The inverse spectral problem for centrally symmetric real analytic domains

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Inverse spectral problems

In this talk we are only interested in inverse spectral problems for plane domains.

- Inverse Δ-spectral problem: Determine a smooth bounded plane domain $\Omega \subset \mathbb{R}^2$ from the eigenvalues $\{\lambda_j^2\}_{j\in\mathbb{N}}$ of $\Delta = -\partial_x^2 - \partial_y^2$.

$$\begin{cases} \Delta\psi_j = \lambda_j^2\psi_j \text{ in } \Omega \\ B\psi_j = 0 \end{cases}$$

  The boundary condition is Dirichlet if $B\psi = \psi|_{\partial\Omega}$, Neumann if $B\psi = \partial_n\psi|_{\partial\Omega}$.

- Inverse length spectral problem: Determine a smooth bounded domain Ω from its length spectrum $\text{Lsp}(\Omega)$, meaning the lengths of periodic billiard trajectories.

Types of inverse spectral problems

Roughly speaking, we wish to understand the inverse image of the maps

$$\Omega \mapsto \mathrm{Spec}(\Delta_\Omega^B) \quad \text{or} \quad \Omega \mapsto \mathrm{Lsp}(\Omega),$$

on a class of domains $\mathcal{D}$ (isometry equivalence classes), such as smooth or analytic domains. We will mainly focus on the map Spec.

Inverse spectral problem: Fix $\Omega \in \mathcal{D}$. Determine the structure of

$$\mathrm{Iso}(\Omega) := \mathrm{Spec}^{-1}\left(\mathrm{Spec}(\Delta_\Omega^B)\right).$$

Questions: Is $\mathrm{Iso}(\Omega)$ equal to $\{\Omega\}$? Is it finite? Is it discrete? Is Ω isolated in $\mathrm{Iso}(\Omega)$? Is there a continuous curve in $\mathrm{Iso}(\Omega)$ passing through Ω, or a C^1 curve, or an analytic curve? Is $\mathrm{Iso}(\Omega)$ compact in $\mathcal{D}$ in a certain topology?

Prior results; uniqueness among all smooth domains

Kac's question: *If two plane domains are isospectral, then are they isometric?* Gordon, Webb, and Wolpert answered Kac's question in the negative.

All the known counterexamples to date consist of non-covex polygons.

- Kac 1966: Disks are spectrally unique among all smooth domains. He used the heat trace invariants to prove that the area and perimeter of a domain are determined by its spectrum, so by the isoperimetric inequality, disks are spectrally determined.
- Watanabe 2000: There are certain nearly circular domains that are spectrally unique among all smooth domains.
- H. and Zelditch 2019: Nearly circular ellipses are spectrally unique among all smooth domains.

Prior results continued; local uniqueness

A weaker inverse spectral problem is to find domains that are locally spectrally unique, meaning that that they can be heard among nearby domains in a certain topology.

- Marvizi-Melrose 1982: Constructed a two-parameter family of planar domains that are locally spectrally unique in the C^∞ topology. The two parameter family consists of domains that are defined by elliptic integrals, and that resemble ellipses.
- Kaloshin and Sorrentino 2018: Ellipses are locally *maked length spectrally* unique among all smooth domains. The marked length spectrum means that we know the number of bounces (and the winding number) of each length in the length spectrum.

Prior result; spectral rigidity

The notion of spectral rigidity of a domain Ω in a class $\mathcal{D}$ is even weaker than local spectral uniqueness. It means that any C^1 one-parameter family of isospectral domains containing Ω and staying within $\mathcal{D}$, must be trivial.

- Colin de Verdière 1984: Analytic domains with the symmetries of the ellipse are length spectrally rigid within the class.
- H.- Zelditch 2012: Ellipses are infinitesimally spectrally rigid among smooth domains with the two axial symmetries of an ellipses.
- Popov and Topalov 2019: Ellipses are Laplace spectrally rigid within the class of analytic domains with the two axial symmetries of an ellipse.
- H. - Zelditch 2019: Using a length spectral rigidity theorem of de Simoi, Kaloshin, and Wei 2017, we prove that nearly circular domains with one axial symmetry are spectrally rigid among such domains.

Prior results; analytic domains and polygonal domains

Another setting is when one tries to show that the Laplace spectrum map Spec is one-to-one in a relatively small class of domains $\mathcal{D}$.

$\mathcal{D}$ is either

infinite dimensional, in which case usually a generic property is added to simplify an otherwise difficult problem.

- Zelditch 2009: Generic analytic domains with an axial symmetry are spectrally distinguishable from each other. Also proved a result for generic analytic domains with dihydral symmetries.

or it is finite dimensional where no genericity assumption is imposed.

- Durso 1988: The shape of a triangle can be heard among other triangles. Later, Grieser-Maronna (2013) gave a much simpler proof.
- H., Lu, and Rowlett 2020: Trapezoidal domains are spectrally unique among themselves.

Centrally symmetric analytic domains

A plane domain $\Omega \subset \mathbb{R}^2$ is called 'centrally symmetric' if it is invariant under the isometric involution $\sigma(x, y) = (-x, -y)$.

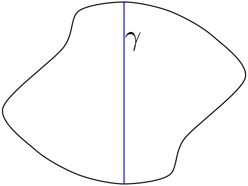


Figure: centrally symmetric domain.

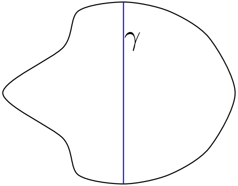


Figure: up-down symmetric domain.

Every simply-connected centrally symmetric bounded plane domain has at least one σ-invariant 'bouncing ball' orbit γ for the billiard flow. We denote $L = L(\gamma)$.

Poincaré map and its eigenvalues

P_γ: the linear Poincaré map of γ.

γ non-degenerate: $\det(I - P_\gamma) \neq 0$.

γ elliptic: eigenvalues of P_γ are of modulus one and of the form $\{e^{i\alpha}, e^{-i\alpha}\}$, $0 < \alpha \leq \pi$.

γ hyperbolic: eigenvalues of P_γ are of the form $\{e^{\alpha}, e^{-\alpha}\}$, $\alpha > 0$.

The class $\mathcal{D}_L$

Locally near the vertices of γ, $\partial\Omega$ consists of two graphs: $\{y = f(x)\} \cup \{y = -f(-x)\}$.

$\mathcal{D}_L$ is the class of simply-connected centrally symmetric real analytic plane domains Ω satisfying:

- There is a non-degenerate bouncing ball orbit γ of length $L_\gamma = 2L$ through the origin.
- The lengths $2L, 4L$ of γ, γ^2, have multiplicity one in the length spectrum $\mathrm{Lsp}(\Omega)$.
- $f^{(3)}(0) \neq 0$.
- In the the elliptic case, the eigenvalues $\{e^{i\alpha}, e^{-i\alpha}\}$ of the linear Poincaré map P_γ satisfy that $\alpha \notin \{0, \frac{\pi}{3}, \pi\}$.

Main theorem

Theorem

For either Dirichlet (or Neumann) boundary conditions B, the map

$$\Omega \longmapsto Spec(\Delta_\Omega^B)$$

is one-to-one on the class $\mathcal{D}_L$.

This result was stimulated by the recent article of Bialy-Mironov.

Bialy-Mironov 2020: A centrally symmetric C^2 convex plane domain which is C^0 foliated in a certain neighborhood of the boundary must be an ellipse. The neighborhood is between an invariant curve of 4-link orbits and the boundary. There is no analyticity assumption.

Poisson relation and wave trace asymptotics

Wave trace:

$$w_\Omega(t) := \operatorname{Tr}\cos\left(t\sqrt{\Delta_\Omega^B}\right) = \sum_{j=1}^{\infty}\cos(t\lambda_j).$$

Poisson relation (Petkov and Stoyanov):

$$\operatorname{SingSupp} w_\Omega(t) \subset \overline{\pm \operatorname{Lsp}(\Omega) \cup \{0\}}$$

Guillemin-Melrose 1979: Let γ be non-degenerate periodic orbit whose length L_γ is simple in $\operatorname{Lsp}(\Omega)$. Let $\hat{\rho} \in C_0^\infty(L_\gamma - \epsilon, L_\gamma + \epsilon)$ with $\hat{\rho} = 1$ on $(L_\gamma - \epsilon/2, L_\gamma + \epsilon/2)$. Then

$$\int_0^\infty \hat{\rho}(t)e^{ikt}w_B^\Omega(t)\,dt \sim F_{B,\gamma}(k)\sum_{j=0}^{\infty} b_{\gamma,j}k^{-j}, \quad k \to \infty$$

$F_{B,\gamma}(k)$: Symplectic pre-factor.

$b_{\gamma,j}$: Wave invariants associated to γ.

The symplectic pre-factor

$$F_{B,\gamma}(k) = C_0 \, (-1)^{\epsilon_B(\gamma)} \frac{L_\gamma^\sharp e^{ikL_\gamma} e^{i\frac{\pi}{4} m_\gamma}}{\sqrt{|\det(I - P_\gamma)|}}.$$

- $\epsilon_B(\gamma)$ is the signed number of intersections of γ with $\partial\Omega$ (the sign depends on the boundary conditions; $+1/-1$ for each bounce for Neumann/Dirichlet boundary conditions).
- m_γ is the Maslov index of γ. The term Maslov index is somewhat ambiguous here, and several authors refer to m_γ as the Gutzwiller Maslov index since it is the exponent arising in the Gutzwiller-Balian-Bloch trace formula.
- $L_\gamma^\sharp$ is the primitive length.
- C_0 is a universal constant.

α and $f''(0)$ as spectral invariants

$$|\det(I - P_\gamma)| = \begin{cases} 2 - 2\cos(\alpha) & \text{(elliptic case)}, \\ 2\cosh(\alpha) - 2 & \text{(hyperbolic case)}. \end{cases}$$

When the domain is up-down or centrally symmetric one has $f''_+(0) = -f''_-(0)$, in which case (see the book of Kozlov-Treshchev):

$$\big(1 + Lf''(0)\big)^2 = \begin{cases} \cos^2(\alpha/2) & \text{(elliptic case)}, \\ \cosh^2(\alpha/2) & \text{(hyperbolic case)}. \end{cases}$$

The roots are given by

$$f''(0) = \begin{cases} \frac{1}{L}(-1 \pm \cos(\alpha/2)) & \text{(elliptic case)}, \\ \frac{1}{L}(-1 \pm \cosh(\alpha/2)) & \text{(hyperbolic case)}. \end{cases}$$

Let $a = -2(1 + Lf''(0))$. Note: $a < 0$ for $+$ and $a > 0$ for $-$.

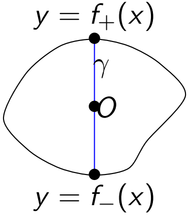
$$\begin{cases} \gamma \text{ is elliptic} & \text{if and only if } |a| < 2 \\ \gamma \text{ is hyperbolic} & \text{if and only if } |a| > 2 \end{cases}$$

Maslov index of iterations of a bouncing ball orbit

The Maslov index m_{γ^r} associated to γ^r is given by

$$m_{\gamma^r} = \ell_r + \operatorname{sgn}(\operatorname{Hess} \mathcal{L}_{+,2r}(0)),$$

where ℓ_r is an integer that depends only on $2r$ and is independent of γ^r.



$$\mathcal{L}_{\pm,2r}(x_1,\ldots,x_{2r}) = \sum_{p=1}^{2r} \left\| \left(x_p, f_{w_\pm(p)}(x_p)\right) - \left(x_{p+1}, f_{w_\pm(p+1)}(x_{p+1})\right) \right\|.$$

Here, $w_+(p)$ (resp. $w_-(p)$) alternates sign starting with $w_+(1) = +$ (resp. $w_-(1) = -$). Also, $x_{2r+1} = x_1$.

Oscillatory integral representation of the wave trace; quick glance

Suppose γ^r is non-degenerate and its length $2rL$ is simple.

Zelditch: Modulus $\sum_j k^{-j}(\mathcal{J}^{2j-2}f)$, one has

$$\int_0^\infty \hat{\rho}(t)e^{ikt}w_B^\Omega(t)\,dt = \sum_\pm \int_{[-\epsilon,\epsilon]^{2r}} e^{ik\mathcal{L}_{\pm,2r}(x_1,\ldots,x_{2r})}a_{\pm,r}^{pr}(k,x_1,\ldots,x_{2r})dx$$

$a_{\pm,r}^{pr}(k,x_1,\ldots,x_{2r})$ can be expressed in terms of $\mathcal{L}_{\pm,2r}$ and the *Hankel amplitude*.

Eigenvalues of $H_{2r} := \text{Hess}\, \mathcal{L}_{+,2r}(0)$

We are only interested in $r = 1$ and $r = 2$. Recall that $a = -2(1 + Lf''(0))$.

$$H_2 = \frac{-1}{L}\begin{pmatrix} a & 2 \\ 2 & a \end{pmatrix} \qquad H_4 = \frac{-1}{L}\begin{pmatrix} a & 1 & 0 & 1 \\ 1 & a & 1 & 0 \\ 0 & 1 & a & 1 \\ 1 & 0 & 1 & a \end{pmatrix}$$

The eigenvalues of H_2: $a + 2, a - 2$.

The eigenvalues of H_4: $a + 2, a, a, a - 2$.

Signature of H_{2r}

It follows that

$$\operatorname{sgn} H_2 = \begin{cases} 0 & \text{(elliptic case)}, \\ 2 & \text{(hyperbolic case and } a > 2), \\ -2 & \text{(hyperbolic case and } a < -2). \end{cases}$$

$$\operatorname{sgn} H_4 = \begin{cases} 2 & \text{(elliptic case) and } 0 < a < 2, \\ -2 & \text{(elliptic case) and } -2 < a < 0, \\ 4 & \text{(hyperbolic case and } a > 2), \\ -4 & \text{(hyperbolic case and } a < -2). \end{cases}$$

This shows that α and $f''(0)$ are spectral invariants. In particular, a, and thus H_{2r}^{-1}, are also spectral invariants.

Wave trace invariants as a corollary of a more general theorem of Zelditch

Theorem

The wave trace invariant for γ^r are given by

$$b_{\gamma^r,j-1} = 4Lr\mathcal{A}_r(0)\Big(2r\widetilde{C}_j\,(h_{2r}^{11})^j\,f^{(2j)}(0)$$

$$-\tfrac{8rL}{a+2}C_j\,(h_{2r}^{11})^j\,f^{(3)}(0)f^{(2j-1)}(0)$$

$$+8r\widehat{C}_j\,(h_{2r}^{11})^{j-2}\textstyle\sum_{q=1}^{2r}(h_{2r}^{1q})^3\,f^{(3)}(0)f^{(2j-1)}(0)\Big)$$

$$+\textit{polynomial in terms of } f^{(\leq 2j-2)}(0).$$

The function $\mathcal{A}_0(r)$ is a non-zero function of r and is independent of Ω and j. $\widetilde{C}_j$, C_j, and $\widehat{C}_j$ are non-zero positive combinatorial constants that depend only on j. The constants h_{2r}^{pq} are the entries of H_{2r}^{-1}.

It follows that,

$$
\begin{aligned}
b'_{\gamma^r, j-1} \quad := \quad & (h^{11}_{2r})^2 \Big(\widetilde{C}_j f^{(2j)}(0) - \tfrac{4L}{a+2} C_j f^{(3)}(0) f^{(2j-1)}(0) \Big) \\
& + \textstyle\sum_{q=1}^{2r} (h^{1q}_{2r})^3 \Big(4 \widehat{C}_j \ \ f^{(3)}(0) f^{(2j-1)}(0) \Big) \\
& + \text{polynomial in terms of } f^{(\leq 2j-2)}(0),
\end{aligned}
$$

is a spectral invariant for each j. We claim that

$$
G(r) := \frac{\sum_{q=1}^{2r} (h^{1q}_{2r})^3}{(h^{11}_{2r})^2},
$$

is non-constant in r. We solve the equation $G(1) = G(2)$:

$$
\tfrac{a^3-8}{(a^2-4)^3} \tfrac{(a^2-4)^2}{a^2} = \tfrac{(a^4-4a^2)^2}{(a^3-2a)^2} \tfrac{a^9-6a^7-2a^6+12a^5}{(a^4-4a^2)^3}.
$$

The roots are $\{0, -1, 2, -2\}$.
Recall that

$$
|a| = 2|1 + Lf''(0)| = \begin{cases} 2\cos(\alpha/2) & \text{(elliptic case)}, \\ 2\cosh(\alpha/2) & \text{(hyperbolic case)}, \end{cases}
$$

Recovering $f^{(3)}(0), f^{(4)}(0), f^{(5)}(0), \ldots$

Allowing $j = 2$, we get that $(f^{(3)}(0))^2$ is a spectral invariant. WLOG we assume $f^{(3)}(0) > 0$.

It then follows that $f^{(4)}(0)$ is determined.

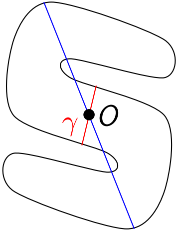
Arguing by induction from $j \to j + 1$, we assume that the $2j - 2$ jet $\mathcal{J}^{2j-2} f(0)$ of f at 0 is known.

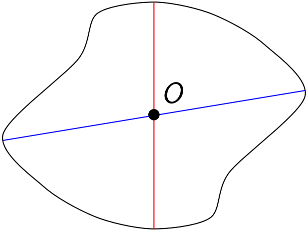
By the decoupling argument we determine $f^{(3)}(0) f^{(2j-1)}(0)$, hence $f^{(2j-1)}(0)$, as long as $f^{(3)}(0) \neq 0$.

Then we can determine $f^{(2j)}(0)$.

Existence of a bouncing ball orbit

Every smooth simply connected centrally symmetric domain Ω has at least one bouncing ball orbit that goes through O. If in addition Ω is star-shaped about the origin O, then Ω has at least two such bouncing ball orbits.





Consider the maximum and minimum points of $D(P) = d(P, -P)^2$ on $\partial\Omega$