Lorentzian Calderón problem under curvature bounds

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Lorentzian Calderón problem

Let (M, g) be a smooth Lorentzian manifold with timelike boundary.
Let $V \in C^\infty(M)$ and consider the Cauchy data set

$$\mathscr{C}(V) = \{(u, \partial_\nu u)|_{\partial M} : u \in C^\infty(M) \text{ and } \Box u + Vu = 0 \text{ on } M\}.$$

Calderón problem. Find V given $\mathscr{C}(V)$.

The classical Calderón problem has the same formulation except that

- (M, g) is a smooth Riemannian manifold with boundary
- $\Box$ is replaced by the Laplacian on (M, g)

The Riemannian/Lorentzian Calderón problem is

- solved when g is the Euclidean/Minkowski metric [Sylvester-Uhlmann'87]/[Stefanov'89]
- open when g is a perturbation of the Euclidean/Minkowski metric

Simple cylinders of negative spatial curvature

A special case of our results reads:

Theorem [Alexakis-Feizmohammadi-L.O.]. Let

$$M = \mathbb{R} \times M_0, \quad g(t,x) = -dt^2 + g_0(x),$$

where (M_0, g_0) is a simple[1] Riemannian manifold with negative curvature. Then the Lorentzian Calderón problem has a unique solution on $(M, \tilde{g})$, where $\tilde{g}$ is a small perturbation of g in $C_0^\infty(M^{\text{int}})$.

The unperturbed case has been solved previously, a particular case of [Feizmohammadi-Ilmavirta-Kian-L.O.'20].

In general, contrary to (M, g), the perturbed Lorentzian manifold $(M, \tilde{g})$ has no symmetries and $\tilde{g}(t,x)$ is not real analytic in t.

[1] That is, a compact, simply connected manifold with strictly convex boundary.

Previous results in the ultrastatic case

Suppose that (M, g) is ultrastatic, that is,

$$M = \mathbb{R} \times M_0, \quad g(t, x) = -dt^2 + g_0(x),$$

where (M_0, g_0) is a compact Riemannian manifold with boundary. Then the Lorentzian Calderón problem has been solved if

- V does not depend on t, using the Boundary Control method [Belishev'87], [Belishev-Kurylev'92], ...
- (M_0, g_0) has injective geodesic ray transform, using geometric optics [Stefanov'89], ..., [Feizmohammadi-Ilmavirta-Kian-L.O.'20]

The "ultrastatic" Riemannian case $g = dt^2 + g_0(x)$, with M as above, has been solved if (M_0, g_0) has injective geodesic ray transform, by using complex geometric optics [Sylvester-Uhlmann'87], ..., [Dos Santos Ferreira-Kurylev-Lassas-Salo'16].

Conformal rescaling

If the Lorentzian Calderón problem is solved on (M, g) of dimension $1 + n$, then it is also solved on (M, cg) for any strictly positive $c \in C^\infty(M)$.

Idea of the proof. We write $\Box_g$ for the wave operator on (M, g), that is,

$$\Box_g u = -\sum_{j,k=0}^{n} |\det g|^{-\frac{1}{2}} \frac{\partial}{\partial x^j} \left(|\det g|^{\frac{1}{2}} g^{jk} \frac{\partial u}{\partial x^k} \right).$$

If a function w satisfies the equation $\Box_{cg} w = 0$, then the function

$$u = c^{(n-1)/4} w$$

satisfies the equation $\Box_g u + Vu = 0$, with $V = c^{-(n-1)/4} \Box_g c^{(n-1)/4}$. $\Box$

This rescaling allows us also to reduce the problem to find a conformal factor to the problem to find a potential, if $n > 1$.

Previous Lorentzian results outside the ultrastatic case

The Boundary Control method generalizes to the case that $M = \mathbb{R} \times M_0$ and both g and V are real analytic in t [Eskin'07].

- The method uses the optimal unique continuation result [Tataru'95]
- Tataru's result does not hold if V is only smooth in t [Alinhac'83]

Geometric optics allows for a reduction to inversion of the light ray transform [Stefanov-Yang'18]. This transform has been inverted only if

- (M, g) is real analytic and satisfies a certain convexity condition [Stefanov'17]
- (M, g) is stationary[1] and satisfies a certain convexity condition [Feizmohammadi-Ilmavirta-L.O.'20]

Contrary to our result, in all the previous results g needs to be *real analytic* with respect to a suitably chosen time variable.

[1] There is a smooth complete timelike Killing vector field

Causality vocabulary

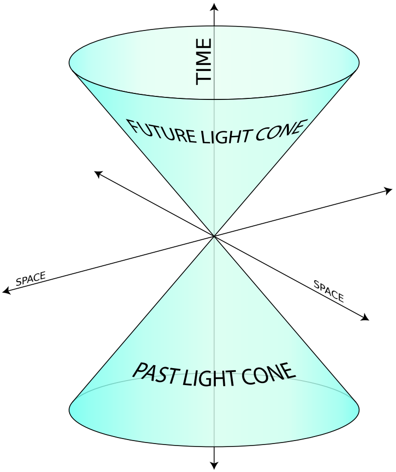
Let (M, g) be a Lorentzian manifold and write $\langle \cdot, \cdot \rangle$ for the inner product.

A tangent vector $v \in T_x M \setminus 0$ is

- spacelike if $\langle v, v \rangle > 0$
- lightlike if $\langle v, v \rangle = 0$
- timelike if $\langle v, v \rangle < 0$
- causal if it is lightlike or timelike

A hypersurface is timelike if its normal vectors are spacelike.

A path γ is causal if its tangent vectors $\dot{\gamma}(s)$ are causal.



Geometric assumptions

We prove that the Calderón problem has a unique solution on a smooth Lorentzian manifold (M, g) that shares some features with simple cylinders of negative spatial curvature.

Roughly speaking, the features are

1. space and time can be split apart
2. (M, g) satisfies a curvature bound
3. (M, g) is "simple"

More precisely, for point 1, we assume that

(H1) There is a smooth, proper, surjective $\tau : M \to \mathbb{R}$ with timelike $\nabla \tau$.

This guarantees that M is diffeomorphic to $\mathbb{R} \times M_0$ with M_0 a compact manifold with boundary [Hintz-Uhlmann'19], and that the direct problem for the wave equation can be solved on (M, g) [Hörmander, Vol 3].

Spacetime curvature bound

Definition [Andersson-Howard'98]. For $K \in \mathbb{R}$, we write $R \le K$ if

$$\langle R(X,Y)Y, X\rangle \le K\left(\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2\right)$$

for all $X, Y \in T_pM$ and $p \in M$. Here R is the curvature tensor on (M, g).

In the Riemannian case, $R \le K$ is equivalent with the sectional curvature bound $\mathrm{Sec}(X, Y) \le K$ for all linearly independent $X, Y \in T_pM$, $p \in M$.

In the Lorentzian case, if $\mathrm{Sec}(X, Y) \le K$ whenever it is well-defined, then the manifold is of constant curvature [Kulkarni'79].

Writing $\mathrm{diam}(M)$ for the supremum of the lengths of inextendible spacelike geodesics on M, we assume that

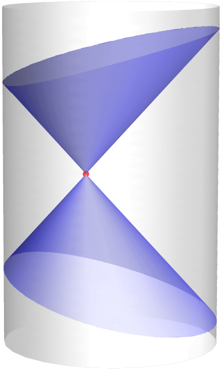
(H2) $R \le K$ for some $K \in \mathbb{R}$, and if $K > 0$ then $\mathrm{diam}(M) < \frac{\pi}{2\sqrt{K}}$.

Spacetime simplicity

For a point $p \in M$, we define $E(p)$ to be the set of points in M that are not causally related to p,

$$E(p) = \{q \in M : q \neq p \text{ and there is no causal path from } p \text{ to } q\}.$$

Figure. p in red, ∂M in gray, $\partial E(p) \cap M^{\text{int}}$ in blue



We assume

(H3) For any lightlike geodesic γ and any points p and q on γ, the only causal path from p to q is along γ. For all $p \in M$, the exponential map $\exp_p$ is a diffeomorphism from the spacelike vectors[1] onto $E(p)$.

(H4) All lightlike geodesics are non-trapped.

(H5) All lightlike geodesics have finite order of contact with ∂M.

[1] in its maximal domain of definition

Solution to the Lorentzian Calderón problem

Theorem [Alexakis-Feizmohammadi-L.O.]. Let a smooth Lorentzian manifold (M, g) with timelike boundary satisfy (H1)–(H5), and let $V_1, V_2 \in C^\infty(M)$. If $\mathscr{C}(V_1) = \mathscr{C}(V_2)$ then $V_1 = V_2$.

In fact, we prove a slightly stronger result that allows for data on a finite time interval and vanishing initial conditions.

The proof is based on

- New optimal unique continuation result
- Construction of focusing solutions: $u|_{t=t_0} = 0$ and $\partial_t u|_{t=t_0} = c\delta_{x_0}$ for a point $(t_0, x_0) \in \mathbb{R} \times M_0 = M$ and a constant $c \in \mathbb{R}$

Unique continuation

Theorem [Alexakis-Feizmohammadi-L.O.]. Let a smooth Lorentzian manifold (M, g) with timelike ∂M satisfy (H1)–(H5). Let $V \in C^\infty(M)$ and let $u \in H^s(M)$ with $s \in \mathbb{R}$. Suppose that $(\Box + V)u = 0$ in $E(p)$ and $u = \partial_\nu u = 0$ on $E(p) \cap \partial M$ for some $p \in M^{\text{int}}$. Then $u = 0$ in $E(p)$.

The Minkowski case, with $u \in C^2(M)$, was known [Alexakis-Shao'15].

The proof is based on

- Carleman estimate
- Microlocal analysis to handle $s < 2$

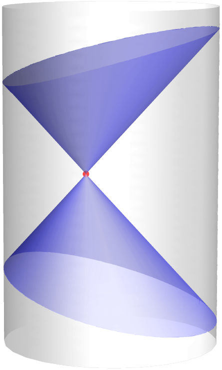


Figure. p in red, ∂M in gray, $\partial E(p) \cap M^{\text{int}}$ in blue

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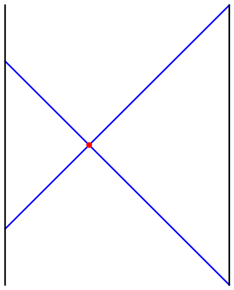

Figure. p in red, ∂M in black, $\partial E(p) \cap M^{\text{int}}$ in blue

Carleman estimate

We prove an estimate of the form

$$\int_\Omega |u|^2 e^{2\ell} \lesssim \int_\Omega |(\Box + V)u|^2 e^{2\ell} + \left| \int_\Omega \operatorname{div} B \right|,$$

where B is a certain vector field, Ω is the exterior of a hyperboloid in normal coordinates at p, and ℓ is a carefully chosen weight function.

The level sets of ℓ are given by the hyperboloids $\{(t,x) \in M : r(t,x) = \rho\}$, $\rho > 0$, where

$$r(t,x) = \left(-t^2 + (x^1)^2 + \cdots + (x^n)^2\right)^{1/2}.$$

Also $\Omega = \{r > \rho_0\}$ for some $\rho_0 > 0$.

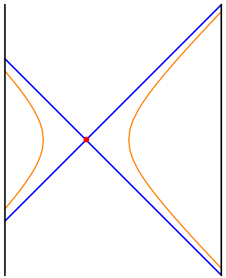

Figure. ∂M in black, $\{r = 0\}$ in blue, $\{r = \rho\}$ in orange

Hyperboloids in the Minkowski space

The function

$$r(t,x) = \left(-t^2 + (x^1)^2 + \cdots + (x^n)^2\right)^{1/2}$$

is a distance function in the sense that $\langle \nabla r, \nabla r \rangle = 1$.

Recall that a function f is pseudoconvex if $\operatorname{Hess} f(X,X) > 0$ for all lightlike vectors X orthogonal to ∇f.

The function r is **not** pseudoconvex:

$$\operatorname{Hess} r(X,X) = \frac{1}{r}\left(\langle X, X\rangle - \langle \nabla r, X\rangle^2\right),$$

and $\operatorname{Hess} r(X,X) = 0$ for all lightlike vectors X orthogonal to ∇r.

The weight function

Fix $\rho > 0$ and set $\ell = F \circ r$ where

$$F(r) = 4 \log(r - \rho) + \tau(r - \rho)^2.$$

Let $\epsilon > 0$ and set $\tau = C\epsilon^{-2}$ for large $C > 0$. Write $\Omega(\rho) = \{r > \rho\}$.

We prove a Carleman estimate on $\Omega(\rho + \epsilon)$.

Note that

$$F(\rho + \epsilon) = 4 \log \epsilon + C \to -\infty$$

as $\epsilon \to 0$.

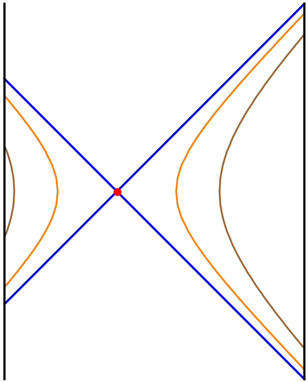

Figure. $\{r = \rho\}$ in orange, $\{r = \rho + \epsilon\}$ in brown

Unique continuation using the Carleman estimate

With $\ell = F \circ r$, we prove

$$\int_{\Omega(\rho+\epsilon)} |u|^2 e^{2\ell} \lesssim \int_{\Omega(\rho+\epsilon)} |(\Box + V)u|^2 e^{2\ell} + \left| \int_{\Omega(\rho+\epsilon)} \operatorname{div} B \right|.$$

Recall that $F(\rho+\epsilon) = 4\log\epsilon + C$. Hence $e^{2\ell} = \mathcal{O}(\epsilon^8)$ on $\{r = \rho+\epsilon\}$, and

$$\int_{\Omega(\rho+\epsilon)} \operatorname{div} B \to 0, \quad \text{as } \epsilon \to 0,$$

assuming that $u = \partial_\nu u = 0$ on $E(p) \cup \partial M$. If also $(\Box + V)u = 0$, then

$$\int_{\Omega(\rho)} |u|^2 (r-\rho)^8 = 0.$$

Thus $u = 0$ on $\Omega(\rho)$. Letting $\rho \to 0$ we conclude that $u = 0$ on $E(p)$.

Existence of focusing solutions

We assume without loss of generality that $M = \mathbb{R} \times M_0$. Let $p = (t_0, x_0) \in M^{\text{int}}$.

There is f such that the solution u of

$$\begin{cases} (\Box + V)u = 0 \quad \text{on } M \\ u|_{\partial M} = f \\ u|_{t \ll 0} = 0 \end{cases}$$

satisfies $u|_{t=t_0} = 0$ and $\partial_t u|_{t=t_0} = \delta_{x_0}$, and $\operatorname{supp}(f) \cap E(p) = \emptyset$.

This follows from our unique continuation result and [Bardos-Lebeau-Rauch'92].

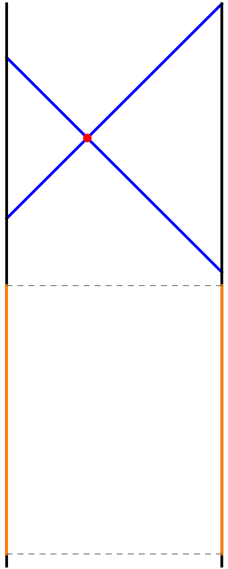

Figure. $\partial E(p) \cap M^{\text{int}}$ in blue, $\operatorname{supp}(f)$ in orange

Conclusion of the proof

Solutions with $u|_{t=t_0} = 0$, $\partial_t u|_{t=t_0} = c\delta_{x_0}$ for some $c \in \mathbb{R}$ can be found by requiring that $u = \partial_\nu u = 0$ on $E(p) \cap \partial M$ and that $u \in H^s(M)$ for suitable $s \in \mathbb{R}$.

Let v be another solution. Then

$$\int_{-\infty}^{t_0} \int_{M_0} v(\Box + V)u - u(\Box + V)v \, dxdt = 0,$$

and an integration by parts gives $cv(t_0, x_0)$. We force c to depend smoothly on (t_0, x_0).

If $c^{-1}(\Box + V)c = \Box + \tilde{V}$ and $c = 1$ on ∂M, then $c = 1$ on M and $\tilde{V} = V$.

Figure. $\partial E(p) \cap M^{\text{int}}$ in blue, $\{t = t_0\}$ in orange