Location and Weyl asymptotics for the eigenvalues of some non self-adjoint operators

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Outline

1. Two spectral problems related to scattering theory
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5. Weyl asymptotics for dissipative eigenvalues close to $\mathbb{R}^-$

Two spectral problems related to scattering theory.

(A) Dissipative eigenvalues . Let $K \subset \mathbb{R}^d$, $d \geq 2$, be a bounded non-empty domain and let $\Omega = \mathbb{R}^d \setminus \bar{K}$ be connected. We suppose that the boundary Γ of K is C^∞. Consider the boundary problem

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}_t^+ \times \Omega, \\ \partial_\nu u - \gamma(x) u_t = 0 \text{ on } \mathbb{R}_t^+ \times \Gamma, \\ u(0,x) = f_0,\ u_t(0,x) = f_1 \end{cases} \tag{1}$$

with initial data $f = (f_1, f_2) \in H^1(\Omega) \times L^2(\Omega) = \mathcal{H}$. Here ν is the unit outward normal to Γ pointing into Ω and $\gamma(x) \geq 0$ is a C^∞ function on Γ. The solution of (1) is given by $V(t)f = e^{tG} f,\ t \geq 0$, where $V(t)$ **is a semi-group** in $\mathcal{H}$ whose generator has a domain $D(G)$ which is the closure in the graph norm of functions $(f_1, f_2) \in C^\infty_{(0)}(\mathbb{R}^n) \times C^\infty_{(0)}(\mathbb{R}^n)$ satisfying the boundary condition $\partial_\nu f_1 - \gamma f_2 = 0$ on Γ. Lax and Phillips proved that the spectrum of G in $\mathrm{Re}\, z < 0$ is formed by isolated eigenvalues with finite multiplicity.

Notice that if $Gf = \lambda f$ with $f = (f_1, f_2) \neq 0$ and $\partial_\nu f_1 - \gamma f_2 = 0$ on Γ, we get

$$\begin{cases} (\Delta - \lambda^2) f_1 = 0 \text{ in } \Omega, \\ \partial_\nu f_1 - \lambda \gamma f_1 = 0 \text{ on } \Gamma. \end{cases} \tag{2}$$

Moreover, $u(t,x) = V(t)f = e^{\lambda t} f(x), \mathrm{Re}\, \lambda < 0$, is a solution of (1) with exponentially decreasing global energy. Such solutions are called asymptotically disappearing and they perturb the inverse scattering problems. We proved that if we have a least one eigenvalue λ of G with $\mathrm{Re}\, \lambda < 0$, then the wave operators $W_\pm$ are not complete, that is $\mathrm{Ran}\, W_- \neq \mathrm{Ran}\, W_+$ and we cannot define the scattering operator S by $S = W_+^{-1} W_-$. We may define S by using another evolution operator. For Maxwell system we study the same problems for the system

$$\begin{cases} \partial_t E = \mathrm{curl}\, B, \qquad \partial_t B = -\mathrm{curl}\, E \quad \text{in} \quad \mathbb{R}_t^+ \times \Omega, \\ E_{tan} - \gamma(x)(\nu \wedge B_{tan}) = 0 \quad \text{on} \quad \mathbb{R}_t^+ \times \Gamma, \\ E(0,x) = e_0(x), \qquad B(0,x) = b_0(x). \end{cases} \tag{3}$$

(B) Interior transmission eigenvalues.

We will study another important spectral problem leading to non self-adjoint operator. The inhomogeneous medium in K is characterised by a smooth function $n(x) > 1$ in $\bar{K}$, called contrast. The inverse scattering problem of the reconstruction of K based on the linear sampling method of Colton and Kress breaks down for frequencies $k \in \mathbb{R}$ which are interior transmission eigenvalues (ITE). This means that we have non-trivial solution $(u, v) \neq 0$ of the problem

$$\begin{cases} \Delta u + k^2 u = 0 \text{ in } K, \\ \Delta v + k^2 n(x) v = 0 \text{ in } K, \\ u = v, \; \partial_\nu u = \partial_\nu v \text{ on } \Gamma \end{cases} \tag{4}$$

Moreover, we may have complex (ITE) and all (ITE) are important for the reconstruction of $n(x)$.

We consider a more general setting. A complex number $\lambda \in \mathbb{C}, \lambda \neq 0$, is called interior transmission eigenvalue (ITE) if the following problem has a non-trivial solution $(u, v) \neq 0$:

$$\begin{cases} \left(\nabla c_1(x)\nabla + \lambda^2 n_1(x)\right) u = 0 \text{ in } K, \\ \left(\nabla c_2(x)\nabla + \lambda^2 n_2(x)\right) v = 0 \text{ in } K, \\ u = v, \;\; c_1\partial_\nu u = c_2\partial_\nu v \text{ on } \Gamma, \end{cases} \tag{5}$$

where ν denotes the exterior unit normal to Γ, $c_j(x), n_j(x) \in C^\infty(\overline{K})$, $j = 1, 2$ are strictly positive real-valued functions. For the analysis of (ITE) one imposes the condition

$$d(x) = c_1(x)n_1(x) - c_2(x)n_2(x) \neq 0, \quad \forall x \in \Gamma. \tag{6}$$

Partial cases: 1) isotropic case: $c_1(x) = c_2(x)$, $\partial_\nu c_1(x) = \partial_\nu v_2(x)$, $\forall x \in \Gamma$.
2) anisotropic case: $c_1(x) \neq c_2(x)$, $\forall x \in \Gamma$. In the isotropic case the celebrated complementing condition of Agmon, Douglas and Nirenberg is not satisfied.

Problems

(I) **Prove the discreteness of the spectrum in some subset $U \subset \mathbb{C}$.**

(II) **For (ITE) find eigenvalues-free domains having the form**

$$|\operatorname{Im}\lambda| \geq C(1+|\operatorname{Re}\lambda|)^{\delta_+}, |\operatorname{Re}\lambda| \geq 1, 0 \leq \delta_+ < 1.$$

$$1 \geq |\operatorname{Re}\lambda| \geq C_N(1+|\operatorname{Im}\lambda|)^{-N},\ |\operatorname{Im}\lambda| \geq 1,\ \forall N \in \mathbb{N}.$$

In some cases we have $\delta_+ = 0$. Find for (A) **similar eigenvalues-free domains.**

(III) **Establish a Weyl asymptotic for (ITE) with remainder** $\mathcal{O}_\epsilon(r^{d-\kappa+\epsilon})$, arbitrary $0 < \epsilon \ll 1$ and $0 < \kappa \leq 1$ for the counting function

$$N(r) = \#\{\lambda_j\text{-}(ITE) : |\lambda_j| \leq r\} = cr^d + \mathcal{O}_\epsilon(r^{d-\kappa+\epsilon}),\ r \to \infty.$$

In this talk we treat the problems (II) and (III). The problem (I) is easier to deal with and we can find an operator A such that $A - z$ is Fredholm one in a suitable regions. For transmission eigenvalues (II) and (III) are connected and $\kappa = 1 - \delta_+$.

Semi-classical Dirichlet-to-Neumann map

Set $\lambda = \frac{\sqrt{z}}{h}$, so $\lambda^2 = \frac{z}{h^2}$. Given $f \in H^s(\Gamma)$, consider the problem

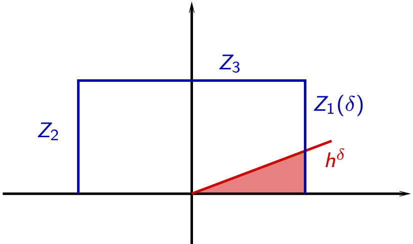
$$\begin{cases} (P(h) - z)u = 0 \text{ in } K, \\ u = f \text{ on } \Gamma. \end{cases} \tag{7}$$

Here $P(h) = -\frac{h^2}{n(x)}\nabla c(x)\nabla$, $0 < h \ll 1$, is a semiclassical parameter and $z \in Z_1 \cup Z_2 \cup Z_3$, where

$$Z_1 = \{\mathrm{Re}\, z = 1, 0 \le \mathrm{Im}\, z \le 1\},\ Z_1(\delta) = Z_1 \cap \{\mathrm{Im}\, z \ge h^\delta\},$$

$$Z_2 = \{\mathrm{Re}\, z = -1, 0 \le \mathrm{Im}\, z \le 1\},\ Z_3 = \{|\mathrm{Re}\, z| \le 1, \mathrm{Im}\, z = 1\}.$$

Figure 1: Contours $Z_1(\delta), Z_2, Z_3$



Region $Z_1(1/2-\epsilon)\cup Z_2\cup Z_3$

Let $D_\nu = -i\partial_\nu$, and let γ_0 denote the trace on Γ. It is important to construct a semi-classical parametrix for the problem (7) for $z \in Z_1(1/2-\epsilon)\cup Z_2\cup Z_3$ with $0<\epsilon\ll 1$ and to study the semi-classical Dirichlet-to-Neumann map (DN)

$$\mathcal{N}(z,h) : H_h^s(\Gamma) \ni f \longrightarrow \gamma_0 h D_\nu u \in H_h^{s-1}(\Gamma)$$

for domains with arbitrary geometry. Here $H_h^s(\Gamma)$ is the semi-classical Sobolev space with norm $\|\langle hD\rangle^s u\|_{L^2(\Gamma)}$. By the estimate of the resolvent $(h^2 G_D - z)^{-1}$ of the Dirichlet Laplacian G_D, it is easy to see that $\mathcal{N}(z,h)$ is a meromorphic function with poles on $\mathbb{R}^+$

G. Vodev (2014) constructed a semi-classsical parametrix for (7) as a FIO with complex phase $\varphi(x,\xi';z)$ in a small neighborhood of the boundary Γ. The eikonal equation and the transport equations can be solved only modulo $\mathcal{O}(x_d^N)$, $\forall N\gg 1$, $x_d = \mathrm{dist}(x,\Gamma)$.

Next we use some $h-$pseudo-differential operators.
Set $x=(x',x_d)$, $\xi=(\xi',\xi_d)$. We say that $a(x',\xi';h)\in S^k_\delta(\Gamma)$ if the following conditions are satisfied:

$$|\partial_x'^\alpha \partial_{\xi'}^\beta a(x,\xi';h)|\leq C_{\alpha,\beta}h^{-\delta(|\alpha|+|\beta|)}\langle\xi'\rangle^{k-|\beta|},\ \forall\alpha,\forall\beta,$$

where $\langle\xi'\rangle=(1+|\xi'|^2)^{1/2}$. For $a\in S^{k,m}_\delta(\Gamma)$, we consider the operator

$$\Big(Op_h(a)f\Big)(x)=(2\pi h)^{-d+1}\int\int e^{i\langle x'-y',\xi'\rangle/h}a(x',\xi';h)f(y')dyd\xi'.$$

We have a calculus for the $h-$pseudodifferential operators with symbols in S^k_δ if $0<\delta<1/2$. In particular, if $a\in S^1_\delta$, $b\in S^{-1}_\delta$, one gets

$$\|Op_h(a)Op_h(b)-Op_h(ab)\|_{L^2(\Gamma)}\leq Ch^{1-2\delta}.$$

Close to the boundary introduce geodesic normal coordinates (x', x_d) in a neighborhood of a point $x_0 \in \Gamma$ with $x_d = 0$ on Γ(we take $x_d = \mathrm{dist}(x, \Gamma)$). Let $n_0(x') = \frac{n(x',0)}{c(x',0)} > 0$. The symbol of $-h^2\Delta$ becomes

$$\xi_d^2 + r(x, \xi') + hq(x, \xi) + h^2 q_0(x)$$

and $r(x', 0, \xi') = r_0(x', \xi')$ is the principal symbol of the Laplace-Beltrami operator $-h^2\Delta|_\Gamma$ on Γ. For $z \in Z_1 \cup Z_2 \cup Z_3$, let

$$\rho(x', \xi', z) = \sqrt{n_0(x')z - r_0(x', \xi')} \in C^\infty(T^*(\Gamma)), \ \mathrm{Im}\,\rho > 0$$

be the root of the equation

$$\rho^2 + r_0(x', \xi') - n_0(x')z = 0$$

.

It is easy to see that $\rho \in S^1_{1/2-\epsilon}$, if $z \in Z_1(1/2 - \epsilon)$, $\rho \in S^1_0$, if $z \in Z_2 \cup Z_3$.

Proposition 1 (Vodev, (2014))

Given $0 < \epsilon \ll 1$, there exists $0 < h_0(\epsilon) \ll 1$ such that for $z \in Z_1(1/2 - \epsilon)$ and $0 < h \leq h_0(\epsilon)$ we have

$$\|\mathcal{N}(z,h) - Op_h(\rho + hb)\|_{L^2(\Gamma)\to H^1_s(\Gamma)} \leq \frac{Ch}{\sqrt{|\operatorname{Im} z|}}, \tag{8}$$

where $C > 0$ is independent of h, z, ϵ and $b \in S^0_0$ does not depend on z, h and the function $n_0(x')$. Moreover, for $z \in Z_2 \cup Z_3$ the above estimate holds with $|\operatorname{Im} z|$ replaced by 1.

The analysis of $\mathcal{N}(z,h)$ in the region

$$\Sigma = \{\operatorname{Re} z = 1,\ C_0 h \leq |\operatorname{Im} z| \leq C_1 h^{1/2-\epsilon}\},$$

is a more difficult problem. Set $r_\#(x',\xi') = n_0^{-1}(x') r_0(x',\xi')$ and introduce

$$\mathcal{H} = \{(x',\xi') \in T^*(\Gamma) :\ r_\#(x',\xi') < 1\},\ \mathcal{G} = \{(x',\xi') \in T^*(\Gamma) :\ r_\#(x',\xi') = 1\},$$

$$\mathcal{E} = \{(x',\xi') \in T^*(\Gamma) :\ r_\#(x'\ \xi') > 1\}.$$

Region $C_0 h \leq |\operatorname{Im} z| \leq C_1 h^{1/2-\epsilon}$

In general, for boundary with arbitrary geometry it is not possible to construct a semi-classical parametrix for data supported in a neighbourhood ω of $\mathcal{G}$. For general obstacles Vodev (2017),(2019) constructed a parametrix in the zone $C_1 h^{2/3-\epsilon} \leq \operatorname{Im} z \leq C_2 h^{1/2-\epsilon}$ and for $z \in \Sigma$ for data supported outside ω. The strictly convex case has been treated previously by Sjöstrand (2014) for $C_1 h^{2/3} \leq \operatorname{Im} z \leq C_2 h^{2/3}$. Let $\chi_\delta^0 \in C_0^\infty(T^*(\Gamma))$ be supported in $\{(x', \xi') : |r_\#(x', \xi') - 1| \leq 2\delta^2\}, \chi_\delta^0 = 1$ for $\{(x', \xi') : |r_\#(x', \xi') - 1| \leq \delta^2\}$.

Theorem 1 (Vodev, (2017))

Let $0 < \epsilon < 1/2$ be arbitrary. Then for every $0 < \delta \ll 1$ there are constants $C_\delta > 1,\ 0 < c_{\epsilon,\delta} \ll 1$ such that we have

$$\|\mathcal{N}(z, h) - Op_h(\rho(1 - \chi_\delta^0) + hb)\|_{L^2(\Gamma)\to H_h^1(\Gamma)} \leq C\delta \tag{9}$$

for $\operatorname{Re} z = 1,\ C_\delta h \leq |\operatorname{Im} z| \leq h^{1/2-\epsilon},\ 0 < h \leq c_{\epsilon,\delta}$, where $C > 0$ is a constant independent of h, δ, ϵ and $b \in S_0^0(\Gamma)$ is independent of h, δ and the function $n_0(x')$.

By energy method Vodev showed that $\|\mathcal{N}(z, h) Op_h(\chi_\delta^0)\|_{L^2(\Gamma)\to H_h^1(\Gamma)} = \mathcal{O}(\delta)$.

Exterior Dirichlet-to-Neumann map

For the analysis of the dissipative eigenvalues we need to apply the exterior DIrichlet-to-Neumann map $\mathcal{N}_{ext}(z, h)$ defined as

$$\mathcal{N}_{ext}(z, h) : H_h^s(\Gamma) \ni f \longrightarrow \gamma_0 h D_\nu u \in H_h^{s-1}(\Gamma),$$

where u is the outgoing solution of the problem

$$(h^2\Delta + n_0(x')z)u = 0 \text{ in } \Omega = \mathbb{R}^d \setminus \bar{K},\ u|_\Gamma = f.$$

The operator $\mathcal{N}_{ext}(z, h)$ is a meromorphic function related to the cut-off outgoing resolvent $\chi(h^2 G_D - z)^{-1}\chi$ with poles in the half-plane $\{\mathrm{Im}\, z < 0\}$. The result similar to Prop.1 was proved by -P. (2016). For strictly convex obstacles K and $\mathrm{Re}\, z \sim 1,\ |\,\mathrm{Im}\, z| \leq c_0 h^{2/3}$ Sjöstrand (2014) obtained results similar to Th. 1. Finally, the case $h^{1/2-\epsilon} \leq \mathrm{Im}\, z \leq c_0 h^{2/3}$ for strictly convex obstacles has been covered by -P. by a semi-classical parametrix construction inspired by that of Vodev. Thus the result of Th. 1 holds for $\mathcal{N}_{ext}(z, h),\ -c_0 h^{2/3} \leq \mathrm{Im}\, z \leq h^{1/2-\epsilon}$ and strictly convex obstacles.

Eigenvalues-free regions

First we treat the problem with dissipative boundary conditions $\partial_\nu u - \gamma(x) u_t = 0$. If $\gamma(x) \equiv 1$ for $x \in \Gamma$ in the case when K is a ball there are no eigenvalues. Thus it is convenient to consider two cases: (i) $1 - \gamma(x) > 0, \forall x \in \Gamma$, (ii) $1 - \gamma(x) < 0, \forall x \in \Gamma$.

Theorem 2 (-P. (2016))

In the case (i) *for every* ϵ, $0 < \epsilon \ll 1$, *the eigenvalues of* G *lie in the region*

$$\Lambda_\epsilon = \{\lambda \in \mathbb{C} : \, |\operatorname{Re}\lambda| \leq C_\epsilon(|\operatorname{Im}\lambda|^{\frac{1}{2}+\epsilon} + 1), \, \operatorname{Re}\lambda < 0\}.$$

In the case (ii) *for every* ϵ, $0 < \epsilon \ll 1$, *and every* $N \in \mathbb{N}$ *the eigenvalues of* G *lie in the region* $\Lambda_\epsilon \cup \mathcal{R}_N$, *where*

$$\mathcal{R}_N = \{|\operatorname{Im}\lambda| \leq C_N(1 + |\operatorname{Re}\lambda|)^{-N}, \, \operatorname{Re}\lambda < -R < 0\}.$$

For strictly convex obstacles K we improve the above result in the case (ii).

Theorem 3 (-P. (2016))

Assume K *strictly convex. In the case* (ii) *for every* $N \in \mathbb{N}$ *the eigenvalues of* G *lie in the region* $\mathcal{R}_N \cup \{|\lambda| < R, \operatorname{Re}\lambda < 0\}$.

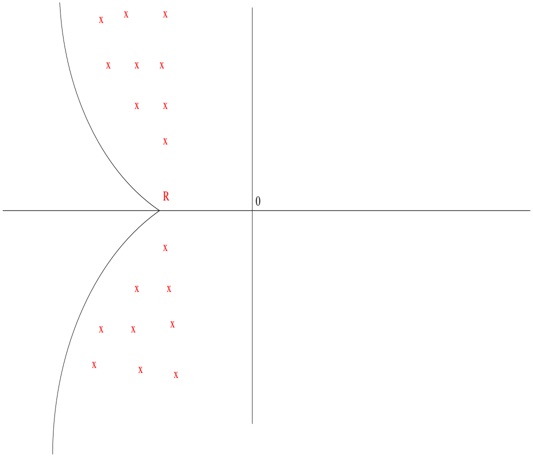


Figure 2: Eigenvalues for $0 < \gamma(x) < 1$

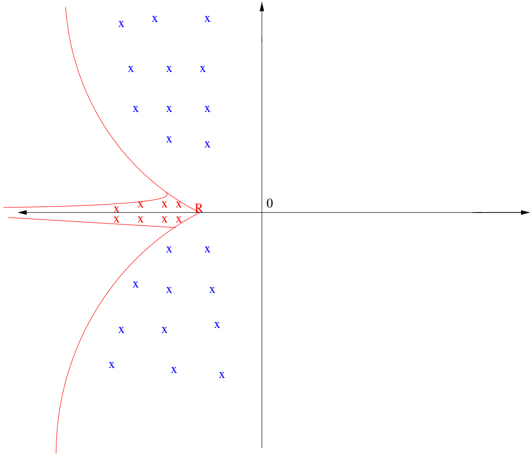


Figure 3: Eigenvalues for $\gamma(x) > 1$

Weaker results have been obtained by A. Majda (1976). For the Maxwell system with dissipative boundary conditions the results of Th. 2 have been established by F. Colombini, -P. and J. Rauch (2017).

Passing to the (ITE), one has the following

Theorem 4 (Vodev, (2014), (2017))

Assume $d(x) = (c_1 n_1 - c_2 n_2)(x) \neq 0, x \in \Gamma$. *Assume either the condition* $(a) : c_2(x) = c_1(x),\ \partial_\nu c_1(x) = \partial_\nu c_2(x),\ n_1(x) \neq n_2(x),\ \forall x \in \Gamma$ *or*

$$(b) : (c_1(x) - c_2(x))d(x) < 0,\ \forall x \in \Gamma.$$

Then there are there are no (ITE) in the region $\{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| \geq 1, |\mathrm{Im}\,\lambda| \geq C\}$. *In the case (a) there are no eigenvalues in* $\{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| \leq 1, |\mathrm{Im}\,\lambda| \geq \tilde{C}\}$. *In the case (b) for every* $N \in \mathbb{N}$ *there are no eigenvalues in*

$$\{1 \geq |\mathrm{Re}\,\lambda| \geq C_N(1 + |\mathrm{Im}\,\lambda|)^{-N}, |\mathrm{Im}\,\lambda| \geq \tilde{C}\}. \tag{10}$$

Assume the conditions

$$(c) :\ (c_1(x) - c_2(x))d(x) > 0,\ \ (c') : \frac{n_1(x)}{c_1(x)} \neq \frac{n_2(x)}{c_2(x)},\ \forall x \in \Gamma.$$

Then there are there are no (ITE) in the region

$$\{|\mathrm{Im}\,\lambda| \geq C \log(|\mathrm{Re}\,\lambda| + 2)\}. \tag{11}$$

• If only $((c_1(x) - c_2(x))d(x) > 0$ is satisfied, Vodev established an eigenvalue-free region

$$\{|\operatorname{Im}\lambda| \geq C(1 + |\operatorname{Re}\lambda|)^{3/5}\}.$$

• Previous parabolic eigenvalues-free region in the case $c_1 \equiv c_2 \equiv n_2 \equiv 1$ and $n_1(x) > 1$ in K has been obtained by M. Hitrik, K. Krypchyk, P. Ola and L. Päivärinta (2011).

• Colton and Leung (2012) examined the case of the ball $\{x \in \mathbb{R}^3 : |x| \leq 1\}$ when $c_1 \equiv c_2 \equiv n_2 \equiv 1$ and $n_1(r)$ depends only on $|x| = r$. Then if $n = const \neq 1$ and $\sqrt{n}$ is rational, there exists an infinite sequence of (ITE) $\lambda_k = \alpha k + \beta,\ k \in \mathbb{N}$ with $\alpha > 0, \operatorname{Im}\beta \neq 0$. Thus the result in (a) is optimal.

• Colton, Leung and Meng (2016) proved in the case above with $n_1(r) \neq const$ that if $n_1(1) = 1,\ \int_0^1 \sqrt{n_1(t)}dt \neq 1$, and either $n_1'(1)$ or $n_1(1)$ are not zero, then for any $C > 0$ the (ITE) are not located in a strip $\{|\operatorname{Im} z| \leq C\}$. Consequently, the parabolic regions above cannot be improved to an eigenvalue-free strip.

First consider the case $c_1 = c_2$, $\partial_\nu c_1 = \partial_\nu c_2$, $x \in \Gamma$. Let $z \in Z_1(1-1/2) \cup Z_2 \cup Z_3$ and let (u,v) be an eigenfunction. Set $f = u|_\Gamma = v|_\Gamma$. Let $\rho_j, j = 1,2$, be the roots of the equations $\rho^2 = -r_0(x', \xi') + n_j(x', 0)z$ with $\mathrm{Im}\,\rho_j > 0$. By using the Prop. 1, the analysis of the location of (ITE) is reduced to prove that the estimate

$$\|Op_h(\rho_1 - \rho_2)f\|_{H_h^1(\Gamma)} \leq \frac{Ch}{\sqrt{|\mathrm{Im}\,z|}}\|f\|_{L^2(\Gamma)}$$

yields $f = 0$. Next for $z \in Z_1(1/2 - \epsilon)$ the symbols ρ_j satisfy the estimates

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta \rho_j| \leq C_{\alpha,\beta}|\mathrm{Im}\,z|^{1/2-|\alpha|-|\beta|},\ |\alpha| + |\beta| \geq 1,$$

provided $|\xi'| \leq C_0$, while for $|\xi'| \geq C_0$ we have $|\partial_{x'}^\alpha \partial_{\xi'}^\beta \rho_j| \leq C_{\alpha,\beta}\langle\xi'\rangle^{1-|\beta|}$. Thus $\rho \in S^1_{1/2-\epsilon}$, since $|\mathrm{Im}\,z| \geq h^{1/2-\epsilon}$. On the other hand, for $z \in Z_2 \cup Z_3$ we have $\rho_j \in S^1_0$.

Moreover,

$$\rho_1 - \rho_2 = \frac{\rho_1^2 - \rho_2^2}{\rho_1 + \rho_2} = \frac{z(n_1(x') - n_2(x))}{\rho_1 + \rho_2}$$

and since $n_1(x') - n_2(x) \neq 0, \forall x \in \Gamma$, the operator $Op_h(\rho_1 - \rho_2)$ is elliptic and $(\rho_1 - \rho_2)^{-1} \in S^1_{1/2-\epsilon}$. Thus

$$\|(Op_h(\rho_1 - \rho_2))^{-1} Op_h(\rho_1 - \rho_2) f\|_{L^2(\Gamma)} \leq \|Op_h(\rho_1 - \rho_2) f\|_{H_h^1(\Gamma)}$$

$$\leq \frac{Ch}{\sqrt{|\operatorname{Im} h|}} \|f\|_{L^2(\Gamma)}$$

and

$$\Big\|\Big((Op_h(\rho_1 - \rho_2))^{-1} Op_h(\rho_1 - \rho_2) - Id\Big) f\Big\|_{L^2(\Gamma)} \leq Ch^{1-2\delta} \|f\|_{L^2(\Gamma)}.$$

For small $0 < h \leq h_0(\epsilon)$ we deduce $f = 0$. Then $h^2 \lambda^2 = z = 1 + \mathbf{i} \operatorname{Im} z$ implies easily that we have no (ITE) in the region

$$\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| \geq C(1 + |\operatorname{Re} \lambda|)^{1/2+\epsilon}\}.$$

The analysis of the case $c_1(x) \neq c_2(x)$ is more complicated. Recall that $d(x) = (c_1 n_1 - c_2 n_2)(x) \neq 0$ on Γ. We must study the symbol

$$\zeta = c_1\rho_1 - c_2\rho_2 = \frac{d[z - \frac{(c_1^2 - c_2^2)}{d} r_0]}{c_1\rho_1 + c_2\rho_2}.$$

For $z \in Z_2$ we have $\mathrm{Re}\, z = -1, 0 \leq \mathrm{Im}\, z \leq 1$. Then, for $\mathrm{Im}\, z = 0$ we have $\zeta = 0$ if

$$-1 - \frac{c_1 + c_2}{d^2}[(c_1 - c_2)d] r_0 = 0.$$

This could happen at some points if $(c_1 - c_2)d < 0$, that is in the case (b). This leads to the existence of infinitely many eigenvalues located very close to the imaginary axis. The same phenomenon appear for the dissipative eigenvalues in Th. 2, Th. 3 above, when $\gamma(x) > 1$ and infinitely many eigenvalues are concentrate around $\mathbb{R}^-$. Both phenomena are completely similar to Rayleigh surface waves for the linear elasticity system.

The analysis of the eigenvalues of G follows a similar argument. In the case (i) we have $0 < \epsilon_0 \leq \gamma(x) \leq 1 - \epsilon_0$, $\epsilon_0 > 0$, $\forall x \in \Gamma$. If $u \neq 0$ is an eigenfunction of G with eigenvalue $\lambda \in \{\operatorname{Re} z < 0\}$, then $f = \gamma_0 u \neq 0$. Set $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$. The boundary condition for the eigenfunction becomes

$$\mathcal{N}_{ext}(z, h) f - \gamma\sqrt{z} f = 0.$$

According to Prop. 1 for $\mathcal{N}_{ext}(z, h)$, for $1 \geq \operatorname{Im} z \geq h^{1/2-\epsilon}$ we have

$$\|Op_h(\rho) f - \gamma\sqrt{z} f\|_{L^2(\Gamma)} \leq C \frac{h}{\sqrt{|\operatorname{Im} z|}} \|f\|_{L^2(\Gamma)}, \tag{12}$$

where for $z \in Z_2 \cup Z_3$ the estimate holds with $|\operatorname{Im} z|$ replaced by 1. Consider the symbol

$$c(x', \xi'; z) = \rho(x', \xi'; z) - \gamma\sqrt{z} = \frac{(1 - \gamma^2) z - r_0}{\rho(x', \xi', z) + \gamma\sqrt{z}}.$$

We show that $c(x', \xi'; z)$ is elliptic and we follow a similar argument. Notice that if $\gamma(x) > 1, \forall x \in \Gamma$, for $z \in Z_2$ we have points, where the symbol c vanishes.

Location of the (ITE) for the inhomogenous Maxwel system

Let $E, \hat{E}, H, \hat{H}$ be vector-values functions in $K \subset \mathbb{R}^3$. We say that $\lambda \in \mathbb{C} \setminus \{0\}$ is a (ITE) if $(E, \hat{E}, H, \hat{H}) \neq 0$ satisfy the system

$$\begin{cases} \mathrm{curl}\, E = \mathbf{i}\lambda\mu H, \ \mathrm{curl}\, H = -\mathbf{i}\lambda\gamma E, \ x \in K, \\ \mathrm{curl}\, \hat{E} = \mathbf{i}\lambda\hat{\mu}\hat{H}, \ \mathrm{curl}\, \hat{H} = -\mathbf{i}\lambda\hat{\gamma}\hat{E}, \ x \in K, \end{cases} \tag{13}$$

with boundary conditions

$$\nu \wedge E = \nu \wedge \hat{E}, \ \nu \wedge H = \nu \wedge \hat{H}, \ x \in \Gamma. \tag{14}$$

Here $\nu(x)$ is the exterior unit normal vector on Γ at $x \in \Gamma$, and $\gamma(x), \hat{\gamma}(x), \mu(x), \hat{\mu}(x)$ are positive smooth functions. We assume that

$$d(x) = \gamma(x)\hat{\mu}(x) - \hat{\gamma}(x)\mu(x) \neq 0 \text{ for } x \in \Gamma.$$

There are many works treating the discreetness of the (ITE) (Haddar (2004), Chesnel (2012), Cakoni and al. (2015), Cakoni and H-M.Nguyen (2020)). In the last work one proves also that $\forall\epsilon > 0$ there are no (ITE) in the region

$$\{\lambda \in \mathbb{C} : |\mathrm{Im}\,\lambda| \geq \epsilon|\mathrm{Re}\,\lambda|, |\mathrm{Re}\,\lambda| \geq C_\epsilon > 0\},$$

assuming $d \neq 0$ and the complementing condition of Agmon, Douglas, Nirenberg $\mu \neq \hat{\mu}$, $\gamma \neq \hat{\gamma}$. We will discuss only the isotropic case.

Theorem 5 (-P. (2020))

Assume $\gamma(x) \neq \hat{\gamma}(x)$, $\mu(x) = \hat{\mu}(x)$, $\partial_\nu \mu(x) = \partial_\nu \hat{\mu}(x)$, $x \in \Gamma$. *Then there exists a constant* $C > 0$ *such that there are no (ITE) in the region*

$$\{\lambda \in \mathbb{C} : |\mathrm{Im}\,\lambda| \geq C(1 + |\mathrm{Re}\,\lambda|)^{3/5}\}.$$

The proof is more complicated than that for the wave equation since we must deal with a (5×5) system for $E_{\mathrm{tan}}|_\Gamma = \hat{E}_{tan}|_\Gamma$, $E_{nor}|_\Gamma$, $\hat{E}_{nor}|_\Gamma$ and the boundary conditions contain $\mathcal{N}(z, h)$ and other pseudodifferential operators.

Weyl asymptotics for the (ITE)

To obtain a Weyl formula introduce the coefficients

$$\tau_j = \frac{\omega_d}{(2\pi)^d} \int_K \left(\frac{n_j(x)}{c_j(x)} \right)^{d/2} dx, \; j = 1, 2,$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. In the anisotropic case $c_1(x) = 1, n_1(x) = 1, c_2(x) \neq 1, c_2(x) n_2(x) \neq 1, \forall x \in \bar{K}$, the asymptotics

$$N(r) = (\tau_1 + \tau_2) r^d + o(r^d), \; r \to +\infty. \tag{15}$$

has been obtained by Lakshatanov and Vainberg (2012) under some additional assumptions which guarantee that the boundary problem is parameter-elliptic. The isotropic case $c_1 = c_2, \partial_\nu c_1 = \partial_\nu c_2, x \in \Gamma$ is more difficult since the corresponding operator A has domain

$$D(A) = \{(u, v) \in L^2(K) \times L^2(K) : \; \Delta u \in L^2(K), \; \Delta v \in L^2(K),$$

$$u - v = 0, \partial_\nu (u - v) = 0 \text{ on } \Gamma\}.$$

and the problem is not parameter elliptic.

In this case M. Fairman (2014) and L. Robbiano (2013) obtained (15) by establishing the asymptotics

$$\sum_j \frac{1}{|\lambda_j|^p + t} = \alpha t^{\frac{d}{2p}-1} + o(t^{\frac{d}{2p}-1}),\ t \to +\infty, \tag{16}$$

where $p \in \mathbb{N}$ is sufficiently large. The formula (15)-(16) have been obtained also in a recent work of H-M. Nguyen and Q-H. Hguyen (2020) for the (ITE) related to the system

$$\begin{cases} \operatorname{div}(A_1(x)\nabla u) - \lambda n_1(x)u = 0, \text{ in } K, \\ \operatorname{div}(A_2(x)\nabla v) - \lambda n_2(x)v = 0, \text{ in } K, \end{cases} \tag{17}$$

where $A_1(x), A_2(x)$ are symmetric positive defined matrices. By using (16), an application of the Tauberian theorem of Hardy-Littlewood yields (15). By this argument one obtains a very week estimate for the remainder. To get better results, it is important to take into account the eigenvalues-free regions and to apply different techniques.

Theorem 6 (-P., Vodev (2017))

Assume $d(x) \neq 0,\ x \in \Gamma$. *Assume either the condition* $c_1(x) = c_2(x),\ \partial_\nu c_1(x) = \partial_\nu c_2(x),\ \forall x \in \Gamma$ *or the condition* $c_1(x) \neq c_2(x),\ \forall x \in \Gamma$. *Then for every* $0 < \epsilon \ll 1$ *we have the asymptotics*

$$N(r) = (\tau_1 + \tau_2) r^d + \mathcal{O}_\epsilon(r^{d-\kappa+\epsilon}),\ r \to +\infty, \tag{18}$$

where $\kappa = 1$ *if* $\frac{n_1(x)}{c_1(x)} \neq \frac{n_2(x)}{c_2(x)},\ \forall x \in \Gamma$. *If the latter condition is not satisfied, the asymptotics* (18) *holds with* $\kappa = 2/5$.

• In fact we prove a more general result with $\kappa = 1 - \delta_+$ assuming that we have a region $\{|\operatorname{Re}\lambda| \ \geq 1,\ |\operatorname{Im}\lambda| \geq C(1 + |\operatorname{Re}\lambda|)^{\delta_+}\},\ 0 \leq \delta_+ < 1$ without eigenvalues and we have a bounded inverse $\|T^{-1}(\lambda)\| \leq C|\lambda|^{-M}$ in this region for the operator $T(\lambda)$ introduced below.

• The optimal result should be to have a remainder $\mathcal{O}(r^{d-1})$ but this is an open problem. This result is known only for the interval $K = \{x \in \mathbb{R} : |x| \leq 1\}$(Silvester, Ha and P. Stefanov (2013)).

• Our proof is inspired by the work of F. Cardoso, G. Popov and G. Vodev (2011) treating the asymptotics of the resonances for exterior transmission problem.

Idea of the proof of Theorem 6

We pass to a semi-classical setting. Set $Z = \{z \in \mathbb{C},\ \frac{1}{2} \leq |\mathrm{Re}\, z| \leq 3,\ |\mathrm{Im}\, z| \leq 1\}$ and consider for $z \in Z$ and $0 < h \ll 1$ the operator

$$hT(z/h^2) := c_1\mathcal{N}_1(z,h) - c_2\mathcal{N}_2(z,h),$$

where the DN-maps $N_j(z,h)$ were defined in the previous section.

Let $G_D^{(j)}$, $j = 1,2$, be the Dirichlet self-adjoint realization of the operator $L_j := -n_j^{-1}\nabla c_j \nabla$ in the space $H_j = L^2(K, n_j(x)dx)$. Set $\mathcal{H} = H_1 \oplus H_2$. Let $R(\lambda)$ be the resolvent of the transmission boundary problem. We omit in the notation $j = 1,2$.

Consider the operators

$$\mathcal{N}(z,h)Op_h(1-\chi)f = \tilde{\mathcal{N}}(z,h)f - \gamma_0 D_\nu (h^2 G_D - z)^{-1}\frac{c}{n}Op_h(p)f,$$

$$F(z,h) = \mathcal{N}(z,h) - \tilde{\mathcal{N}}(z,h) = \mathcal{N}(z,h)Op_h(\chi) - \gamma_0 D_\nu (h^2 G_D - z)^{-1}\frac{c}{n}Op_h(p),$$

where $\chi(x',\xi') = \Phi(\delta_0 r_0(x',\xi'))$ with $\Phi(\sigma) = 1$ for $|\sigma| \leq 1$ and $\Phi(\sigma) = 0$ for $|\sigma| \geq 2$, while $0 < \delta_0 \ll 1$ is small enough. Here $\tilde{\mathcal{N}}(z,h)$ is the parametrix of the DN operator $\mathcal{N}(z,h)Op_h(1-\chi)$ in the domain where $r_0(x',\xi') > \frac{1}{\delta_0}$ and p is some symbol with behavior $\mathcal{O}(h^N)$, $N \gg 1$.

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The operator $F(z,h)$ is meromorphic with values in trace class operators and we denote by $\mu_j(F(z,h))$ its characteristic eigenvalues.

Lemma 1

If z does not belong to spec $h^2 G_D$, then for every integer $0 \le m \le N/4$ we have

$$\mu_j(F(z,h)) \le \frac{C}{\delta(z,h)} \left(hj^{1/(d-1)}\right)^{-2m}, \ \forall j,$$

where $\delta(z,h) := \min\{1, \mathrm{dist}\ \{z, \mathrm{spec}\ h^2 G_D\}\} > 0$ and $C > 0$ depends on m and N but is independent of z, h, j.

Let

$$T(\lambda) := c_1 \gamma_0 D_\nu K_1(\lambda) - c_2 \gamma_0 D_\nu K_2(\lambda),$$

where $K_j(\lambda) f = u$, and u is the solution of the problem

$$\begin{cases} \left(L_j - \lambda\right) u = 0 \text{ in } K, \\ u = f \text{ on } \Gamma. \end{cases}.$$

Theorem 7

Assume that $T(\lambda)^{-1}$ is a meromorphic function with residue of finite rank. Let $\delta \subset \mathbb{C}$ be a simple closed positively oriented curve which avoids the eigenvalues of $G_D^{(j)}$, $j = 1, 2$, as well as the poles of $T(\lambda)^{-1}$. Then we have the identity

$$\mathrm{tr}_{\mathcal{H}}\,(2\pi i)^{-1}\int_{\delta} R(\lambda)d\lambda = \sum_{j=1}^{2}\mathrm{tr}_{H_j}\,(2\pi i)^{-1}\int_{\delta}(G_D^{(j)}-\lambda)^{-1}d\lambda$$
$$-\mathrm{tr}_{L^2(\Gamma)}\,(2\pi i)^{-1}\int_{\delta} T(\lambda)^{-1}\frac{dT(\lambda)}{d\lambda}d\lambda. \qquad (19)$$

Let us mention that if $R(\lambda)$ is an operator-valued meromorphic function with residue of finite rank, the multiplicity of a pole $\lambda_k \in \mathbb{C}$ of $R(\lambda)$ is defined by

$$\mathrm{mult}\,(\lambda_k) = \mathrm{rank}\,(2\pi i)^{-1}\int_{|\lambda-\lambda_k|=\epsilon} R(\lambda)d\lambda,\ 0 < \epsilon \ll 1.$$

On the left hand side of (20), the rank is equal to the trace and we obtain the sum of the mutiplicities of the (ITE) lying in the domain $\omega_\delta \subset \mathbb{C}$ bounded by δ. The terms with $(G_D^{(j)} - \lambda)^{-1}$ yield the sum of the multiplicities of eigenvalues of $G_D^{(j)}$ in ω_δ

It is possible to construct invertible, bounded operator $E(z,h) : H_h^s \to H_h^{s+1}$ with bounded inverse $E(z,h)^{-1} : H_h^s \to H_h^{s-1}$, $\forall s \in \mathbb{R}$ so that

$$hT(z/h^2) = E^{-1}(z,h)(I + \mathcal{K}(z,h)),$$

$$(hT(z/h^2))^{-1} = (I + \mathcal{K}(z,h))^{-1} E(z,h)$$

with a trace class operator $\mathcal{K}(z,h)$. Moreover, the operators $E(z,h), E^{-1}(z,h)$ are holomorphic with respect to z in Z, while $\mathcal{K}(z,h)$ is a memoromorphic operator-valued function in this region.

Set $g_h(z) := \det(I + \mathcal{K}(z, h))$ and denote by $M_\delta(h)$ the number of the poles $\{\lambda_k\}$ of $R(\lambda)$ such that $h^2\lambda_k$ are in ω_δ. Similarly, we denote by $M_\delta^{(j)}(h)$ the number of the eigenvalues ν_k of $G_D^{(j)}$ such that $h^2\nu_k \in \omega_\delta$. Then using the analyticity of $E^{-1}(z, h)$ and the well-known formula

$$\mathrm{tr}\,(I + \mathcal{K}(x, h))^{-1} \frac{\partial \mathcal{K}(z, h)}{\partial z} = \frac{\partial}{\partial z} \log\det(I + \mathcal{K}(z, h)),$$

we get from (19) the following

Lemma 2

Let $\delta \subset Z$ be closed positively oriented curve which avoid the eigenvalues of $h^2 G_D^{(j)}, j = 1, 2$ as well as the poles of $T(z/h^2)^{-1}$. Then we have

$$M_\delta(h) = M_\delta^{(1)}(h) + M_\delta^{(2)}(h) + \frac{1}{2\pi i} \int_\delta \frac{d}{dz} \log g_h(z) dz. \quad (20)$$

Observe that $z_0 \in Z \setminus \mathrm{spec}\,(h^2 G_D^{(1)}) \cup \mathrm{spec}\,(h^2 G_D^{(2)})$ is a zero of $g_h(z)$ if and only if z_0 is a pole of $R(z/h^2)$ and hence z_0/h^2 is an (ITE).

Lemma 3

Let $0 < \kappa \leq 1$ be as in Theorem 6. Then, given any $0 < \epsilon \ll 1$, the operator $I + \mathcal{K}(z, h)$ is invertible on $L^2(\Gamma)$ for $z \in Z$, $|\operatorname{Im} z| \geq h^{\kappa - \epsilon}$ and the inverse operator satisfies in this region the estimate

$$\left\| (I + \mathcal{K}(z, h))^{-1} \right\|_{L^2 \to L^2} \leq C_\epsilon h^{-\ell}$$

with constants $C > 0$, $\ell > 0$. For these values of z we have

$$\log \frac{1}{|g_h(z)|} \leq C_\epsilon h^{1-d-\epsilon}, \; 0 < \epsilon \ll 1. \tag{21}$$

Moreover, for these z the function $g_h(z)$ is holomorphic and we have

$$\left| \frac{d}{dz} \log g_h(z) \right| \leq \frac{C_\epsilon h^{1-d-2\epsilon}}{|\operatorname{Im} z|} \tag{22}$$

for $z \in W := \{z \in \mathbb{C} : \; 2/3 \leq |\operatorname{Re} z| \leq 5/2, \; 2h^{\kappa-\epsilon} \leq |\operatorname{Im} z| \leq 1/2\}$.

Proposition 2

For every $0 < \epsilon \ll 1$ *and* $A > 0$, *independent of* h, *we have the asymptotics*

$$I(h) := \sharp \left\{ z_k,\ z_k/h^2 \text{ is (ITE)}:\ 1 - Ah^{\kappa-\epsilon} \le |\mathrm{Re}\, z_k| \le 2 + Ah^{\kappa-\epsilon},\ |\mathrm{Im}\, z_k| \le h^{\kappa-\epsilon} \right\}$$

$$= (2^{d/2} - 1)(\tau_1 + \tau_2)h^{-d} + \mathcal{O}_{\epsilon,A}(h^{-d+\kappa-3\epsilon}). \tag{23}$$

We will discuss only the case of (ITE) with $\mathrm{Re}\, z_k > 0$, since the case $\mathrm{Re}\, z_k < 0$ is similar (and even simpler since the function $g_h(z)$ does not have poles in $\mathrm{Re}\, z < 0$). Consider the points

$$w_1^{\pm} = 1 - Ah^{\kappa-\epsilon} \pm \frac{\mathbf{i}}{3},\ w_2^{\pm} = 2 + Ah^{\kappa-\epsilon} \pm \frac{\mathbf{i}}{3},$$

$$\widetilde{w}_1^{\pm} = 1 - Ah^{\kappa-\epsilon} \pm \mathbf{i}3h^{\kappa-\epsilon},\ \widetilde{w}_2^{\pm} = 2 + Ah^{\kappa-\epsilon} \pm \mathbf{i}3h^{\kappa-\epsilon}$$

and set

$$\Theta_1 = \left\{ z \in \mathbb{C} : 1 - 2(A+1)h^{\kappa-\epsilon} \le \mathrm{Re}\, z \le 1 + h^{\kappa-\epsilon},\ |\mathrm{Im}\, z| \le 4h^{\kappa-\epsilon} \right\},$$

$$\Theta_2 = \left\{ z \in \mathbb{C} : 2 - h^{\kappa-\epsilon} \le \mathrm{Re}\, z \le 2 + 2(A+1)h^{\kappa-\epsilon},\ |\mathrm{Im}\, z| \le 4h^{\kappa-\epsilon} \right\}.$$

Lemma 4

There exist positively oriented piecewise smooth curves $\widetilde{\gamma}_1 \subset \Theta_1$ *and* $\widetilde{\gamma}_2 \subset \Theta_2$*, where* $\widetilde{\gamma}_1$ *connects the point* $\widetilde{w}_1^-$ *with* $\widetilde{w}_1^+$*, while* $\widetilde{\gamma}_2$ *connects the point* $\widetilde{w}_2^+$ *with* $\widetilde{w}_2^-$*, such that*

$$\left| \mathrm{Im} \int_{\widetilde{\gamma}_j} \frac{d}{dz} \log g_h(z) dz \right| \leq C_\epsilon h^{-d+\kappa-2\epsilon}, \quad j = 1, 2. \tag{24}$$

Now we apply Lemma 2 with a contour $\delta = \gamma_1 \cup \gamma_3 \cup \gamma_2 \cup \gamma_4$, where $\gamma_3 \subset W$ is the segment $[w_1^+, w_2^+]$ on the line passing through the points w_1^+ and w_2^+, and $\gamma_4 \subset W$ is the segment $[w_2^-, w_1^-]$ on the line passing through the points w_2^- and w_1^-. Next, $\gamma_1 = [w_1^-, \widetilde{w}_1^-] \cup \widetilde{\gamma}_1 \cup [\widetilde{w}_1^+, w_1^+]$, $\gamma_2 = [w_2^+, \widetilde{w}_2^+] \cup \widetilde{\gamma}_2 \cup [\widetilde{w}_2^-, w_2^-]$ (see Figure 4). Since $\gamma_j \subset W$, $|\gamma_j| = \mathcal{O}(1)$, $j = 3, 4$, by (22) we have

$$\left| \int_{\gamma_j} \frac{d}{dz} \log g_h(z) dz \right| \leq \int_{\gamma_j} \left| \frac{d}{dz} \log g_h(z) \right| |dz|$$

$$\leq C_\epsilon h^{-d+1-2\epsilon} \int_{\gamma_j} |dz| \leq C_\epsilon h^{-d+1-2\epsilon}, \quad j = 3, 4.$$

Applying (22) once more for $j = 1, 2$, we have

$$\left| \int_{[w_j^{\pm}, \tilde{w}_j^{\pm}]} \frac{d}{dz} \log g_h(z) dz \right| \leq C_\epsilon h^{-d+1-2\epsilon} \int_{3h^{\kappa-\epsilon}}^{1/3} \frac{d\sigma}{\sigma} \leq C_\epsilon h^{-d+1-3\epsilon}.$$

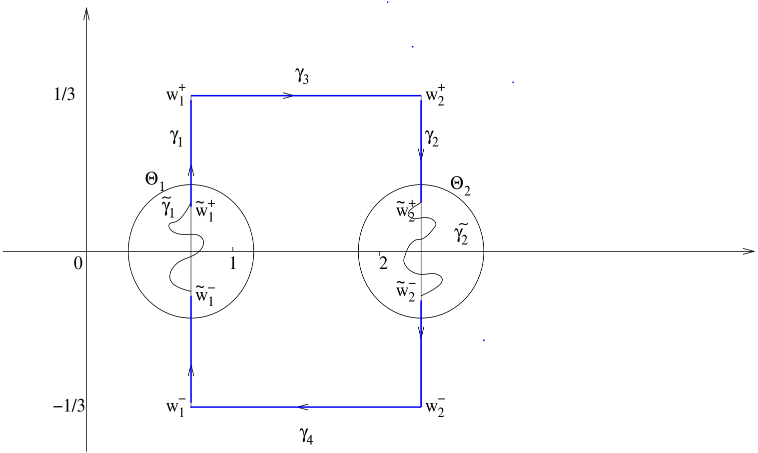

Figure 4: Contour $\delta = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \delta_4$

On the other hand, since the counting function of the eigenvalues of $G_D^{(j)}$ satisfies the Weyl law, we deduce

$$M_{\gamma_0}^{(j)}(h) \leq \sharp \left\{ \nu_k \in \operatorname{spec} G_D^{(j)} : 1 - 2Ah^{\kappa-\epsilon} \leq h^2 \nu_k \leq 2 + 2Ah^{\kappa-\epsilon} \right\}$$

$$= (2^{d/2} - 1)\tau_j h^{-d} + \mathcal{O}_{\epsilon,A}(h^{-d+\kappa-\epsilon})$$

and similarly

$$M_{\gamma_0}^{(j)}(h) \geq \sharp \left\{ \nu_k \in \operatorname{spec} G_D^{(j)} : 1 + 2Ah^{\kappa-\epsilon} \leq h^2 \nu_k \leq 2 - 2Ah^{\kappa-\epsilon} \right\}$$

$$= (2^{d/2} - 1)\tau_j h^{-d} - \mathcal{O}_{\epsilon,A}(h^{-d+\kappa-\epsilon}),$$

hence

$$M_{\gamma_0}^{(j)}(h) = (2^{d/2} - 1)\tau_j h^{-d} + \mathcal{O}_{\epsilon,A}(h^{-d+\kappa-\epsilon}),\ j = 1, 2. \tag{25}$$

Taking together the above estimates and applying Lemma 4, we obtain

$$M_{\gamma_0}(h) = (2^{d/2} - 1)(\tau_1 + \tau_2)h^{-d} + \mathcal{O}_{\epsilon,A}(h^{-d+\kappa-3\epsilon}). \tag{26}$$

This implies easily the statement of Theorem 6.

Illustration of the idea of the proof of Lemma 4

We treat the case $j = 1$. Let $B_1^{\pm}(\theta) = \{z \in \mathbb{C} : |z - \tilde{w}_1^{\pm}| \leq \theta h^{\kappa - \epsilon}\}$, $\Theta_1 \subset B_1^{+}(\theta) \cup B_1^{-}(\theta), 0 < \theta \leq \theta_0$. Denote by $\mathcal{M}_1$ and by $\mathcal{M}_2$ the set of the zeros and the poles of $g_h(z)$ for $z \in B_1^{+}(2\theta_0) \cup B_1^{-}(2\theta_0)$, respectively. Introduce the holomorphic function $\zeta_h(z) := g_h(z) \prod_{w \in \mathcal{M}_1}(z - w)^{-1} \prod_{w \in \mathcal{M}_2}(z - w)$, where the zeros and the poles are repeated with their multiplicities. Write

$$\frac{d}{dz} \log g_h(z) = \frac{d}{dz} \log \zeta_h(z) + \sum_{w \in \mathcal{M}_1} (z - w)^{-1} - \sum_{w \in \mathcal{M}_2} (z - w)^{-1}.$$

We prove an estimate $\log |\zeta_h(z)| \leq C_\epsilon h^{-d+1-2\epsilon}$, $\forall z \in B_1^{+}(2\theta_0) \cup B_1^{-}(2\theta_0)$. For the functions $f_{\pm}(x) = \log \frac{\zeta_h(z)}{\zeta_h(\tilde{w}_1^{\pm})}$ we have $\mathrm{Re}\ f_{\pm}(z) = \log |\zeta_h(z)| - \log |\zeta_h(\tilde{w}_1^{\pm})|$. Since $f_{\pm}(\tilde{w}_1^{\pm}) = 0$, we can apply Caratheodory theorem to obtain a bound of $|f_{\pm}(z)|$ in a smaller disks $B_1^{\pm}(\frac{3}{2}\theta_0)$.

This implies $\left|\frac{d}{dz}\log\zeta_h(z)\right| \leq C_\epsilon h^{-d-\kappa+1-\epsilon}$ in $B_1^\pm(\theta_0)$ and

$$\left|\int_{\tilde{\gamma}_1} \frac{d}{dz}\log\zeta_h(z)dz\right| = \left|\int_{\tilde{w}_1^-}^{\tilde{w}_1^+} \frac{d}{dz}\log\zeta_h(z)dz\right| \leq C_\epsilon h^{-d+1-2\epsilon},$$

since $[\tilde{w}_1^-, \tilde{w}_1^+]$ has length $6h^{\kappa-\epsilon}$. Let $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$. By using the Jensen formula for the zeros of $F_h(z) = g_h(z)\prod_{w\in\mathcal{M}_2}(z-w)$, we estimate

$$\#\{w : w \in \mathcal{M}_1\} \leq C_\epsilon h^{-d+\kappa-2\epsilon}.$$

A similar estimates holds for $\#\{w : w \in \mathcal{M}_2\}$. Next for $M \gg d$ introduce

$$U = \bigcup_{w\in\mathcal{M}} \{z \in \mathbb{C} : \ |z-w| \leq h^M\} = \bigcup_{\nu} U_\nu, \ U_\nu \cap U_\mu = \emptyset, \ \nu \neq \mu.$$

If $U_\nu \cap \partial B_1^\pm(2\theta_0) \neq \emptyset$, we modify the arc of $U_\nu \cap \partial B_1^\pm(2\theta_0)$ by arc of ∂U_ν, so $\mathrm{d}ist\,(w, \partial U_\nu) \geq h^M, \forall w \in \mathcal{M}$.

For $w \notin [\alpha, \beta]$, by using suitable change of variables with $w = \pm\sigma_0, \sigma_0 > 0$, we get

$$\left| \operatorname{Im} \int_\alpha^\beta (z - w)^{-1} dz \right| = \int_\alpha^\beta \frac{\sigma_0 dt}{\sigma_0^2 + t^2} \leq \int_{-\infty}^{\infty} \frac{dt}{1 + t^2} = \pi.$$

On the other hand, $\int_\omega (z - w)^{-1} dz$ and $\int_\alpha^\beta (z - w)^{-1} dz$ are different by $2k\pi i, k = 0, 1$ and we obtain $\left| \operatorname{Im} \int_\omega (z - w)^{-1} dz \right| \leq 3\pi, \forall w \in \mathcal{M}$.

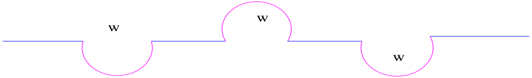

Figure 5: arcs of ∂U_ν

Passing to a limit, we obtain the same for $w \in [\alpha, \beta]$. By using the estimate for the number of point $w \in \mathcal{M}$, we deduce Lemma 4.

Weyl asymptotics for dissipative eigenvalues close to $\mathbb{R}^-$

Consider the Maxwell system in $K = B^3 = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ with dissipative boundary conditions on $|x| = 1$. For $\gamma \equiv 1$ on Γ there are no eigenvalues.

Proposition 3 (Colombini, -P., Rauch (2016))

Assume that $\gamma \in \mathbb{R}^+ \setminus \{1\}$ is a constant and let $\gamma_0 = \max\{\gamma, \frac{1}{\gamma}\}$. Then G has an infinite number of real eigenvalues. All real eigenvalues λ satisfy the estimate

$$\lambda \leq -\frac{1}{\max\{(\gamma_0 - 1), \sqrt{\gamma_0 - 1}\}} = -c_0. \tag{27}$$

Theorem 8 (-P., Colombini (2018))

Assume the conditions of Prop. 3. Then the counting function $N(r) = \#\{\lambda_j \in R^- : |\lambda_j| \leq r\}$ for the ball B^3 has the asymptotic

$$N(r) = (\gamma_0^2 - 1)r^2 + \mathcal{O}_\gamma(r), \; r \geq r(\gamma) > c_0$$

For the wave equation and strictly convex obstacles in the case $\gamma(x) > 1, \forall x \in \Gamma$, by Th. 3 we know that for $\forall N \in \mathbb{N}$ all eigenvalues of the generator G of the semi-group $V(t) = e^{tG}$ are in a small neighbourhood of the negative real axis

$$\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leq C_N(1 + |\operatorname{Re}\lambda|)^{-N}, \operatorname{Re}\lambda < -R\} \cup \{|\lambda| \leq R, \operatorname{Re}\lambda < 0\}.$$

For $\gamma(x) \equiv \gamma_0 > 1$ in a work in progress -P. proved in this case that the counting function $N(r)$ of the eigenvalues has the asymptotics

$$N(r) = \frac{\omega_{d-1} r^{d-1}}{(2\pi)^{d-1}} \int_\Gamma (\gamma_0^2 - 1)^{\frac{d-1}{2}} dx + \mathcal{O}_\gamma(r^{d-2}),\ r \geq c(\gamma_0) > 0,$$

ω_{d-1} being the volume of $\{x \in \mathbb{R}^{d-1} : |x| \leq 1\}$. We expect that this result is true when $\gamma(x) > 1$ is not constant. The proof is based on a trace formula

$$\operatorname{tr}_{\mathcal{H}} \frac{1}{2\pi \mathbf{i}} \int_\delta (\lambda - G)^{-1} d\lambda = \operatorname{tr}_{L^2(\Gamma)} \frac{1}{2\pi \mathbf{i}} \int_\delta C^{-1}(\lambda) \frac{\partial C}{\partial \lambda}(\lambda) d\lambda,\ \delta \subset \{\operatorname{Re}\lambda < 0\},$$

where $C(\lambda) = \mathcal{N}(\lambda) - \lambda\gamma = \mathcal{N}(\lambda)\Big(Id - \lambda \mathcal{N}^{-1}(\lambda)\gamma\Big)$, $\operatorname{Re}\lambda < 0$ and $\mathcal{N}(\lambda)f$ is the (exterior) Dirichlet-to-Neumann map related to $(\Delta - \lambda^2)u = 0,\ u|_\Gamma = f$.