The fixed angle inverse scattering problem

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University of Delaware

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Overview

Two results.

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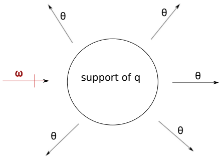
- 

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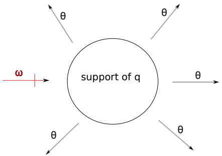
- 

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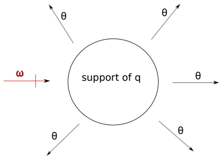
- 

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- Back-scattering problem: Invert

$$\mathcal{B} : q \to \alpha(-\omega, k, \omega)|_{|\omega|=1, k\in\mathbb{R}}.$$

More difficult; cannot show $\mathcal{B}$ distinguishes $q = 0$ from $q \neq 0$.

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- Greenleaf-Uhlmann '93, Ruiz '01 - recovery of 'principal singularities' of q.

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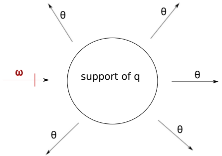
- Merono, P-Machado, Salo '20: Extended results to $\partial_t^2 - (\nabla - A)^2 + q$; about recovering curl A, q.

Idea - convert to equivalent time domain problem



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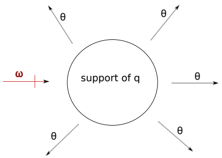
- 

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$$(\Box + q)U(x,t) = 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R},$$
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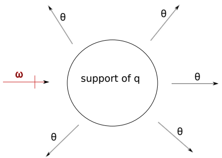
- 

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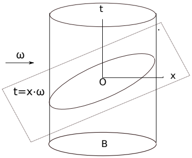

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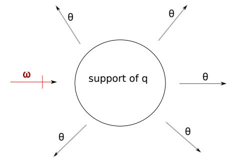
- 

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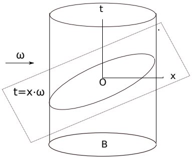

- $\alpha(\theta, k, \omega)|_{|\theta|=1, k\in\mathbb{R}}$ (fixed ω) equivalent to $u(x,t,\omega)|_{\partial B\times\mathbb{R}}$.
What is time domain equivalent for fixed θ, ω? Conjectures.

Result for equivalent time domain problem

- $$(\Box + q)u_q(x,t,\omega) = 0, \quad t \geq x\cdot\omega,$$
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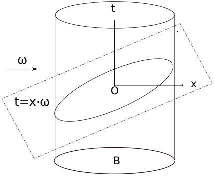

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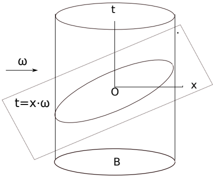

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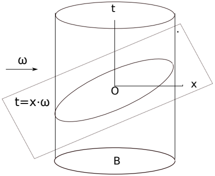

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$$\|p\|_{L^2(B)} \preccurlyeq \|u_{q+p} - u_q\|_{H^1(\partial B\times[-1,T])} + \|u_{q+p} - u_q\|_{H^1(E)}$$

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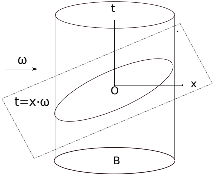

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- Uses Bukhgeim-Klibanov method as modified by Yamamoto, Puel, Imanuvilov, Isakov, Bellasoued. More later.

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- $q(x,t)$ smooth function on $B \times [0,T]$;
- Let $u(x,t)$ be solution of IBVP

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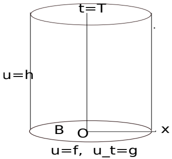

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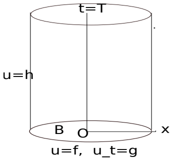

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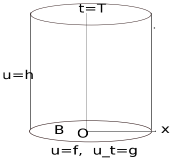

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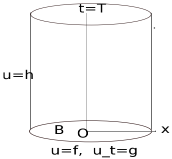

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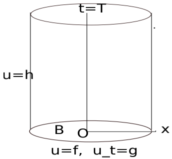

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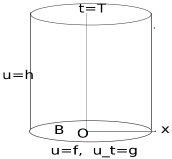

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- Overdetermined if $n > 1$. Results obtained using products of geometric optics solutions. Lassas et al '20 - semilinear - undetermined problem.

Our problem and result (with Krishnan & Senapati)

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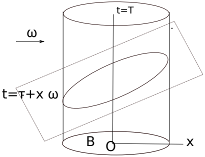

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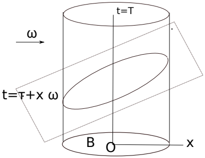

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  Goal : Invert $q \to (u|_L, u|_H, u_t|_H)_{\tau\in\mathbb{R}}$
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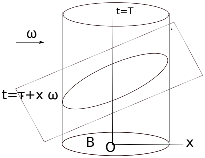

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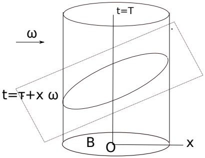

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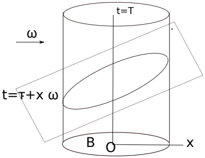

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- Used adapted version of Bukhgeim-Klibanov method.

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- Bukhgeim and Klibanov (1981): Carleman estimates and inverse problems. Isakov; Imanuvilov and Yamamoto (2001): adapted [B-K 1981]. We adapt [I-Y 2001].

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- D-N type problems: work with products $p(x)v(x,t)w(x,t)$ - use many solutions v, w. Of course - no $p(x)$ in initial data.

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- (B-K) Use weighted (Carleman) estimate - weight $e^{\sigma\phi(x,t)}$, $\phi(x,t)$ strictly decreasing in $|t|$.
- Since $\phi(x,0) > \phi(x,t)$ for $t > 0$, weighted $\|p(x)\|_B$ on LHS dominates weighted $\|p(x)v(x,t)\|_{B\times[0,T]}$ on RHS.