Determination of black holes by boundary measurements

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Inverse boundary value problems

Let $(x_0, x_1, x_2, ..., x_n) \in \mathbb{R} \times \mathbb{R}^n$, $x_0 \in \mathbb{R}$ be the time variable, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. Consider the Lorentzian metric in $\mathbb{R} \times \mathbb{R}^n$

$$\sum_{j,k=0}^{n} g_{jk}(x) dx_j dx_k \tag{1}$$

with the signature $(+1, -1, ..., -1)$. Let $g(x) = \det[g_{jk}(x)]_{j,k=0}^n$ and let $[g^{jk}(x)]_{j,k=0}^n$ be the inverse to the metric tensor $[g_{jk}(x)]_{j,k=0}^n$. We assume that the metric does not depend on $x_0 \in \mathbb{R}$.
Let

$$Lu = \sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n g(x)}} \frac{\partial}{\partial x_j}\Big(\sqrt{(-1)^n g(x)} g^{jk}(x) \frac{\partial}{\partial x_k} u(x)\Big) = 0 \tag{2}$$

be the wave equation corresponding to the metric (1).

Define the Dirichlet-to-Neumann operator Λf as

$$\Lambda f = \sum_{j,k=0}^{n} g^{jk}(x) \frac{\partial u}{\partial x_j} \nu_k(x) \Big(\sum_{p,r=0}^{n} g^{pr}(x) \nu_p(x) \nu_r(x) \Big)^{-\frac{1}{2}} \Big|_{\mathbb{R}\times\partial\Omega}, \tag{3}$$

$u(x_0, x)$ is the solution of (2), $\nu_0 = 0, (\nu_1, ..., \nu_n)$ is the unit outward normal to $\partial\Omega$.

Let Γ_0 be a subdomain of $\partial\Omega$. The inverse boundary value problem on $\mathbb{R} \times \Gamma_0$ consists of determining the metric (1) knowing the Λf on $\mathbb{R} \times \Gamma_0$ for all f with the compact support in $\mathbb{R} \times \overline{\Gamma}_0$.

A powerful boundary control method for solving hyperbolic inverse problems for equation

$$\frac{\partial^2 u}{\partial x_0^2} + \Delta_h u = 0,$$

where Δ_h is the Laplace-Beltrani operator, was discovered by M.Belishev, and further developed by Belyshev and Kurylev, Kurylev and Lassas, Katchalov, Kurylev and Lassas, and others. In the author's works a localized variant of the boundary control method was developed. The method was extended in [E, 2008] to include inverse boundary value problems for the equation (2) with Lorentz metric. Also it was extended to the case of the inverse problems for the equation of the form (1) with time-dependent coefficients ([E, 2017]) under two extra conditions. The Bardos-Lebeau-Rauch condition and the condition of the analyticity of the metric in the time variable.
The method of [E, 2008] allows to recover the metric in a neighborhood of any point of Ω where the spatial part of the wave operator is elliptic. These results are the basis of the present paper.

Let $y = \varphi(x)$ be a diffeomorphism of $\overline{\Omega}$ on some bounded smooth domain $\overline{\Omega}_0 \subset \mathbb{R}^n$ and let $a(x) \in C^\infty(\Omega),\ a(x) = 0$ on $\overline{\Gamma}_0$. Consider the map

$$(y_0, y) = \Phi(x_0, x) = (x_0 + a(x), \varphi(x)) \tag{4}$$

of $\overline{\Omega} \times \mathbb{R}$ onto $\overline{\Omega}_0 \times \mathbb{R}$ such that $\varphi(x) = x$ and $a(x) = 0$ on Γ_0. Note that change of variables $y = \varphi(x), y_0 = x_0 + a(x)$ does not change the DN operator Λ.

Ergoregions and black holes

The domain $\Delta \subset \Omega$ is called the ergoregion if

$$g_{00}(x) \leq 0 \quad \text{on} \quad \Delta. \tag{5}$$

We assume that $g_{00}(x) > 0$ in the exterior of Δ. Let

$$\Delta(x) = \det[g^{jk}(x)]_{j,k=1}^{n}. \tag{6}$$

Then $g_{00}(x) = g^{-1}(x)\Delta(x)$. Thus (5) is equivalent to the inequality

$$\Delta(x) \leq 0. \tag{7}$$

We assume that $\Delta(x) = 0$ is a smooth surface in $\mathbb{R}^n$, $\frac{\partial \Delta(x)}{\partial x} = \left(\frac{\partial \Delta(x)}{\partial x_1}, ..., \frac{\partial \Delta(x)}{\partial x_n}\right) \neq 0$ when $\Delta(x) = 0$.

Now we shall define the black hole.
Let $S(x) = 0$ be a closed surface in $\mathbb{R}^n$ and Ω_{int} be the interior of the surface $S(x) = 0$. We call the region Ω_{int} a black hole if no signal (disturbance) inside $S(x) = 0$ can reach the exterior of $S(x) = 0$. Let $S(x) = 0$ be a characteristic surface for the equation (2), i.e.

$$\sum_{j,k=0}^{n} g^{jk}(x) S_{x_j}(x) S_{x_k}(x) = 0 \quad \text{when} \quad S(x) = 0. \tag{8}$$

It was proven that $S(x) = 0$ in a boundary of a black hole if $S(x) = 0$ is a characteristic surface and

$$\sum_{j=1}^{n} g^{j0}(x) S_{x_j}(x) < 0 \quad \text{when} \quad S(x) = 0. \tag{9}$$

The boundary $S(x) = 0$ of the black hole is called the black hole event horizon.

Analogue metrics

When the metric (1) is not a solution of the Einstein equation, i.e. the metric (1) is not related to the general theory of relativity, the black hole Ω_{int} is called an analogue black hole. In physical applications the analogue black holes appear when one studies the propagation of waves in a moving medium.
An example of the analogue metric is the following acoustic metric : Consider a fluid flow in a vortex with the velocity field

$$v = (v^1, v^2) = \frac{A}{r}\hat{r} + \frac{B}{r}\hat{\theta}, \tag{10}$$

where $r = |x|$, $\hat{r} = \left(\frac{x_1}{|x|}, \frac{x_2}{|x|}\right)$, $\hat{\theta} = \left(-\frac{x_2}{|x|}, \frac{x_1}{|x|}\right)$, A and B are constants, $A < 0$. When $B \neq 0$ (10) is a rotating flow.
The inverse metric tensor $[g^{jk}]_{j,k=1}^2$ has the form

$$g^{00} = \frac{1}{\rho c},\ g^{0j} = g^{j0} = \frac{1}{\rho c} v^j,\ 1 \leq j \leq 2, \tag{11}$$

$$g^{jk} = \frac{1}{\rho c}(-c^2\delta_{ij} + v^J v^k),\ 1 \leq j, k \leq 2.$$

Here c is the sound speed, ρ is the density. We shall assume, for the simplicity, that $\rho = 1,\ c = 1$. It was shown that $\{r \le \sqrt{A^2 + B^2}\}$ is the ergoregion and $\{r < |A|\}$ is the black hole.

We consider also another example of analogue metric, the Gordon metric, that arise when one studies the propagation of light in a moving dielectric medium:

Let $w = (w_1(x), w_2(x), w_3(x))$ be the velocity of the flow. The Gordon metric has the form

$$\sum_{j,k=0}^{3} g_{jk}(x) dx_j dx_k, \tag{12}$$

where $g_{jk}(x) = \eta_{jk} + \big(n^{-2}(x) - 1\big) v_j v_k,\ \ n(x)$ is the index of the refraction,

$$v_0 = \big(1 - \frac{|w|^2}{c^2}\big)^{-\frac{1}{2}},\ v_j(x) = \big(1 - \frac{|w|^2}{c^2}\big)^{-\frac{1}{2}} \frac{w_j(x)}{c},\ 1 \le j \le 3, \tag{13}$$

η_{jk} is the Lorentz metric.

Recovery of the ergosphere from the boundary measurements

Let Γ' be any small subset of $\partial\Omega$ and $P_0 \in \Gamma'$.
It was proven in [E, 2008], Theorem 3.1, (see also [E, 2017], Theorem 6.2), that knowing boundary data on $[0, +\infty) \times \Gamma'$ one can recover, modulo change of variables (4), the metric (1) in $[0. +\infty) \times V(P_0)$ where $V(P_0)$ is a neighborhood of P_0 in $\overline{\Omega}$. The key condition for the the validity of Theorem 3.1 is that the spatial part of the equation (2) is elliptic in $V(P_0)$. Thus $V(P_0)$ is outside the ergosphere.
Next taking $P_1 \in V(P_0)$ one can recover (1) in $[0, \infty) \times V(P_1)$ where $V(P_1) \subset \overline{\Omega}$ is a neighborhood of P_1. Repeating this argument infinitely many times we can recover the metric outside the ergosphere $\Delta(x) = 0$ when $\Delta(x)$ is the same as in (6). Taking the limit we can recover the metric on $\Delta(x) = 0$ too. Thus we have the following theorem:

Theorem

Knowing the DN operator (3) on $\mathbb{R} \times \Gamma_0$ we can determinate the ergosphere $\Delta(x) = 0$, and the metric (1) on $\Delta(x) = 0$.

There are three cases:
a) $e(x)$ is orthogonal to the surface $\Delta(x) = 0$ for all x, i.e. $\Delta(x) = 0$ is characteristic at any $x \in \Delta$.
b) $e(x)$ is not orthogonal to the surface $\Delta(x) = 0$ for any x, i.e. is not characteristic for any $x \in \Delta$.
c) $e(x)$ is orthogonal to $\Delta(x) = 0$ only on some nonempty subset of $\Delta(x) = 0$.
In the case a) we have that $e(x)$ is collinear to the gradient $\frac{\partial \Delta(x)}{\partial x}$, $\Delta(x) = 0$, for all x. Therefore

$$\sum_{k=1}^{n} g^{jk}(x)\Delta_{x_k}(x) = 0, \ 1 \leq j \leq n, \ \Delta(x) = 0. \tag{14}$$

Multiplying (14) by Δ_{x_j} and summing in j we get

$$\sum_{j,k=1}^{n} g^{jk}(x)\Delta_{x_j}\Delta_{x_k} = 0 \ \text{ when } \quad \Delta(x) = 0. \tag{15}$$

Then $\Delta(x) = 0$ is the boundary of a black hole if

$$\sum_{j=1}^{n} g^{0j}(x)\frac{\partial \Delta}{\partial x_j} < 0 \quad \text{when} \quad \Delta(x) = 0. \tag{16}$$

Zero energy null-geodesics in the case of two space dimension

The case of not Schwartzschield type metrics is more difficult. We shall study only the case of two dimensions. The underlying idea in analyzing the black holes in two space dimensions is the following: Consider the Hamiltonian

$$H(x_1, x_2, \xi_0, \xi_1, \xi_2) = \sum_{j,k=0}^{2} g^{jk}(x)\xi_j\xi_k. \tag{17}$$

Let

$$\frac{dx_k}{ds} = \frac{\partial H(x,\xi)}{\partial \xi_k}, \quad \frac{d\xi_k}{ds} = -\frac{\partial H}{\partial x_k}, \quad 0 \le k \le 3, \tag{18}$$
$$x_k(0) = y_k, \quad \xi_k(0) = \eta_k$$

be the equation of null-biocharacteristics. Thus

$$H\big(x_1(s), x_2(s), \xi_0(s), \xi_1(s), \xi_2(s)\big) = 0 \ \text{ for all } \ s. \tag{19}$$

Since H is independent of x_0 we have that $\frac{d\xi_0}{ds} = 0$, i.e. $\xi_0(s) = \eta_0$ is a constant. We choose $\xi_0 = 0$, and we shall call the null-bicharacteristics with $\xi_0 = 0$ the zero energy null-bicharacteristics. The projection of zero energy null-bicharacterstcs on the (x_1, x_2)-space is called the zero-energy null-geodesics. Therefore we have

$$\sum_{j,k=1}^{2} g^{jk}(x(s))\xi_j(s)\xi_k(s) \equiv 0, \quad \forall s. \tag{20}$$

This equation is a quadratic equation in $\xi_j(s)$, $1 \leq j \leq 2$, and therefore we have two families of solutions

$$\xi_j^{\pm}(s) = p_j^{\pm}(x(s)), \quad x = (x_1, x_2), \quad j = 1, 2. \tag{21}$$

If substitute $\xi_j^{\pm}$ in (18) and choose x_0 as a parameter instead of s we obtain two 2×2 system of differential equations in (x_1, x_2):

$$\frac{dx_j^{\pm}}{dx_0} = \frac{g^{j1}(x^{\pm})p_1^{\pm}(x^{\pm}) + g^{j2}(x^{\pm})p_2^{\pm}(x^{\pm})}{g^{01}(x^{\pm})p_1^{\pm}(x^{\pm}) + g^{02}(x^{\pm})p_2^{\pm}(x^{\pm})}, \quad j = 1, 2. \tag{22}$$

Therefore the solution of 4×4 system of null-bicharacteristics (18) is reduced to the solution of two 2×2 systems (22). This reduction substantially simplifies the study of black holes.

Description of the black hole inside the ergosphere in the case of two space dimensions

Let $\Delta(x) = 0, n = 2$, be the ergosphere. Assume that the normal to that ergosphere is not characteristic for any $x \in \{\Delta(x) = 0\}$

$$\sum_{j,k=1}^{2} g^{jk}(x)\nu_j\nu_k \neq 0 \quad \text{for all} \quad x \in \{\Delta(x) = 0\}, \tag{23}$$

where (ν_1, ν_2) is the unit normal to $\Delta = \{\Delta(x) = 0\}$. We assume that the ergosphere $\Delta(x) = 0$ is smooth, i.e. $\frac{\partial\Delta}{\partial x} = \left(\frac{\partial\Delta}{\partial x_1}, \frac{\partial\Delta}{\partial x_2}\right) \neq 0$ when $\Delta(x) = 0$.
Introduce coordinates (ρ, θ) where $\rho = 0$ is the equation of $\Delta(x) = 0, \rho = -\Delta(x)$ near $\rho = 0$. For the convenience we extend $\theta \in [0, 2\pi]$ to $\theta = \mathbb{R}/2\pi\mathbb{Z}$. We have $0 \leq \rho \leq \rho_0(\theta)$ where $\rho = \rho_0(\theta)$ is the black hole event horizon, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Since the set $\{\Delta(x) = \det[g^{jk}(x)]_{j,k=1}^2 < 0\}$ is inside the ergosphere, there are two characteristics $S^{\pm}(x)$ such that

$$\sum_{j,k=1}^{2} g^{jk}(x) S^{\pm}_{x_j} S^{\pm}_{x_k} = 0, \quad \Delta(x) < 0,$$

or, in (ρ, θ) coordinates,

$$\hat{g}^{\rho\rho}\left(\hat{S}^{\pm}_{\rho}\right)^2 + 2\hat{g}^{\rho\theta}\hat{S}^{\pm}_{\rho}\hat{S}^{\pm}_{\theta} + \hat{g}^{\theta\theta}\left(\hat{S}^{\pm}_{\theta}\right)^2 = 0, \tag{24}$$

where $\begin{bmatrix} \hat{g}^{\rho\rho} & \hat{g}^{\rho\theta} \\ \hat{g}^{\rho\theta} & \hat{g}^{\theta\theta} \end{bmatrix}$ is the matrix $\begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix}$ in (ρ, θ) coordinates. We assume that $\hat{S}^{\pm}(\rho, \theta)$ satisfy the following boundary conditions

$$\hat{S}^{\pm}(0, \theta) = \theta \quad \text{for any} \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}. \tag{25}$$

Solving the quadratic equation (24) we get

$$\hat{S}^{\pm}_{\rho}(\rho, \theta) = \frac{-\hat{g}^{\rho\theta} \pm \sqrt{-\tilde{\Delta}}}{\hat{g}^{\rho\rho}} \hat{S}_{\theta}(\rho, \theta), \tag{26}$$

where $\tilde{\Delta} = \hat{g}^{\rho\rho}\hat{g}^{\theta\theta} - (\hat{g}^{\rho\theta})^2$.

Consider the equations for the null-bicharacteristics (null geodesics). When we use the time variable as a parameter we have two families of null-geodesics:
the $(+)$ null-geodesics $(\rho^+(x_0), \theta^+(x_0)),\ \rho^+(0) = 0,\ \theta^+(0) = \theta_0,\ x_0 \geq 0$, and
the $(-)$ null-geodesics $(\rho^-(x_0), \theta^-(x_0)),\ \rho^-(0) = 0,\ \theta^-(0) = \theta_0,\ x_0 \leq 0$.
Note that condition (23) is equivalent that both $(+)$ and $(-)$ null-geodesics are not tangent to $\Delta(x) = 0$.
Assume that the ergoregion $\{\Delta(x) \leq 0\}$ contains a trapped region O_ε, i.e. a region that both $(+)$ and $(-)$ null-geodesics reach when $x_0 \to +\infty$ and stay there. One of the examples of trapped region is a neighborhood of a singularity of the metric similar to the singularity of the acoustic metric at $x = 0$.
When $x_0 \to +\infty$ $(\rho^+(x_0), \theta^+(x_0))$ reaches the trapped region O_ε and remains there for all large x_0.

The null-geodesics $\gamma_- = (\rho^-(x_0), \theta^-(x_0))$, $x_0 < 0$, ends at $((0, \theta_0)$ when $x_0 = 0$ and $(\rho^-(x_0), \theta^-(x_0))$ cannot reach O_ε when $x_0 \to -\infty$. Thus $(\rho^-(x_0), \theta^-(x_0))$ has no limit points in $\{\Delta(x) \leq 0\} \setminus O_\varepsilon$. Therefore the limiting set of the trajectories $\{\rho^-(x), \theta^-(x_0), x_0 < 0\}$ is inside $\Delta \setminus O_\varepsilon$. By the Poincare-Bendixson theorem (cf. [13]) there exists a limit cycle, i.e. closed periodic solution $\gamma_0 = \{(\rho_0^-(x_0), \theta_0^-(x_0))\}$ that is a black hole event horizon. The solution γ_- approaches γ_0 spiraling around γ_0. All other $(-)$ null-geodesics also approach γ_0 spiraling.
Denote by Π the infinite strip

$$\Pi = \{0 \leq \rho < \rho_0(\theta),\ -\infty < \theta < +\infty\}, \tag{27}$$

where $\{\rho = \rho_0(\theta), \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$ is the equation of the black hole event horizon γ_0. Therefore Π is traced by all $(-)$ null-geodesics, and γ_0 is the boundary of Π: $\gamma_0 \subset \partial\Pi$. Thus the following theorem holds:

Theorem

Let (23) holds, i.e. the ergosphere $\Delta(x) = 0$ *is not characteristic for all* $x \in \{\Delta(x) = 0\}$*. Suppose the ergoregion* $\{\Delta(x) \leq 0\}$ *contains a trapped region* O_ε*. Then there exists a black hole* $\overline{\Pi} = \{0 \leq \rho \leq \rho_0(\theta),\ -\infty < \theta < +\infty\}$ *and* $\{\rho = \rho_0(\theta), \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$ *is the black hole event horizon.*

Recovery of the black hole knowing the boundary data on the ergosphere

Consider the characteristic equations (24). Note that

$$\tilde{\Delta}(\rho,\theta) = g^{\rho\rho}g^{\varphi\varphi} - (g^{\rho\varphi})^2 \equiv -C_1\rho, \ \ C_1 > 0. \tag{28}$$

Let $\rho = \rho^{\pm}(x_0), \ \theta = \theta^{\pm}(x), \ x_0 \geq 0$, be the zero energy null-geodesics. Then

$$S^{\pm}(\rho^{\pm}(x_0), \theta^{\pm}(x_0)) \equiv S^{\pm}(\rho(0), \ \theta(0)), \rho(0) = 0, \ \theta(0) = \theta_0. \tag{29}$$

We have that $(\rho^+(x_0), \theta^+(x_0))$ crosses the black hole horizon $\gamma_0 = \{\rho = \rho_0(\theta), \theta \in \mathbb{R})$ at some point $x_0 = x_0^{(0)}$ and remains inside the black hole. The null-geodesics $(\rho^-(x_0), \theta^-(x_0))$ approach the black hole event horizon γ_0 when $x_0 \to -\infty$. Thus γ_0 is the limit set of $(\rho^-(x_0), \theta^-(x_0)), x_0 < 0$.
Periodically extending $\theta \in [0, 2\pi]$ to $\theta \in (-\infty, +\infty)$ we have that the $(-)$ null-geodesics cover the strip $\Pi = \{0 \leq \rho < \rho_0(\theta), -\infty < \theta < +\infty\}$ when $0 \leq \rho(x_0) < \rho_0, \ \theta(x_0) = \theta, \ -\infty < \theta < +\infty$.

Let

$$\sigma = S^{+}(\rho,\theta), \quad \tau = S^{-}(\rho,\theta) \tag{30}$$

when $(\rho,\theta) \in \Pi$, $\Pi = \{0 \le \rho < \rho_0(\theta), \theta \in \mathbb{R}^1\}$, $\rho = \rho_0(\theta)$ is the black hole event horizon.

Note that $Lu = 0$ has the form

$$\frac{\partial^2 u}{\partial\sigma\partial\tau} = 0 \tag{31}$$

in coordinates (σ,τ).

Make a new change of variables

$$\begin{gathered} y_1 = \tfrac{\sigma+\tau}{2} = \tfrac{S^{+}(\rho,\theta)+S^{-}(\rho,\theta)}{2} \\ y_2 = \tfrac{\sigma-\tau}{2}, \quad y_1\Big|_{\rho=0} = \theta, \quad y_2\Big|_{\rho=0} = 0. \end{gathered} \tag{32}$$

It follows from (31) that

$$\frac{\partial^2 u}{\partial y_1^2} - \frac{\partial^2 u}{\partial y_2^2} = 0 \text{ in } \overline{\mathbb{R}}_+^2 = \{(y_1,y_2), y_1 \ge 0, y_2 \in (-\infty,+\infty)\}. \tag{33}$$

We have that $(y_1,y_2) = \Phi(\sigma,\tau)$ is the map of Π onto the half-plane $\overline{\mathbb{R}}_+^2 = \{(y_1,y_2), y_1 \ge 0, y_2 \in (-\infty,+\infty)\}$. Note that Φ is a homeomorphism of Π onto $\overline{\mathbb{R}}_+^2$, Φ is the identity on $\rho = 0$.

The closure $\overline{\Pi} = \{0 \le \rho \le \rho_0(\theta), \theta \in \mathbb{R}^1\}$ has the form $\overline{\Pi} = \Pi \cup \gamma_0$. Thus the black hole event horizon γ_0 belongs to the closure of Π.
Suppose we have another metric g_1 in Ω such that $\Lambda_1 f\big|_{\mathbb{R}\times\Gamma_0} = \Lambda f\big|_{\mathbb{R}\times\Gamma_0}$ for all $f \in \mathbb{R}\times\Gamma_0$ where Λ_1, Λ are two DN operators for g_1 and g, respectively. Then, as in §4, $g = g_1$ modulo of the change of variables (4) outside of the ergosphere.
Therefore, without loss of generality, we can assume the ergosphere $\Delta = 0$ for metrics g, g_1 is the same and the metrics g, g_1 are equal on $\Delta = 0$.
Let $\varphi^{\pm}$ be the solutions of characteristic equation of the form (24) with $[\hat{g}^{\rho\theta}]_{j,k=1}^2$ replaced by $[g_1^{\rho\theta}]_{j,k=1}^2$.
We assume, as in (29), that

$$\varphi^{\pm}(\rho_1^{\pm}(x_0), \theta_1^{\pm}(x_0)) = \varphi^{\pm}(\rho_1(0), \theta_1(0)),\ \rho_1(0) = 0,\ \theta_1(0) = \hat{\theta}_1. \tag{34}$$

Make change of variables as in (30)

$$\sigma' = \varphi^+(\rho, \theta), \quad \tau' = \varphi^-(\rho, \theta), \tag{35}$$

where

$$(\rho', \theta') \in \Pi', \quad \Pi' = \{0 \leq \rho' < \rho_0'(\theta'), \theta' \in \mathbb{R}^1\}, \tag{36}$$

$\rho = \rho_0'(\theta'), \theta' \in \mathbb{R}$ is the black hole event horizon for the metric g_1. Note that $L'u' = 0$ has the form $\frac{\partial^2 u'}{\partial \sigma' \partial \tau'} = 0$ in (σ', τ') coordinates, where L' is the operator of the form (2) with g replaced by g_1.
As in (32), make a change of variables

$$\begin{aligned} y_1' &= \frac{\sigma' + \tau'}{2} = \frac{\varphi^+(\rho, \theta) + \varphi^-(\rho, \theta)}{2}, \\ y_2' &= \frac{\sigma' - \tau'}{2} = \frac{\varphi^+ - \varphi^-}{2}, \\ y_1'\big|_{\rho=0} &= \theta, \quad y_2'\big|_{\rho=0} = 0. \end{aligned} \tag{37}$$

Note that $L'u' = 0$ has the form $\left(\frac{\partial^2}{\partial y_1'^2} - \frac{\partial^2}{\partial y_2'^2}\right)u = 0$, $(y_1', y_2') \in \mathbb{R}_+^2$.

Let Φ_1 be the map (37), Φ_1 is a homeomorphism of Π' onto $\overline{\mathbb{R}}_+^2$. Take $y_1 = y_1', y_2 = y_2'$ and consider the composition $\Phi_0 = \Phi^{-1}\Phi_1$. Note that $\Phi_0 = \Phi^{-1}\Phi_1$ is a homeomorphism of Π' onto Π. Note also that $\gamma_0' = \partial\Pi'$ is the black hole event horizon for the metric g_1. Since the closure $\overline{\Phi}_0$ maps $\overline{\Pi}'$ onto $\overline{\Pi}$ we have that γ_0' is mapped onto γ_0. Thus the event horizon γ_0 can be recovered up to a change of variables. Therefore the following theorem holds:

Theorem

Suppose we have two wave equations $Lu = 0$, $L'u' = 0$ such that corresponding DN operators are equal on $\mathbb{R} \times \Gamma_0$. Suppose the ergosphere $\Delta(x) = 0$ of $Lu = 0$ is not characteristic for any $\Delta(x) = 0$. Let $O_\varepsilon \subset \Delta$ be the trapped region. Then the ergosphere $\Delta'(x) = 0$ of $L'u' = 0$ is also non-characteristic for all $\Delta'(x') = 0$ and has a trapped region O_ε'. Moreover, if $\overline{\Pi}$ is the black hole of $Lu = 0$ and $\overline{\Pi}'$ is the black hole of $L'u' = 0$, then $\overline{\Pi}$ and $\overline{\Pi}'$ are homeomorphic.

Remark 7.1

Since the kernel of $[g^{jk}]_{j,k=1}^2$ is one-dimensional for all x belonging to the ergosphere Δ, there exists a vector $e(x) \in \ker [g^{jk}]_{j,k=1}^2$ smoothly depending on $x \in \Delta$ (cf. §4). Above we considered two cases when $e(x)$ is normal to $\Delta(x) = 0$ for all $x \in \Delta$ (case a)) and when $e(x)$ is not normal to Δ for all $x \in \Delta$ (case b)). The last condition is equivalent to the conditions that the null-geodesics on Δ are not tangent to Δ.
There is also a case c) when $e(x)$ is normal to $\Delta = 0$ only on some subset of $\Delta = 0$. Black holes in the case c) were studied in [11].
In the case c) black holes also exist and the boundary of the black hole consists of segments of "plus" or "minus" zero energy null-geodesics. In some cases the boundary of black hole may have corners when "plus" null-geodesics and "minus" null-geodesics intersect.
We will not consider the inverse problems for the black holes in the case c).

THANK YOU VERY MUCH FOR YOUR ATTENTION!