Microlocal Analysis of Doppler Synthetic Aperture Radar

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Overview

Joint work with Raluca Felea, Romina Gaburro and Cliff Nolan

- Conventional (monostatic) SAR
- Doppler SAR
- Main results
- Future directions
- Supported by NSF DMS-1362271 and -1906186, and a SQuaRE collaboration at the American Institute of Mathematics. Thanks to Margaret Cheney for suggesting this problem and numerous discussions.

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Conventional synthetic aperture radar (SAR)

- EM waves are **transmitted** by air/spaceborne antenna.
- Waves **scatter** from features on Earth's surface.
- **Monostatic** SAR: reflected waves are measured by a receiver **co-located** with transmitter.
  (In **bistatic** SAR, receiver is separated from transmitter.)
- Transmitted wave is comprised of a train of **complex signals,** designed to improve temporal resolution, improve signal-to-noise ratio after DSP by receiver. Necessarily **wide bandwidth**.
- Allows high resolution estimates of two-way travel times and hence **distances (ranges)** between antenna and surface.
- Construct image of surface features via **filtered backprojection.**

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- Large literature. Will focus on **microlocal** techniques, analyzing effect of **flight path geometry,** characterizing **artifacts** and suggesting ways to remove or avoid them, e.g., beam forming.

- A few (of many) papers: Nolan-Cheney (2002,2004); Felea (2007); Felea-Gaburro-Nolan (2013); Stefanov-Uhlmann (2013).

- Microlocal analysis a reasonable tool to use for conventional SAR: complex waveforms employed have **high frequency** content.

- Want to apply MLA to Doppler SAR, using relative **velocity** data rather than distance/range data. Since DSAR is **narrowband**, not obvious that MLA should be useful, but obtain positive results.

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Doppler SAR: Overview

- **Goal:** To image a stationary scene located on a flat Earth, using DSAR data collected from a flight path at constant height.

- Eventually focus on two model flight paths: linear and circular. Analyze artifacts and describe artifact-free regions.

- Our analysis involves multiple approximations, the most important of which are:

    1. The Born (**single scattering**) approximation.
    2. The form of **reflectivity function**.
    3. **Start-stop** approximation (and a linear approximation).

- Modifying these to make the model more accurate might result in different conclusions, but we have evidence that our results are robust to altering the start-stop approximation.

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Doppler and SAR: History

- Using **Doppler radar** (frequency shift) data, collected by a stationary receiver, to image moving or rotating objects is familiar from weather radar and has been used in planetary astronomy (Thomson-Ponsonby (1968)).

- Alternatively, can try to **combine** Doppler and SAR to image fixed objects from a moving platform. Proposed by Borden-Cheney (2005), studied further for bistatic data and named Doppler SAR by Wang-Yazici (2012,2014). Related approaches by Coetzee-Baker-Griffiths (2006), Sun-Feng-Lu (2010), etc.

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Doppler SAR: Setup

- Antenna **transmits** along flightpath $\gamma(t)$ at constant altitude, h, and co-located receiver measures waves scattered by surface features.

- To obtain high-resolution Doppler shift measurements, transmit a **single frequency** continuous wave (CW) signal.

- Ignore polarization and model the wave using **scalar** equation,

$$\left(\nabla^2 - \frac{1}{c^2(\mathbf{y})}\partial_t^2\right) E(t, \mathbf{y}) = f(t, \mathbf{y}), \tag{1}$$

where $\mathbf{y} = (\mathbf{y}', y_3) \in \mathbb{R}^3$, E is (one component of) the electric field, f describes the **source**, and $c(\mathbf{y})$ is the wave propagation speed.

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Doppler SAR: Setup

- CW source modeled as

$$f(t, \mathbf{y}) = e^{-i\omega_0 t}\delta(\mathbf{y} - \gamma(t))$$

- Assumes radiating isotropically. For beam forming, is anisotropic; affects amplitudes but not phase functions or geometry, and will suppress.

- **Assumption 1.** Between the flight path and the ground, $c(\mathbf{y}) = c_0$, the constant speed of light in dry air.

  Def. The *reflectivity function* is $\frac{1}{c_0^2} - \frac{1}{c^2(\mathbf{y})}$.

- **Assumption 2.** The reflectivity function is of the form

  $V(\mathbf{y}')\delta(y_3)$, $\mathbf{y}' \in \mathbb{R}^2$. Call V the **scene.**

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Doppler SAR: Setup

Formal linearization:

- Write $c = c_0 + \delta c, E = E_0 + \delta E \implies$

$$\left(\nabla^2 - \frac{1}{c_0^2}\partial_t^2\right)\delta E(t, \mathbf{y}) = -V(\mathbf{y})\partial_t^2 E_0(t, \mathbf{y}) \quad (2)$$

  where the **incident field** E_0 satisfies (1) with c replaced by c_0.

- **Assumption 3. Start-stop approximation:** The speed of wave propagation is so much $>> |\dot{\gamma}|$ that the point where the scattered wave is detected by the receiver is **the same** as where the antenna transmitted it.

- The start-stop approx is widely used in radar, incl conventional SAR. For CW waves, harder to justify, but can do this *ex post facto*, since (i) for linear γ, further correction leads to a similar result for the FIO geometry, and (ii) for circular γ, the FIO geometry is structurally stable.

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Doppler SAR: Setup

- For point on the surface, $\mathbf{x} \in \mathbb{R}^2$, let

$$\mathbf{R}(t) := (\mathbf{x}, 0) - \gamma(t) \quad \text{and} \quad R(t) := |\mathbf{R}(t)|.$$

  Suppress dependence on $\mathbf{x}$.

- Under the start-stop approximation, total time of travel for a transmitted wave is $T_{tot} \approx 2c_0^{-1}R(t)$.

- Using Green's function for the free wave equation,

$$G(t, \mathbf{y}) = \frac{\delta(t - |\mathbf{y}|/c)}{4\pi|\mathbf{y}|},$$

  the scattered wave measured on the antenna is

$$d(t) := \delta E(t, \gamma(t)) \approx \int_{\mathbb{R}^2} \frac{e^{-i\omega_0(t - 2R(t)/c_0)}}{(4\pi R(t))^2} p(t, \mathbf{x}) V(\mathbf{x}) d\mathbf{x}. \tag{3}$$

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Doppler SAR: Windowed Fourier transform

We now convert the one-dim signal $d(t)$ to a function of two variables.

- Like most radar problems, multiple time scales.
- $c_0 \approx 3 \cdot 10^8$ m/sec >> plane $\approx 3 \cdot 10^2$ m/sec, satellite $\approx 10^4$ m/sec.
- Also frequencies involved are large, typically $\omega_0 \geq 1$ GHz $= 10^9$/sec.
- Analyze data $d(t)$ by introducing a "slow time" s and a frequency ω: multiply d by a windowing function, whose duration is
  1. small relative to the antenna motion (*i.e.*, the distance the antenna travels during the window is small relative to a wavelength), but
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- We take the window about $s = t$ to be of the form $\ell\left(\omega_0\left(t-s\right)\right)$, where $\ell(t)$ is smooth, identically equal to 1 for $|t| \leq L$ and supported in $|t| \leq 2L$, for some appropriately chosen $L > 0$.

- Although of compact support, it is natural to consider $\ell(\cdot)$ as being a symbol of order zero (see below), as the functions $\ell(\omega_0 \cdot)$ are symbols of order 0 uniformly in ω_0 and L.

- Thus, using the expression (3) for $d(t)$, we form

$$W_0(s,\omega) := \int e^{i\omega(t-s)}\ell(\omega_0(t-s))d(t)dt$$
$$= \int e^{i\omega(t-s)}\ell(\omega_0(t-s)) \int \frac{e^{-i\omega_0(t-2R(t)/c_0)}}{\left(4\pi R\left(t\right)\right)^2} V\left(\mathbf{x}\right)d\mathbf{x}dt \quad (4)$$

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- We take the window about $s = t$ to be of the form $\ell\left(\omega_0\left(t - s\right)\right)$, where $\ell(t)$ is smooth, identically equal to 1 for $|t| \leq L$ and supported in $|t| \leq 2L$, for some appropriately chosen $L > 0$.

- Although of compact support, it is natural to consider $\ell(\cdot)$ as being a symbol of order zero (see below), as the functions $\ell(\omega_0 \cdot)$ are symbols of order 0 uniformly in ω_0 and L.

- Thus, using the expression (3) for $d(t)$, we form

$$W_0(s,\omega) := \int e^{i\omega(t-s)}\ell(\omega_0(t-s))d(t)dt$$
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- Taylor expand $R(t)$ about $t = s$,

$$R(t) = R(s) + \dot{R}(s)(t - s) + \cdots$$

(Dot denotes differentiation with respect to time.)

- Keep only the linear terms, insert into (4), change variables $t \mapsto \tau = t - s$. Get

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$$W(s,\omega) := \int \Big[\int e^{i\tau(\omega - \omega_0 + 2\omega_0 \dot{R}(s)/c_0)} \times \underbrace{(\ell(\omega_0 \tau) e^{i\omega_0(2R(s)/c_0 - s)})/((4\pi R(s))^2)}_{a(s,\omega_0,\mathbf{x};\tau)} \, d\tau \Big] V(\mathbf{x}) \, d\mathbf{x} \qquad (5)$$

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Doppler SAR: The DSAR transform

Define the **DSAR transform** as the map $\mathcal{F} : V(\mathbf{x}) \to W(s, \omega)$.
The phase function $\phi(s, \omega, \mathbf{x}; \tau) := \tau\big(\omega - \omega_0 + 2\omega_0 \dot{R}(s)/c_0\big)$ is an *nondegenerate operator phase function* in the sense of Hörmander, the amplitude is of order 0 in τ, and we have

Theorem

Under assumptions above, the mapping $\mathcal{F}$ is a Fourier integral operator of order $-1/2$ associated with the canonical relation

$$\mathcal{C} = \left\{ \left(s, \omega_0 - 2\omega_0 \dot{R}/c_0, 2\omega_0 \tau \ddot{R}/c_0, \tau; \mathbf{x}, \underbrace{-2\tau\omega_0 d_{\mathbf{x}} \dot{R}/c_0}_{\xi} \right) : s \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^2, \tau \in \mathbb{R} \setminus 0 \right\}$$

$$\subset (T^*\mathbb{R}^2_{s,\omega} \setminus \mathbf{0}) \times (T^*\mathbb{R}^2_{\mathbf{x}} \setminus \mathbf{0}).$$

Doppler SAR: The DSAR transform

Well known from applications of microlocal analysis to linear or linearized inverse problems that one should study the projections from $\mathcal{C}$:

- The left projection, $\pi_L : \mathcal{C} \to T^*\mathbb{R}^2$, defined by

$$\pi_L : (s, \omega, \sigma, \tau; \mathbf{x}, \xi) \mapsto (s, \omega, \sigma, \tau), \tag{6}$$

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- The right projection, $\pi_R : \mathcal{C} \to T^*\mathbb{R}^2$, given by

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Doppler SAR: The DSAR transform

- At any point $\lambda_0 \in \mathcal{C}$, if $det(d\pi_R) \neq 0$, then $det(d\pi_L) \neq 0$ as well, and $\mathcal{C}$ is a local canonical graph.

- Adding the condition that π_L is 1-1 ($\leftrightarrow$ Bolker condition in tomography, travel time injectivity condition in linearized seismology) gives

$$(\mathcal{F})^* \mathcal{F} \in \Psi^{-1}\left(\mathbb{R}^2_{\mathbf{x}}\right),$$

with ellipticity $\leftrightarrow$ an illumination condition. This leads to imaging by filtered backprojection; can determine singularities of $V(\mathbf{x})$ from sings of $W(s, \omega)$.

- So: for model flightpaths γ, want to find $\Sigma := \{det(d\pi_R) = 0\} \subset \mathcal{C}$ and, if $\Sigma \neq \emptyset$, study structure of π_R, π_L at Σ, and globally.

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Main results: linear flight path

Theorem

If the flight path γ is a straight, horizontal line, then the canonical relation $\mathcal{C}$ is a fold/ blowdown. I.e., where $\mathcal{C}$ is not a local canonical graph, π_L has a Whitney fold singularity and π_R is a blowdown.

There is a left-right artifact about the projection, Γ of the flight path γ onto the ground. By beam forming to the left or right, the artifact can be eliminated, in which case the processed data W determines any scene V to the left or right of Γ, up to a possible C^∞ smooth error.

- This is the same structure of $\mathcal{C}$ as arises in conventional monostatic SAR for a linear flightpath.

Main results: linear flight path

- The Schwartz kernel of $(\mathcal{F})^*\mathcal{F}$ is then known to be a paired Lagrangian distribution, with WF relation not only on the diagonal, but on another canonical graph, resulting in a strong artifact unless one beam forms (Felea (2007)):

- $WF\left((\mathcal{F})^*\mathcal{F}\right) \subseteq \Delta_{T^*\mathbb{R}^2} \cup Gr(\chi)$, where χ is the canonical involution induced by the reflection $(x_1, x_2) \to (x_1, -x_2)$ about Γ.

- $Gr(\chi)$ intersects Δ cleanly in codimension 2, and

$$(\mathcal{F})^*\mathcal{F} \in I^{-1,0}\left(\Delta, Gr(\chi)\right),$$

which has the same order -1 on both the Δ and $Gr(\chi)$.

- This results in a strong, nonremovable artifact if the transmitter is isotropic, but artifacts are avoidable via beam forming.

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Main results: circular flight path

Theorem

If the flight path is circular, the DSAR map $\mathcal{F}$ is associated to a canonical relation $\mathcal{C}$ with a fold/cusp degeneracy. I.e., π_L is a Whitney fold and π_R has singularities of up to and including simple cusp type. For a scene V with suitable, explicitly describable support, or by suitable beam forming, the associated artifact can be eliminated, and then the processed data W determines V up to a possible smooth C^∞ error.

- This is **worse** than what happens in monostatic SAR for a circular flight path, which is a fold/fold (Nolan-Cheney (2004), Felea (2007)).
- A similar geometry comes up in monostatic SAR for γ with a simple inflection point. There is a strong, nonremovable artifact but the non-diagonal part of the WF relation is an open umbrella, which is not even a smooth canonical relation (Felea-Nolan (2015), cf. Felea-Greenleaf (2010)).

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Main results: circular flight path

- If $\gamma(t) = (\rho\cos(t), \rho\sin(t), \rho h)$, then we show that the surface region

$$\left\{\mathbf{x} \in \mathbb{R}^2 \;:\; |\mathbf{x}| < \rho(1+h^2)/2\right\}$$

  is artifact free.

- The DSAR artifacts are disjoint from the monostatic SAR artifacts for the same circular flight path, raising the possibility of combining DSAR and conventional SAR data for artifact free imaging.

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Main results: correction to start-stop approximation

- Recall range $R(t) = |\mathbf{x} - \gamma(t)|$ and T_{tot} = total travel time for wave, transmitter-scatterer-receiver. T_{tot} is determined implicitly by

$$c_0 T_{tot} = R(t) + R\left(t + T_{tot}\right).$$

- The start-stop approximation is used to assume

$$R(t + T_{tot}) = R(t) \implies T_{tot} \approx \frac{2}{c_0} R(t).$$

- We refine this, expanding

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- Solving for T_{tot} and ignoring terms $\mathcal{O}(c_0^{-3})$, we obtain the first order refinement of the start-stop approximation, namely that the total travel time is
$$T_{tot} \approx 2c_0^{-1}R(t) + 2c_0^{-2}R(t)\dot{R}(t)$$
- Under this refined approximation, the scattered wave arrives at time
$$t_{sc} = t + 2c_0^{-1}R(t) + 2c_0^{-2}R(t)\dot{R}(t).$$
- Using this, the refinement of the windowed W_0 is
$$W_1(s,\omega) = \int e^{i\omega(t-s)}\ell(\omega_0(t-s)) \quad \times$$
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Main results: correction to start-stop approximation

- Into this we substitute the linear terms of the Taylor expansions

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- Calculations show that the canonical relation $\mathcal{C}_{mod}$ parametrized by the new phase function ϕ is **still** of fold/blowdown type, with the same implications for artifacts and beam forming as were obtained under the start-stop approximation.

- Folds **are** structurally stable, so it is not surprising that a refined model results in a canonical relation whose projection π_L is still a Whitney fold.

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Future directions

Some open questions:

- For linear flight path, can we completely remove start-stop approximation, working with the exact, implicitly defined T_{tot}? For a circular path, even the first order correction described above remains to be done.

- Can we obtain the fold/cusp structure, and a reasonable description of an artifact-free zone, for a general flightpath with nonzero curvature?

- Can we make the model more physically realistic? What are the effects of multiple scattering? Can we quantify the degradation from a transmitted wave not being pure single-frequency? Etc.

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