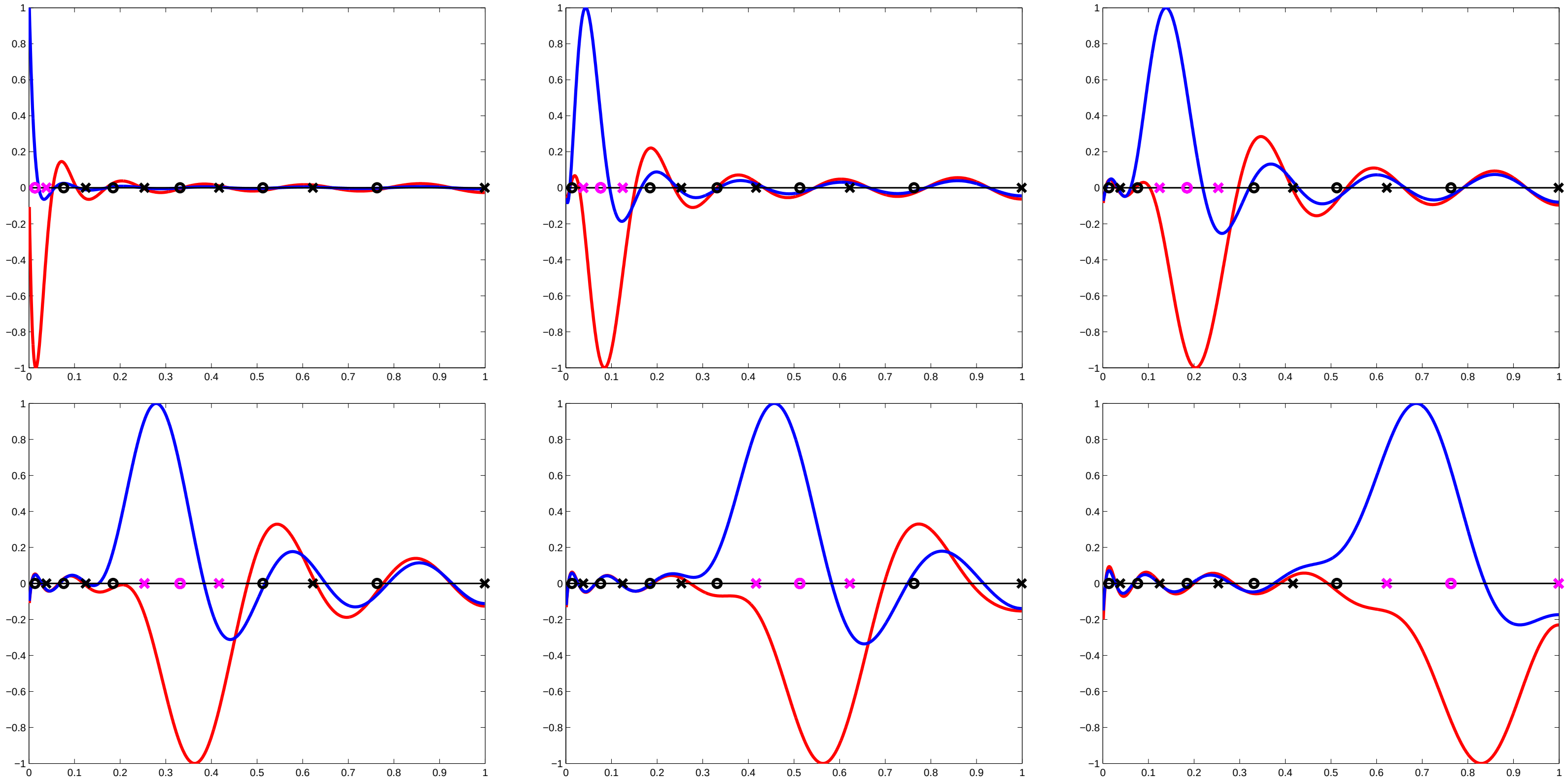
ROM eq. $(\omega \boldsymbol{I} + \widetilde{\boldsymbol{A}}^{\text{ROM}})\widetilde{\mathbf{u}}^{\text{ROM}}(\omega) = \widetilde{\boldsymbol{b}}^{\text{ROM}}$ is finite difference scheme for

$$(\omega I + \boldsymbol{A})u(\omega, \boldsymbol{x}) = \boldsymbol{b}(x)$$

with 3−point stencil in depth

• Transformation to block tridiagonal is via the Lanczos iteration

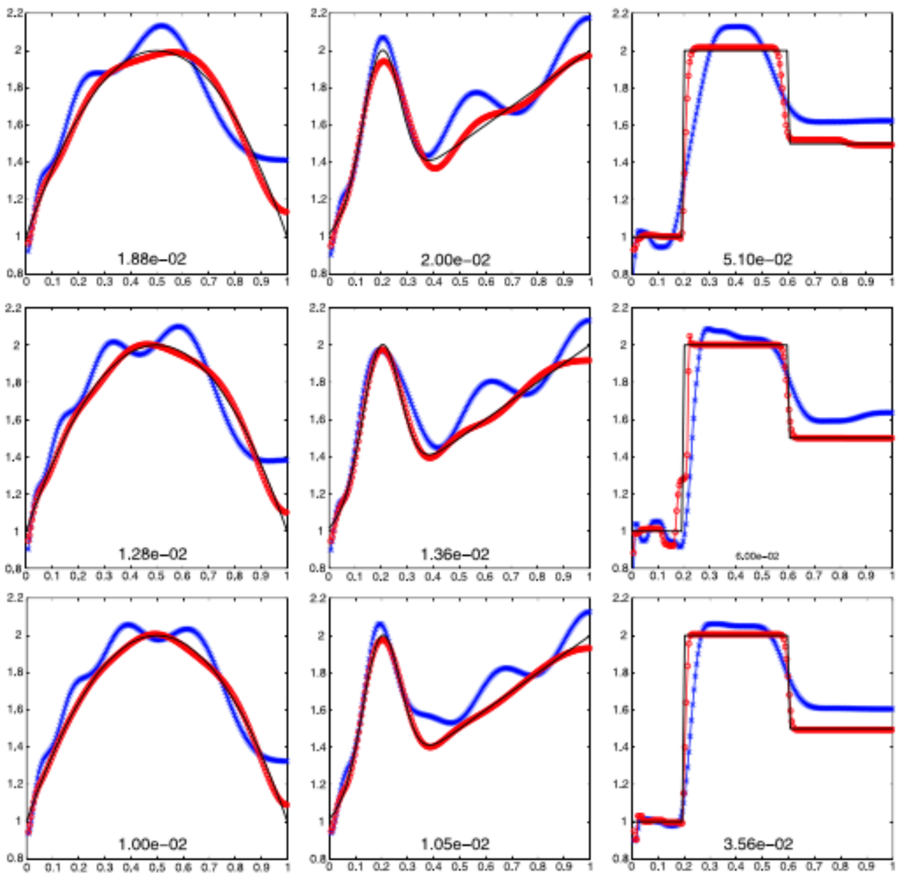
Sensitivity of map $q \mapsto$ entries in $\widetilde{A}^{\text{ROM}}$ in $1-D$



- Sensitivities are large in grid cells calculated for $q = 0$

- This grid tells us if we have good frequency samples

Inversion results (initial guess $q \equiv 1$)



Black: True q

Blue: 1st iteration

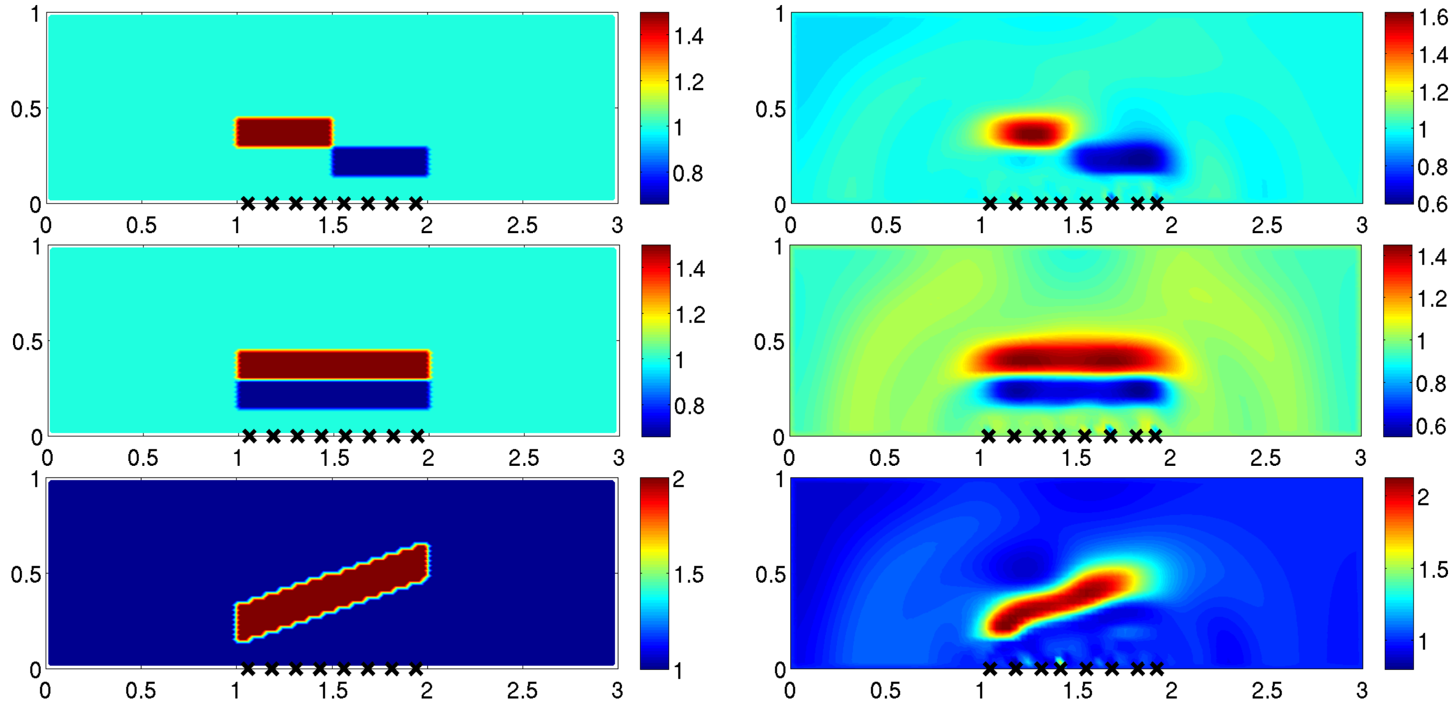
Red: 5th iteration

ROM (top to bottom) size is $n = 4, 5, 6$

Relative error displayed in each plot

• Instead of Least Squares data fit, we find coefficient q by minimizing misfit between data driven $\widetilde{\boldsymbol{A}}^{\text{ROM}}$ and the ROM calculated for the trial q.

Inversion results 2-D (initial guess $q \equiv 1$)



• Other results based on estimates $u_j(\boldsymbol{x}) = \boldsymbol{V}(\boldsymbol{x})\mathbf{u}_j^{\text{ROM}} \approx \boldsymbol{V}_0(\boldsymbol{x})\mathbf{u}_j^{\text{ROM}}$ were discussed in Shari Moskow's talk.

Hyperbolic PDE: Causal ROM construction

- Find coefficient $q(\boldsymbol{x})$ of self-adjoint, positive definite A in

$$\partial_t^2 \mathbf{u}(t,\boldsymbol{x}) + A\mathbf{u}(t,\boldsymbol{x}) = 0$$
$$\mathbf{u}(0,\boldsymbol{x}) = \boldsymbol{b}(\boldsymbol{x}), \quad \partial_t \mathbf{u}(0,\boldsymbol{x}) = 0$$

with homogeneous boundary conditions.

- Data are

$$\boldsymbol{D}(t_j) = \langle \boldsymbol{b}, u(j\tau,\cdot)\rangle = \left\langle \boldsymbol{b}, \cos(j\tau\sqrt{A})\boldsymbol{b}\right\rangle, \quad j = 0,\ldots,2n-1.$$

- **ROM** is not for A but for **wave propagator** $\mathcal{P} = \cos\left(\tau\sqrt{A}\right)$

Note: $\mathbf{u}_j(\boldsymbol{x}) := \mathbf{u}(j\tau,\boldsymbol{x}) = \cos(j\tau\sqrt{A})\boldsymbol{b}(\boldsymbol{x}) = \cos\left(j \arccos \mathcal{P}\right)\boldsymbol{b}(\boldsymbol{x})$

$\mathcal{T}_j(\mathcal{P}) = \cos\left(j \arccos \mathcal{P}\right)$ = Tchebyshev polynomial of first kind

Dynamical system for propagation and Galerkin projection

- Recurrence relation of Chebyshev polynomials $\rightsquigarrow$

$$\begin{aligned} \mathbf{u}_{j+1}(\boldsymbol{x}) &= 2\mathcal{P}\mathbf{u}_j(\boldsymbol{x}) - \mathbf{u}_{j-1}(\boldsymbol{x}), \quad j \geq 0, \quad \boldsymbol{x} \in \Omega \\ \mathbf{u}_0(\boldsymbol{x}) &= \boldsymbol{b}(\boldsymbol{x}) \\ \mathbf{u}_1(\boldsymbol{x}) &= \mathbf{u}_{-1}(\boldsymbol{x}) \end{aligned}$$

- Galerkin space: range $\boldsymbol{U}(\boldsymbol{x})$, $\boldsymbol{U}(\boldsymbol{x}) := \Big(\mathbf{u}_0(\boldsymbol{x}), \ldots, \mathbf{u}_{n-1}(\boldsymbol{x})\Big)$

$\mathbf{u}_j(\boldsymbol{x}) \approx \boldsymbol{U}(\boldsymbol{x})\boldsymbol{g}_j$, with $\boldsymbol{g}_j \in \mathbb{R}^{nm \times m}$ satisfying

$$\boldsymbol{U}^T\Big(\boldsymbol{U}\boldsymbol{g}_{j+1} - 2\mathcal{P}\boldsymbol{U}\boldsymbol{g}_j + \boldsymbol{U}\boldsymbol{g}_{j-1}\Big) = 0, \quad \boldsymbol{U}^T\boldsymbol{U} = \Big(\big\langle \mathbf{u}_j, \mathbf{u}_l \big\rangle\Big)_{0 \leq j,l \leq n-1}$$

First n coefficients: $\boldsymbol{g}_0 = \boldsymbol{e}_1, \; \ldots \;, \; \boldsymbol{g}_{n-1} = \boldsymbol{e}_n$

- Galerkin eq. $\boldsymbol{M}\boldsymbol{g}_{j+1} = 2\boldsymbol{S}\boldsymbol{g}_j - \boldsymbol{M}\boldsymbol{g}_{j-1}, \quad j \geq 0$
  $\boldsymbol{g}_j = \boldsymbol{e}_{j+1}, \quad 0 \leq j \leq n-1$

- Data $\mapsto$ $nm \times nm$ matrices $\boldsymbol{M} = \boldsymbol{U}^T\boldsymbol{U}$ and $\boldsymbol{S} = \boldsymbol{U}^T\mathcal{P}\boldsymbol{U}$:

$$\boldsymbol{M}_{lj} = \left\langle \mathbf{u}_l, \mathbf{u}_j \right\rangle = \left\langle \mathcal{T}_l(\mathcal{P})\boldsymbol{b}, \mathcal{T}_j(\mathcal{P})\boldsymbol{b} \right\rangle$$

$$= \left\langle \boldsymbol{b}, \mathcal{T}_l(\mathcal{P})\mathcal{T}_j(\mathcal{P})\boldsymbol{b} \right\rangle = \frac{1}{2}\left\langle \boldsymbol{b}, \left[\mathcal{T}_{l+j}(\mathcal{P}) + \mathcal{T}_{|l-j|}(\mathcal{P})\right]\boldsymbol{b} \right\rangle$$

$$= \frac{1}{2}\left(\boldsymbol{D}_{l+j} + \boldsymbol{D}_{|l-j|}\right)$$

Similarly,

$$\boldsymbol{S}_{lj} = \left\langle \mathbf{u}_l, \mathcal{P}\mathbf{u}_j \right\rangle = \frac{1}{4}\left(\boldsymbol{D}_{l+j+1} + \boldsymbol{D}_{|j-l+1|} + \boldsymbol{D}_{|j-l-1|} + \boldsymbol{D}_{|l+j-1|}\right)$$

ROM dynamical system

- Square root $\boldsymbol{M} = \boldsymbol{R}^T \boldsymbol{R}$ (Cholesky) $\rightsquigarrow$ $\boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} & \dots & \boldsymbol{R}_{1n} \\ 0 & \boldsymbol{R}_{22} & \dots & \boldsymbol{R}_{2n} \\ \vdots & & & \\ 0 & 0 & \dots & \boldsymbol{R}_{nn} \end{pmatrix}$

- Multiply Galerkin eq. $\boldsymbol{M}\boldsymbol{g}_{j+1} = 2\boldsymbol{S}\boldsymbol{g}_j - \boldsymbol{M}\boldsymbol{g}_{j-1}$ by $\boldsymbol{R}^{-T}$ on left

$$\begin{aligned} \mathbf{u}_{j+1}^{\text{ROM}} &= 2\mathcal{P}^{\text{ROM}}\mathbf{u}_j^{\text{ROM}} - \mathbf{u}_{j-1}^{\text{ROM}} = 0, \quad j \geq 0 \\ \mathbf{u}_0^{\text{ROM}} &= \boldsymbol{b}^{\text{ROM}} \\ \mathbf{u}_1^{\text{ROM}} &= \mathbf{u}_{-1}^{\text{ROM}} \end{aligned}$$

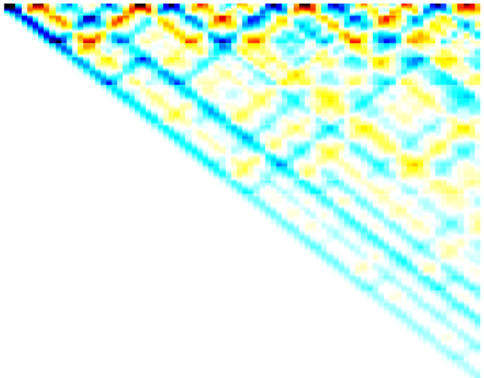
- ROM snapshots: $\mathbf{u}_j^{\text{ROM}} = \boldsymbol{R}\boldsymbol{g}_j$ and propagator $\mathcal{P}^{\text{ROM}} = \boldsymbol{R}^{-T}\boldsymbol{S}\boldsymbol{R}^{-1}$

- Gram-Schmidt: $\boldsymbol{U}(\boldsymbol{x}) = \boldsymbol{V}(\boldsymbol{x})\boldsymbol{R}$ $\rightsquigarrow$ $\mathbf{u}_j^{\text{ROM}} = \boldsymbol{R}\boldsymbol{g}_j = \boldsymbol{V}^T \underbrace{\boldsymbol{U}\boldsymbol{g}_j}_{\approx \mathbf{u}_j}$ and

$$\mathcal{P}^{\text{ROM}} = \boldsymbol{R}^{-T}\boldsymbol{S}\boldsymbol{R}^{-1} = \boldsymbol{R}^{-T}\boldsymbol{U}^T\mathcal{P}\boldsymbol{U}\boldsymbol{R}^{-1} = \boldsymbol{V}^T\mathcal{P}\boldsymbol{V}$$

Preserving causality is important

• 1-D ilustration (1 source/receiver): snapshots $\mathbf{u}(j\tau, \boldsymbol{x})$ plotted vs. $j = 0, 1, \ldots$ (horizontal axis) and $\boldsymbol{x}$ (vertical axis)

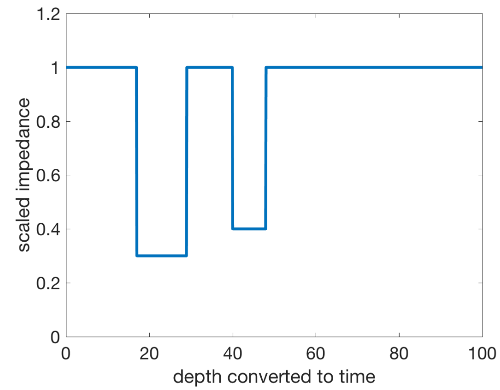


• ROM snapshots: $(\mathbf{u}_0^{\text{ROM}}, \ldots, \mathbf{u}_{n-1}^{\text{ROM}}) = \boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} & \ldots & \boldsymbol{R}_{1n} \\ \mathbf{0} & \boldsymbol{R}_{22} & \ldots & \boldsymbol{R}_{2n} \\ \vdots & & & \\ \mathbf{0} & \mathbf{0} & \ldots & \boldsymbol{R}_{nn} \end{pmatrix}$

– In higher dimensions we have $m \times m$ blocks (m sources/receivers). Rows of blocks model advancement of wavefront.

• Gram-Schmidt $\boldsymbol{U}(\boldsymbol{x}) = \boldsymbol{V}(\boldsymbol{x})\boldsymbol{R}$ maps nearly triangular $\boldsymbol{U}(\boldsymbol{x})$ to $\boldsymbol{R} \rightsquigarrow \boldsymbol{V}(\boldsymbol{x})$ weakly dependent of reflectivity.

Illustration for sound waves in 1-D



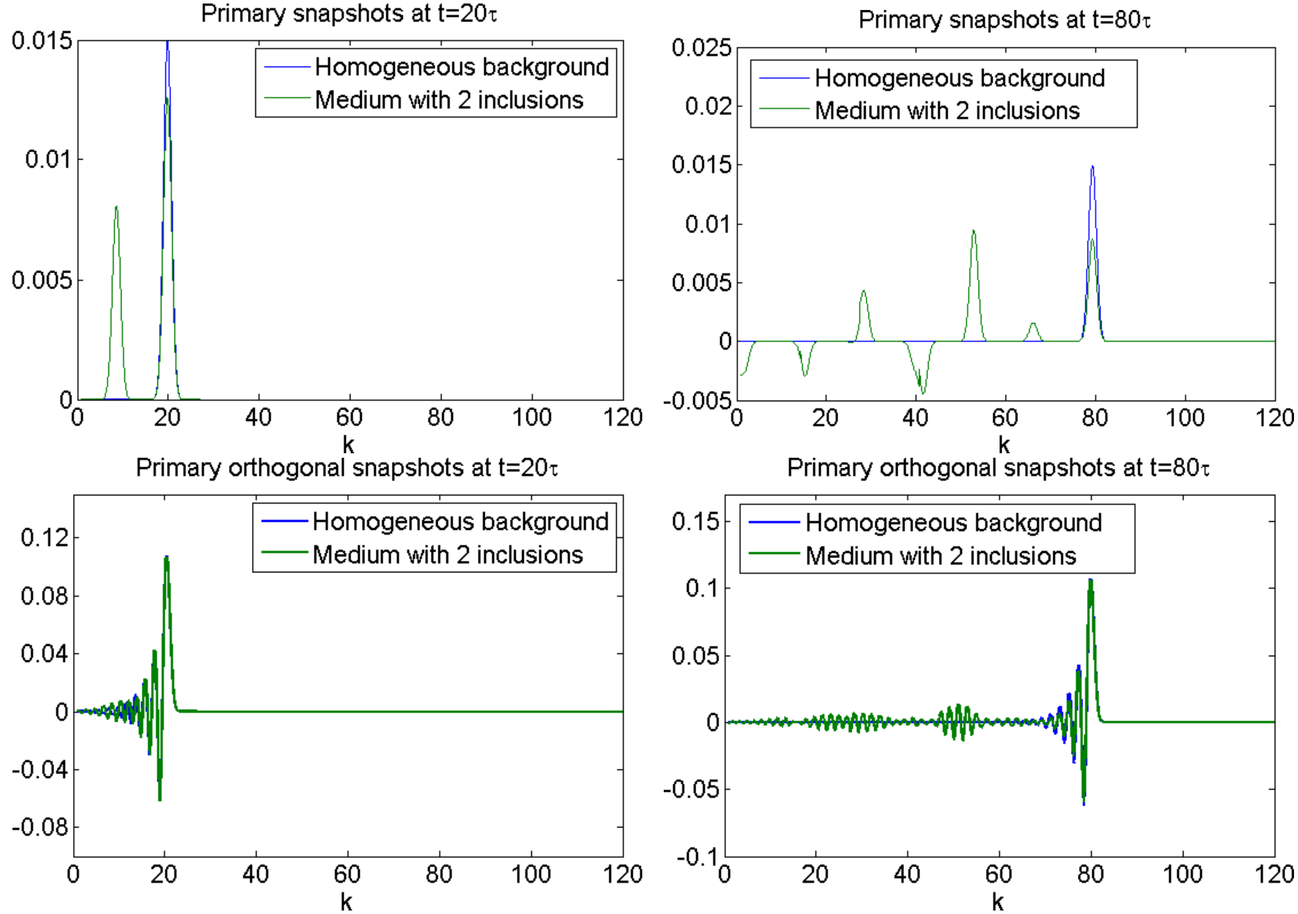
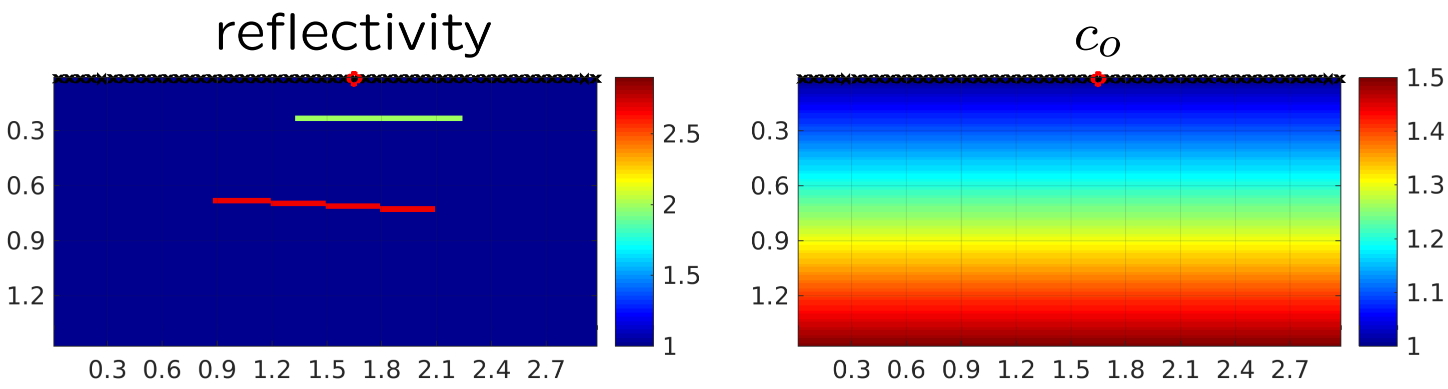
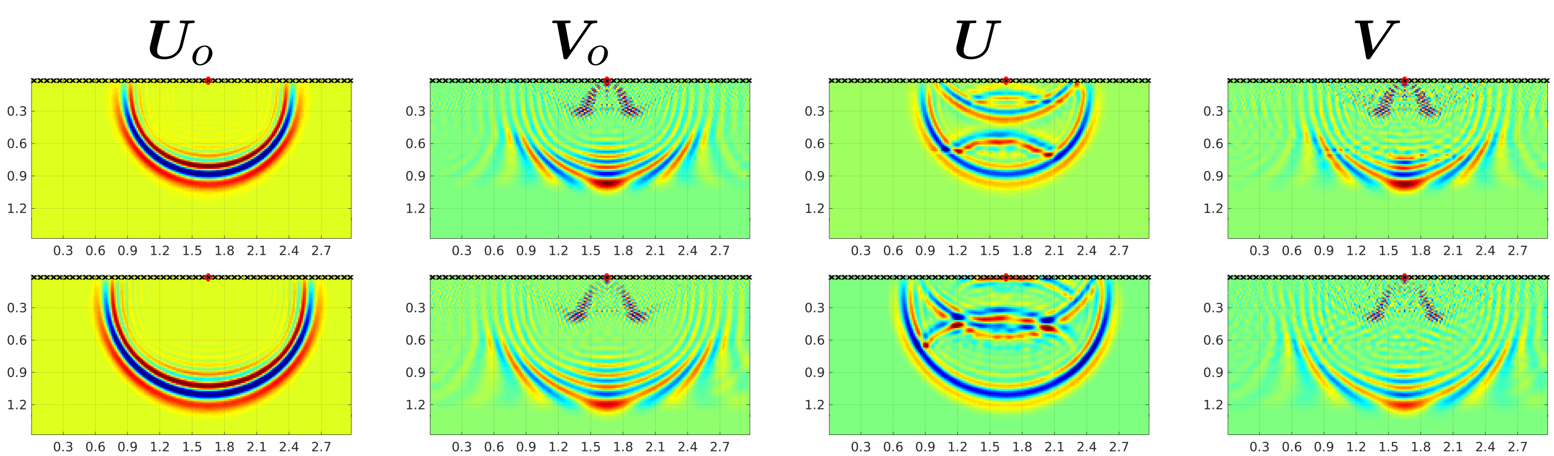


Illustration for sound waves in 2-D





Array with $m = 50$ sensors $\times$
Snapshots plotted for a single source $\circ$

Qualitative imaging*

- Consider internal "fictitious source" at $\boldsymbol{y} \in \Omega$:

$$\delta_{\boldsymbol{y}}(\boldsymbol{x}) := \boldsymbol{V}\boldsymbol{V}^T\delta(\boldsymbol{x}-\boldsymbol{y}) = \sum_{j=0}^{n-1} v_j(\boldsymbol{x}) \left\langle v_j, \delta(\cdot - \boldsymbol{y})\right\rangle = \boldsymbol{V}(\boldsymbol{x})\boldsymbol{V}^T(\boldsymbol{y})$$

- Generated wave after one time step τ :

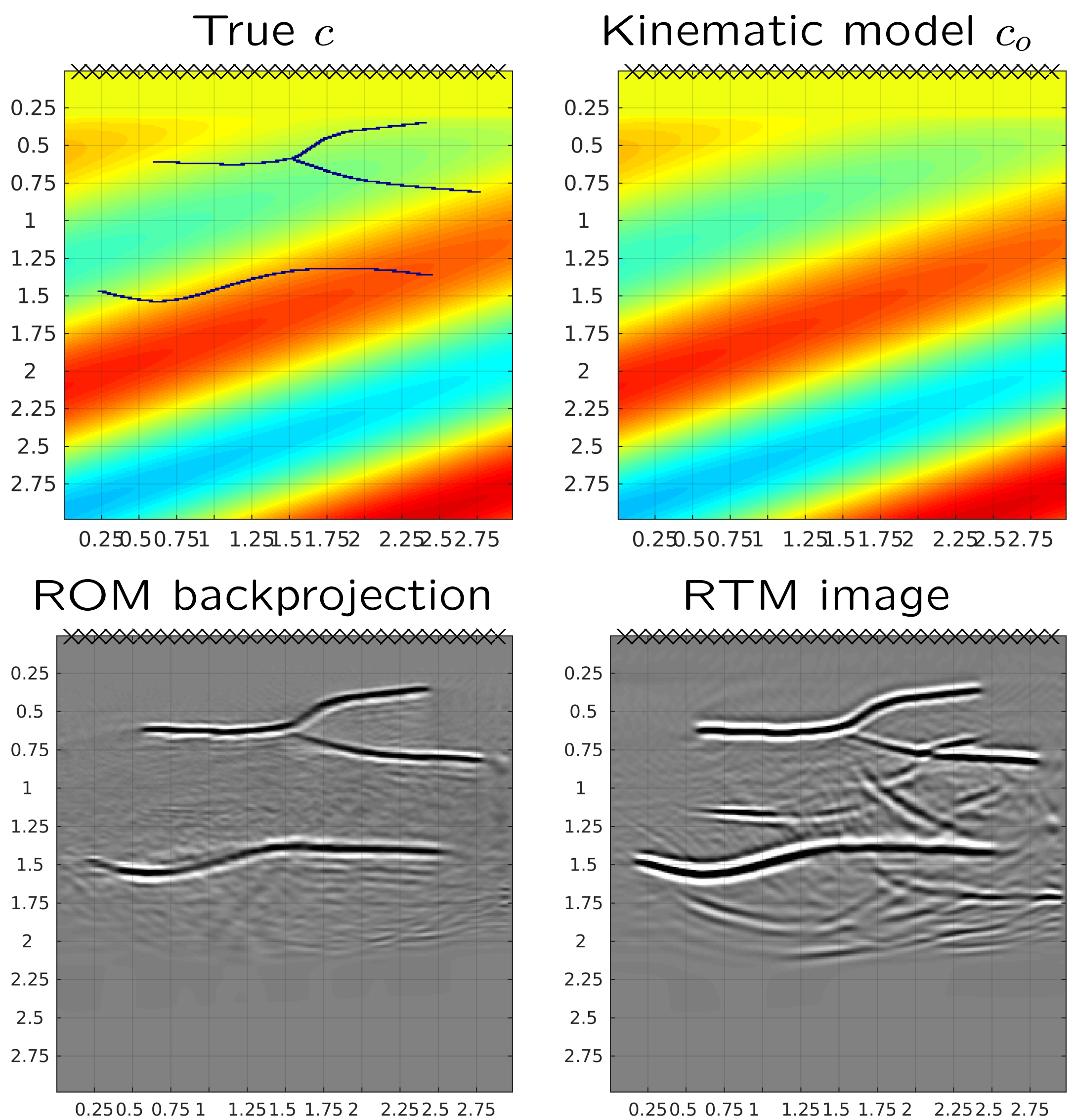
$$\mathcal{P}\delta_{\boldsymbol{y}}(\boldsymbol{x}) \approx \boldsymbol{V}\boldsymbol{V}^T\mathcal{P}\delta_{\boldsymbol{y}}(\boldsymbol{x}) = \boldsymbol{V}\boldsymbol{V}^T\mathcal{P}\boldsymbol{V}\boldsymbol{V}^T\delta(\boldsymbol{x}-\boldsymbol{y}) = \boldsymbol{V}(\boldsymbol{x})\mathcal{P}^{\text{ROM}}\boldsymbol{V}^T(\boldsymbol{y})$$

⇝ Backprojection imaging function:

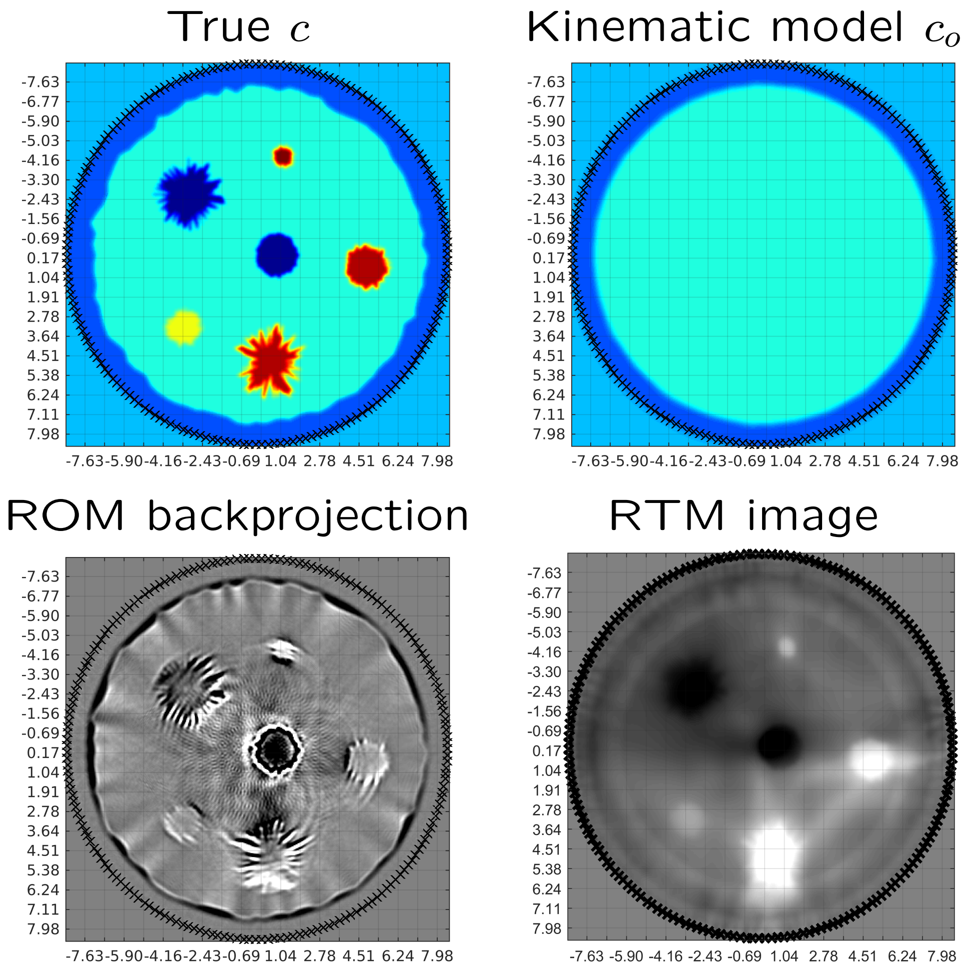
$$\mathcal{I}(\boldsymbol{y}) := \boldsymbol{V}_o(\boldsymbol{y})\Big(\mathcal{P}^{\text{ROM}} - \mathcal{P}_o^{\text{ROM}}\Big)\boldsymbol{V}_o^T(\boldsymbol{y}), \qquad \forall \boldsymbol{y} \in \Omega.$$

*Results for acoustic wave equation: Druskin, Mamonov, Zaslavsky - SIAM Imaging Sci., 2018

Backprojection vs Reverse Time Migration



Backprojection vs Reverse Time Migration



Other inversion results

• Quantitative inversion for reflectivity (rough part of wave speed):

B., Druskin, Mamonov, Zaslavsky, Zimmerling, *Reduced Order Model Approach to Inverse Scattering*, SIAM Imaging Sciences 13 (2), 2020, p. 685-723.

• ROM used to approx. derivative of reflectivity $\mapsto \{\boldsymbol{D}_j\}_{0\le j\le 2n-1}$ i.e., linearized (Born) model assumed in conventional imaging:

B., Druskin, Mamonov, Zaslavsky, *Untangling the nonlinearity in inverse scattering with data-driven reduced order models*, Inverse Problems 34 (6), 2018, p. 065008

B., Druskin, Mamonov, Zaslavsky, *Robust nonlinear processing of active array data in inverse scattering via truncated reduced order models*, Journal Comp. Physics 381, 2019, p. 1-26.

• With Josselin Garnier we are working on other applications and on velocity estimation.